This homework is due September 12, 2016, at Noon.

1. Homework process and study group

(a) Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.)

Solution: I worked on this homework with...

(b) How long did you spend working on this homework? How did you approach it?

Solution: I spent a total of X hours working on this assignment.

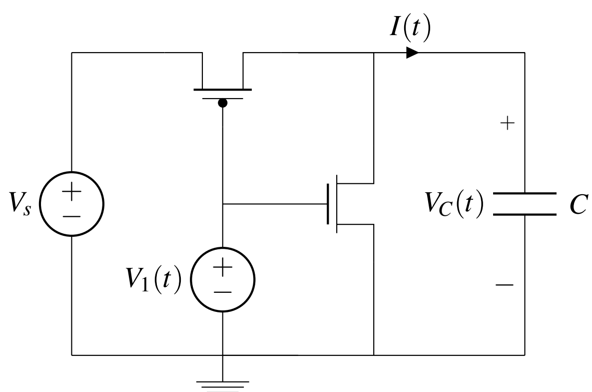
I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

For self-grading, be sure to enter the number of hours alone in the comment field for this problem. This will help us better automatically collect this information.

2. Transistors as switches

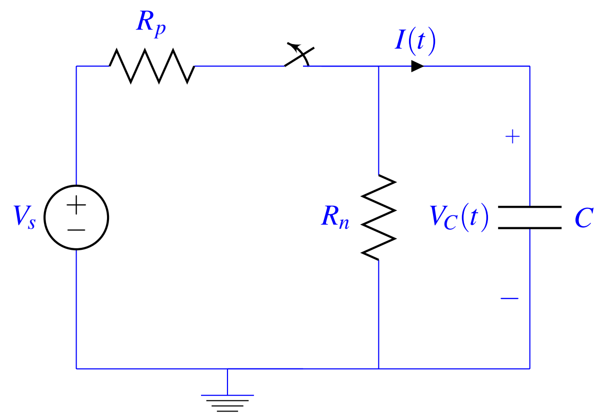
Consider the transistor and capacitor switch circuit below:



In this circuit, the source of the PMOS transistor is connected to the positive node of V_s and the source of the NMOS transistor is connected to ground. $|V_{th}| = 2\text{V}$ for both transistors. $R_p = R_n = 1k\Omega$ for the transistor resistances and $V_s = 5\text{V}$. $V_1(t)$ is defined as follows:

$$V_1(t) = \begin{cases} 5\text{V for } t \leq 0 \\ 0\text{V for } t > 0 \end{cases}$$

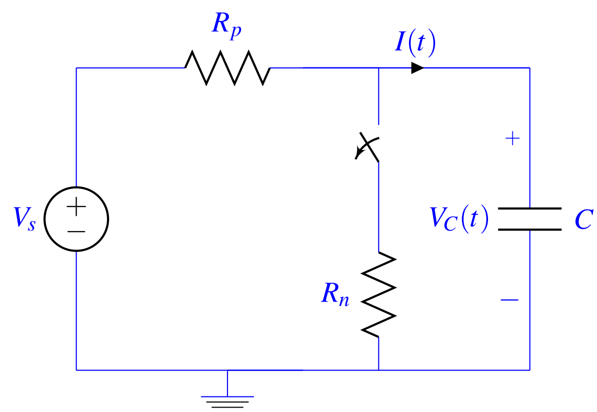
(a) For $t \leq 0$, Redraw the circuit with the transistors replaced with their resistor switch models. Assuming this circuit has reached steady state, what is $V_C(0)$?



Solution: $t \leq 0$

Since this circuit is a capacitor connected to ground via a resistor, at steady state after a long period of time, the capacitor should completely discharge. Therefore, $V_C(0) = 0$

(b) For $t > 0$, Redraw the circuit with the transistors replaced with their resistor switch models.



Solution: $t > 0$

(c) Find the differential equation that models this circuit for $t > 0$.

Solution: First, we start with the current through a capacitor:

$$I(t) = C\frac{dV_C(t)}{dt}$$

We can also find an equation for $I(t)$ from ohms law:

$$I(t) = \frac{V_s - V_C(t)}{R_p}$$

Therefore:

$$\frac{V_s - V_C(t)}{R_p} = C\frac{dV_C(t)}{dt}$$

$$\frac{dV_C(t)}{dt} + \frac{V_C(t)}{R_pC} = \frac{V_s}{R_pC}$$

(d) Solve for $V_C(t)$. Starting from $t = 0$, how long does it take for $V_C(t)$ to be within 1% of its final value?

Solution: We see this is an inhomogeneous differential equation, so we can use the method from lecture where we find the steady state and initial voltages and write it in the form:

$$V_C(t) = V_C(\infty) + [V_C(0) - V_C(\infty)]e^{-\frac{t}{R_pC}}$$

Eventually, this capacitor will be fully charged, so we know $V_C(\infty) = V_s$. Plugging in our values for this circuit:

$$V_s + [0 - V_s]e^{-\frac{t}{R_pC}}$$

$$5(1 - e^{-\frac{t}{1000C}})\ \text{V}$$

An RC circuit will take 5τ to reach 1% of its final value. $\tau = 1000C$ for this circuit, so it will take $5000C$ s. For example, if C were 1 pF, it would take 1 ns to settle.

