

This homework is due November 21, 2016, at Noon.

1. Homework process and study group

- (a) Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.)
- (b) How long did you spend working on this homework? How did you approach it?

2. Orthonormal proofs

In this problem, we ask you to establish several important properties of orthonormal bases. This is designed to both sharpen your understanding of these properties as well as practice doing proofs/derivations.

Let $U = [\vec{u}_0 \ \vec{u}_1 \ \cdots \ \vec{u}_{n-1}]$ be an n by n matrix, where its columns $\vec{u}_0, \vec{u}_1, \dots, \vec{u}_{n-1}$ form an orthonormal basis. One example of this is the DFT basis.

- (a) Show that $U^{-1} = U^*$, where U^* is the conjugate transpose of U .
- (b) Show that U preserves inner products, i.e. if \vec{v}, \vec{w} are vectors of length n , then

$$\langle \vec{v}, \vec{w} \rangle = \langle U\vec{v}, U\vec{w} \rangle.$$

Recall that the inner-product is defined to be $\langle \vec{v}, \vec{w} \rangle = \vec{v}^* \vec{w}$.

Also remember that for any matrices A, B or appropriate size so that their multiplication makes sense, that $(AB)^* = B^*A^*$. (This latter fact can be seen by looking at the entry in the i th row and j th column of $(AB)^*$. This is complex conjugate of the entry in the j th row and i th column of AB . This entry is just $A_j \vec{b}_i$ where A_j is the j th row of A and \vec{b}_i is the i th column of B . Notice that the complex conjugate of this is just $\vec{b}_i^* A_j^*$ since the complex conjugate of a product is just the product of complex conjugates (most easily seen in polar form) and the complex conjugate of a sum is the sum of complex conjugates. This is thus the i th row and j th column of the product B^*A^* .)

This fact is called Parseval's relation when applied in signal processing contexts and it helps us see orthogonality as well as think about energy in different bases.

- (c) Show that $\vec{u}_0, \dots, \vec{u}_{n-1}$ must be linearly independent.
(Hint: Suppose $\vec{w} = \sum_{i=0}^{n-1} \alpha_i \vec{u}_i$, then first show that $\alpha_i = \langle \vec{u}_i, \vec{w} \rangle$. From here ask yourself whether a nonzero linear combination of the $\{\vec{u}_i\}$ could ever be identically zero.)
This basic fact shows how orthogonality is a very nice special case of linear independence.
- (d) Let M be a matrix which can be diagonalized by U , i.e. $M = U\Lambda U^*$, where Λ is a diagonal matrix with the eigenvalues $\lambda_0, \dots, \lambda_{n-1}$ along the diagonal. Show that M^* has the same set of eigenvectors U , while the eigenvalues of M^* are $\lambda_0^*, \dots, \lambda_{n-1}^*$. (Aside: think about in which other problem on this HW this fact might be useful.)
- (e) Let V be another n by n matrix, where the columns also form an orthonormal basis. Show that the columns of the product, UV , also form an orthonormal basis. (This will turn out to be very helpful when we are defining a two-dimensional DFT or thinking image processing generally.)

3. DFT properties

- (a) Suppose the DFT of a length- N sequence $x(t)$, $t = 0, 1, \dots, N - 1$, is given by $X(k)$, $k = 0, 1, \dots, N - 1$ and a new sequence is defined as

$$y(t) = e^{jm\frac{2\pi}{N}t}x(t).$$

Show that

$$Y(k) = X((k - m) \bmod N)$$

- (b) Let $N = 4$ and $y(t) = (-1)^t x(t)$, $t = 0, 1, 2, 3$. Write $Y(0)$, $Y(1)$, $Y(2)$, $Y(3)$ in terms of $X(0)$, $X(1)$, $X(2)$, $X(3)$.

4. Denoising signals using the DFT

Professor Maharbiz is sad. He just managed to create a beautiful audio clip consisting of a couple pure tones with beats and he wants Professor Arcaak to listen to it. He calls Professor Arcaak on a noisy phone and plays the message through the phone. Professor Arcaak then tells him that the audio is very noisy and that he is unable to truly appreciate the music. Unfortunately, Professor Maharbiz has no other means of letting Professor Arcaak listen to the message. Luckily, they have you! You propose to implement a denoiser at Professor Arcaak's end.

- (a) In the IPython notebook, listen to the noisy message. Plot the time signal and comment on visible structure, if any.
- (b) Take the DFT of the signal and plot the magnitude. In a few sentences, describe what the spikes you see in the spectrum are.
- (c) There is a simple method to denoise this signal: Simply threshold in the DFT domain! Threshold the DFT spectrum by keeping the coefficients whose absolute values lie above a certain value. Then take the inverse DFT and listen to the audio. You will be given a range of possible values to test. Write the threshold value you think works best.

Yay, Professor Maharbiz is no longer sad!

5. Amplitude modulation (AM)

In electronic communication, transmission is achieved by varying *some aspect* of a *higher frequency signal*, (“carrier signal”), with some information-bearing waveform (“base signal”) to be sent, such as an audio or video signal, a process known as “**modulation**”. Amplitude modulation is a specific scheme of modulation, where the amplitude of the carrier oscillations is varied. In this question, you will be led through the mixing of some simple base and carrier signals so you know the magic behind your traditional AM radio.

