EECS 16B Designing Information Devices and Systems II
Fall 2016 Murat Arcak and Michel Maharbiz Homework 11

This homework is due November 14, 2016, at Noon.

1. Homework process and study group

(a) Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.)

(b) How long did you spend working on this homework? How did you approach it?

2. Lagrange interpolation by polynomials

Given n distinct points and the corresponding evaluations/sampling of a function $f(x)$, $(x_i, f(x_i))$ for $0 \leq i \leq n-1$, the Lagrange interpolating polynomial is the polynomial of the least degree which passes through all the given points.

Given n distinct points and the corresponding evaluations, $(x_i, f(x_i))$ for $0 \leq i \leq n-1$, the Lagrange polynomial is

$$P_n(x) = \Sigma_{i=0}^{i=n-1} f(x_i) L_i(x),$$

where

$$L_i(x) = \Pi_{j=0;j\neq i}^{j=n-1} \frac{(x-x_j)}{(x_i-x_j)} = \frac{(x-x_0)}{(x_i-x_0)} \cdots \frac{(x-x_{i-1})}{(x_i-x_{i-1})} \frac{(x-x_{i+1})}{(x_i-x_{i+1})} \cdots \frac{(x-x_{n-1})}{(x_i-x_{n-1})}$$

.

Here is an example: for two data points, $(x_0, f(x_0))$ = (0, 4), $(x_1, f(x_1))$ = (-1, -3), we have

$$L_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-(-1)}{0-(-1)} = x+1$$

and

$$L_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-(0)}{(-1)-(0)} = -x$$

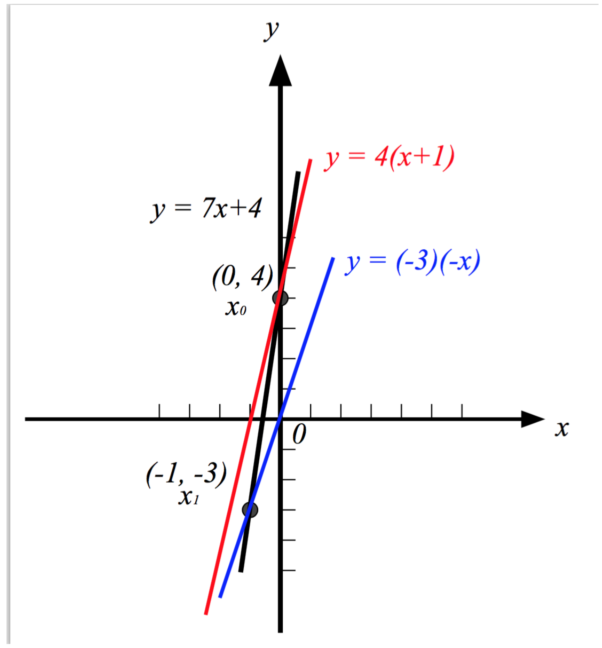
. Then

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) = 4(x+1) + (-3)(-x) = 7x+4$$

We can sketch those equations on the 2D plane as follows:

(a) Given three data points, (2, 3), (0, -1) and (-1, -6), find a polynomial $f(x) = ax^2 + bx + c$ fitting the three points. Do this by solving a system of linear equations for the unknowns a, b, c. Is this polynomial unique?

(b) Like the monomial basis $\{1, x, x^2, x^3, \ldots\}$, the set $\{L_i(x)\}$ is a new basis for the subspace of degree n or lower polynomials. $P_n(x)$ is the sum of the scaled basis polynomials. Find the $L_i(x)$ corresponding to the three sample points in (a). Show your steps.



(c) Find the Lagrange polynomial $P_n(x)$ for the three points in (a). Compare the result to the answer in (a). Are they different from each other? Why or why not?

(d) Sketch $P_n(x)$ and each $f(x_i)L_i(x)$ on the 2D plane.

(e) Show that the Lagrange interpolating polynomial must pass through all given points. In other words, show that $P_n(x_i) = f(x_i)$ for all x_i. Do this in general, not just for the example above.

3. The vector space of polynomials

A polynomial of degree at most n on a single variable can be written as

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots p_n x^n$$

where we assume that the coefficients $p_0, p_1, \ldots, p_n$ are real. Let P_n be the vector space of all polynomials of degree at most n.

(a) Consider the representation of $p \in P_n$ as the vector of its coefficients in $\mathbb{R}^{n+1}$.

$$\vec{p} = \begin{bmatrix} p_0 & p_1 & \ldots & p_n \end{bmatrix}^T$$

Show that the set $\mathscr{B}_n = \{1, x, x^2, \ldots, x^n\}$ forms a basis of P_n, by showing the following.

- Every element of P_n can be expressed as a linear combination of elements in $\mathscr{B}_n$.
- No element in $\mathscr{B}_n$ can be expressed as a linear combination of the other elements of $\mathscr{B}_n$. (*Hint*: Use the aspect of the fundamental theorem of algebra which says that a nonzero polynomial of degree n has at most n roots, and use a proof by contradiction.)

(b) Suppose that the coefficients $p_0, \ldots, p_n$ of p are unknown. To determine the coefficients, we evaluate p on $n+1$ points, $x_0, \ldots, x_n$. Suppose that $p(x_i) = y_i$ for $0 \le i \le n$. Find a matrix V in terms of the x_i, such that

$$V \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

(c) For the case where $n = 2$, compute the determinant of V and show that it is equal to

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i).$$

Conclude that if $x_0, \ldots, x_n$ are distinct, then we can uniquely recover the coefficients $p_0, \ldots, p_n$ of p. This holds for $n > 2$ in general, but consider only the case where $n = 2$ for now.

(d) (optional) Argue using Lagrange interpolation that indeed such matrices V above must always be invertible if the x_i are distinct.

(e) We can define an inner product on P_n by setting

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Show that this satisfies the following properties of a real inner product. (We would have to put in a complex conjugate on p if we wanted a complex inner product.)

- $\langle p, p \rangle \ge 0$, with equality if and only if $p = 0$.
- For all $a \in \mathbb{R}$, $\langle ap, q \rangle = a\langle p, q \rangle$.
- $\langle p, q \rangle = \langle q, p \rangle$.

(f) Now that we have an inner product on P_n, we can consider orthonormality. If $\mathscr{B} = \{b_0, b_1, \ldots, b_n\}$ is a basis for P_n, we say that it is an *orthonormal* basis if

- $\langle b_i, b_j \rangle = 0$ if $i \neq j$.
- $\langle b_i, b_i \rangle = 1$.

We can also compute projections. For any $p, u \in P_n, u \neq 0$, the projection of p onto u is

$$\text{proj}_u p = \frac{\langle p, u \rangle}{\langle u, u \rangle} u.$$

Consider the case where $n = 2$. From part (a), we have the basis $\{1, x, x^2\}$ for P_2. Convert this into an orthonormal basis using the Gram-Schmidt process.

(g) (optional) An alternative inner-product could be placed upon real polynomials if we simply represented them by a sequence of their evaluations at $0, 1, \ldots, n$ and adopted the standard Euclidean inner product on sequences of real numbers. Can you give an example of an orthonormal basis with this alternative inner product?

4. Sampling a continuous-time control system to get a discrete-time control system

The goal of this problem is to help us understand how given a linear continuous-time system:

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$
$$\vec{y}(t) = C\vec{x}(t)$$

we can sample it every T seconds and get a discrete-time form of the control system. The discretization of the state equations is a *sampled* discrete time-invariant system given by

$$\vec{x}_d(k+1) = A_d\vec{x}_d(k) + B_d\vec{u}_d(k) \tag{1}$$
$$\vec{y}_d(k) = C_d\vec{x}_d(k) \tag{2}$$

Here, the $\vec{x}_d(k)$ denotes $\vec{x}(kT)$. This is a snapshot of the state. Similarly, the output $\vec{y}_d(k)$ is a snapshot of $\vec{y}(kT)$.

The relationship between the discrete-time input $\vec{u}_d(k)$ and the actual input applied to the physical continuous-time system is that $\vec{u}(t) = \vec{u}_d(k)$ for all $t \in [kT, (k+1)T)$.

While it is clear from the above that the discrete-time state and the continuous-time state have the same dimensions and similarly for the control inputs, what is not clear is what the relationship should be between the matrices A, B and the matrices A_d, B_d. By contrast it is immediately clear that $C_d = C$.

(a) Argue intuitively why if the continuous-time system is stable, the corresponding discrete-time system should be stable too. Similarly, argue intuitively why if the discrete-time system is unstable, then the continuous-time system should also be unstable.

(b) Consider the scalar case where A and B are just constants. What are the new constants A_d and B_d? *(HINT: Think about solving this one step at a time. Every time a new control is applied, this is a simple differential equation with a new constant input. How does $\dot{x}(t) = \lambda x(t) + u$ evolve with time if it starts at $x(0)$? Notice that $x(0)e^{\lambda t} + \frac{u}{\lambda}(e^{\lambda t} - 1)$ seems to solve this differential equation.)*

(c) Consider now the case where A is a diagonal matrix and B is some general matrix. What is the new matrix A_d and B_d?

(d) Consider the case where A is a diagonalizable matrix. Use a change of coordinates to figure out the new matrix A_d and B_d.

(e) Consider a general diagonal matrix A with distinct eigenvalues and a vector $B = \vec{b}$ that consists of all 1s. Is the pair $(A, \vec{b})$ necessarily controllable? Prove that it must be or show a case where it isn't. *(HINT: Polynomials)*

(f) Now consider a 2×2 diagonal matrix A that has the same eigenvalue repeated twice and a vector $B = \vec{b}$. Is it ever possible for the pair $(A, \vec{b})$ to be controllable? Show such a case or prove that it cannot exist.

(g) Now consider the case of complex eigenvalues for a diagonal matrix A (with all the eigenvalues distinct) with a vector $B = \vec{b}$ that consists of all 1s. Can you find a case in which $(A, \vec{b})$ is controllable but (A_d, B_d) is not controllable? What has to be true about the sampling period T in relation to the eigenvalues for this to happen?

5. Aliasing intuition in continuous time

The concept of "aliasing" is intuitively about having a signal of interest whose samples look identical to a different signal of interest — creating an ambiguity as to which signal is actually present.

While the concept of aliasing is quite general, it is easiest to understand in the context of sinusoidal signals.

(a) Consider two signals,

$$x_1(t) = a\cos(2\pi f_0 t + \phi)$$

and

$$x_2(t) = a\cos(2\pi(-f_0 + m f_s)t - \phi)$$

where $f_s = 1/T_s$. Are these two signals the same or different when viewed as functions of continuous time t?

(b) Consider the two signals from the previous part. These will both be sampled with the sampling interval T_s. What will be the corresponding discrete-time signals $x_{d,1}[n]$ and $x_{d,2}[n]$? (The $[n]$ refers to the nth sample taken — this is the sample taken at real time nT_s.) Are they the same or different?

(c) What is the sinusoid $a\cos(\omega t + \phi)$ that has the smallest $\omega \geq 0$ but still agrees at all of its samples (taken every T_s seconds) with $x_1(t)$ above?

Contributors:

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