Recap of DFT: Consider a sampled signal that is a function of discrete time $x[t]$. We can represent it as a vector of discrete samples over time \vec{x} , of length n .

$$\vec{x} = [x[0] \quad \dots \quad x[n-1]]^T \quad (1)$$

Let $\vec{X} = [X[0] \quad \dots \quad X[n-1]]^T$ be the signal \vec{x} represented in the frequency domain, that is

$$\vec{X} = U^{-1}\vec{x} = U^*\vec{x} \quad (2)$$

where U is a matrix of the DFT basis vectors.

$$U = \begin{bmatrix} | & & | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{n-1} & & \\ | & & | & & | \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{\frac{2\pi i}{n}} & e^{\frac{2\pi i(2)}{n}} & \cdots & e^{\frac{2\pi i(n-1)}{n}} \\ 1 & e^{\frac{2\pi i(2)}{n}} & e^{\frac{2\pi i(4)}{n}} & \cdots & e^{\frac{2\pi i2(n-1)}{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{\frac{2\pi i(n-1)}{n}} & e^{\frac{2\pi i2(n-1)}{n}} & \cdots & e^{\frac{2\pi i(n-1)(n-1)}{n}} \end{bmatrix} \quad (3)$$

Alternatively, we have that $\vec{x} = U\vec{X}$ or more explicitly

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \quad (4)$$

In other words, \vec{x} can be always represented as a linear combination of the DFT basis signals \vec{u}_m with coefficients $X[m]$.

- (a) Let n be the number of samples in a vector. Suppose we have a signal that is a complex exponential, $s[t] = e^{i\frac{2\pi Kt}{n}}$, which has a frequency of $\frac{K}{n}$ for some integer K and the number of samples n . In vector form it is given by:

$$e^{i\frac{2\pi Kt}{n}} \Rightarrow \left[1 \quad e^{i\frac{2\pi K}{n}} \quad e^{i\frac{2\pi K(2)}{n}} \quad \cdots \quad e^{i\frac{2\pi K(n-1)}{n}} \right]^T = \sqrt{n}\vec{u}_K \quad (5)$$

Note, that this is a scaled version of the K th DFT basis vector.

Mixing two signals is defined to be *the process of element-wise multiplying them together*, i.e. computing $e^{i\frac{2\pi Kt}{n}} x[t]$. This amounts to *element-wise multiplication* \odot :

$$e^{i\frac{2\pi Kt}{n}} x[t] = (\sqrt{n}\vec{u}_K \odot \vec{x})[t] \quad (6)$$

The idea is, instead of transmitting the signal \vec{x} directly, we would transmit the mixed signal. The mixing process is known as modulation. There are two questions you probably have are: how can we demodulate a signal? and, what does amplitude modulation do for us? You will know both answers by the end of this problem. But you don't have to answer them now.

Let's start simple. Suppose our signal $x(t)$ is a simple complex exponential with frequency $\frac{k}{n}$, that is $\vec{x} = \vec{u}_k$.

What is \vec{x} written in the frequency domain? Now mix \vec{x} with the complex signal $\sqrt{n}\vec{u}_K$. What is $\sqrt{n}\vec{u}_K \odot \vec{x}$? What is this new signal written in the frequency domain? What does mixing with a complex exponential do to the frequency of the signal?

- (b) Now suppose that \vec{x} is a cosine signal, that is $x[t] = \cos(2\pi\frac{k}{n}t)$. Mix this new \vec{x} with the complex exponential $\vec{s} = \sqrt{n}\vec{u}_K$. What is the frequency domain representation of this new signal and how does it differ from \vec{x} ?
- (c) With the above $x[t] = \cos(2\pi\frac{k}{n}t)$, let's go one step further by mixing it with a carrier signal \vec{s} , where $s[t] = \cos(2\pi\frac{K}{n}t)$.

This is practically important because complex-exponentials need to be realized in a physical way that is real to build real-world AM radios.

What is the frequency domain representation of this new signal and how does it differ from \vec{x} ?

Assume here that K is much bigger than k and in-turn, n is much bigger than K .

- (d) Now that you see the effect of mixing the base signal with a cosine wave as the carrier signal. It is natural to wonder how can we undo this process. The most obvious answer — divide through by the cosine — suffers from the problem of having to divide by numbers that might be close to zero. Dividing by very small numbers is generally a bad idea in engineering contexts.
- The technique that is used is called **de-modulation**. Now, mix $\cos\left(\frac{2\pi Kt}{n}\right)$ with the signal you get from the previous question. How many resulting terms do you get in the frequency domain? Considering the fact that $K \gg k$, what are the frequencies relative positions on the frequency spectrum? Have we recovered the original signal?
- If not, what additional steps would we have to take? (*Hint: is there a subspace in which we have correctly recovered the original signal? How can we just keep that subspace?*)
- (e) Refer to the iPython notebook for the implementations. Comment on what you see.
- (f) Now Assume that we want to send two signals, $x_1[t] = \cos(2\pi\frac{k_1}{n}t)$ and $x_2[t] = \cos(2\pi\frac{k_2}{n}t)$. Assume that the carrier frequency $K \gg k_1$ and $K \gg k_2$. If we modulate the first signal by $s_1[t] = \cos(2\pi\frac{K}{n}t)$ and the second signal by $s_2[t] = \cos(2\pi\frac{2K}{n}t)$ and add them together, can we recover both signals?

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