3. Matrix differential equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{1}$$

where x, y are variables depending on t, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, and A is a 2×2 matrix with constant coefficients. We call (1) a matrix differential equation.

(a) Suppose we have a system of ordinary differential equations

$$x(t)' = 8x(t) + 7y(t) \tag{2}$$
$$y(t)' = -4x(t) - 3y(t) \tag{3}$$

Write this in the form of (1).

Solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ -4 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4}$$

(b) Compute the eigenvalues and eigenvectors of the matrix A from the previous part.

Solution: The characteristic polynomial of A is

$$\det \begin{pmatrix} 8-\lambda & 7 \\ -4 & -3-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4).$$

Thus the eigenvalues of A are $\lambda = 1, 4$, with corresponding eigenvectors $(1, -1)^T$ and $(7, -4)^T$ respectively.

(c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_0 e^{\lambda_0 t} \vec{v}_0 + c_1 e^{\lambda_1 t} \vec{v}_1,$$

where c_0, c_1 are constants, and λ_0, λ_1 are the eigenvalues of A with eigenvectors $\vec{v}_0, \vec{v}_1$ respectively. Suppose that the initial conditions are $x(0) = 1, y(0) = 1$. Solve for the constants c_0, c_1.

Solution: Substituting the eigenvalues and eigenvectors computed in the previous part, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_0 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_1 e^{4t} \begin{pmatrix} 7 \\ -4 \end{pmatrix}.$$

At $t = 0$, we have

$$1 = c_0 + 7c_1 \tag{5}$$
$$1 = -c_0 - 4c_1 \tag{6}$$

This gives $c_0 = -\frac{11}{3}$ and $c_1 = \frac{2}{3}$. Thus we have

$$x(t) = -\frac{11}{3}e^t + \frac{14}{3}e^{4t} \tag{7}$$
$$y(t) = \frac{11}{3}e^t - \frac{8}{3}e^{4t} \tag{8}$$

(d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (2), (3).

Solution: We compute the derivative of x with respect to t in (7) to get

$$x'(t) = -\frac{11}{3}e^t + \frac{56}{3}e^{4t}.$$

The right hand side of (2) is

$$8x + 7y = -\frac{88}{3}e^t + \frac{112}{3}e^{4t} + \frac{77}{3}e^t - \frac{56}{3}e^{4t} = -\frac{11}{3}e^t + \frac{56}{3}e^{4t}$$

hence our solution for $x(t)$ satisfies (2).

Similarly, we compute the derivative of y with respect to t in (8) to get

$$y'(t) = \frac{11}{3}e^t - \frac{32}{3}e^{4t}.$$

The right hand side of (3) is

$$-4x - 3y = \frac{44}{3}e^t - \frac{56}{3}e^{4t} - \frac{33}{3}e^t + \frac{24}{3}e^{4t} = \frac{11}{3}e^t - \frac{32}{3}e^{4t}$$

hence our solution for $y(t)$ satisfies (3).

(e) We now apply the method above to solve a second order ordinary differential equation. Suppose we have the system

$$z''(t) - 5z'(t) + 6z(t) = 0 \tag{9}$$

Write this in the form of (1), by using the change of variables $x(t) = z(t)$, $y(t) = z'(t)$.

Solution: If we set $x(t) = z(t)$, $y(t) = z'(t)$, then we have

$$x'(t) = z'(t) = y(t) \tag{10}$$
$$y'(t) = z''(t) = 5z'(t) - 6z(t) = 5y(t) - 6x(t) \tag{11}$$

We can write this in the form of (1) as follows

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{12}$$

(f) Solve the system in (9) with the initial conditions $z(0) = 1$, $z'(0) = 1$, using the method developed in parts (b) and (c).

Solution: We first compute the eigenvalues and eigenvectors of the matrix from the previous part. The characteristic polynomial is

$$\det\begin{pmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)\,.$$

Thus the eigenvalues are $\lambda = 2, 3$, with corresponding eigenvectors $(1,2)^T$ and $(1,3)^T$ respectively. From part (c), the solution for $x(t), y(t)$ is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_0 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_1 e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\,.$$

At $t = 0$, we have

$$1 = c_0 + c_1 \tag{13}$$
$$1 = 2c_0 + 3c_1 \tag{14}$$

This gives $c_0 = 2$ and $c_1 = -1$. Thus we have

$$x(t) = 2e^{2t} - e^{3t} \tag{15}$$
$$y(t) = 4e^{2t} - 3e^{3t} \tag{16}$$

Hence we have the solution

$$z(t) = 2e^{2t} - e^{3t}\,.$$

4. Solving second-order differential equations

Solve the following second-order differential equation:

$$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) - 3y(t) = 0$$

where the initial conditions are $y(0) = 0$, $\frac{d}{dt}y(0) = 1$.

(a) Firstly, please write the above equations into the matrix form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$. How do you define $\vec{x}(t)$, A, and the initial conditions $\vec{x}(0)$?

Solution:

We can define the vector variable $\vec{x}(t) = \begin{bmatrix} y(t) \\ \frac{d}{dt}y(t) \end{bmatrix}$, because then if we take the first derivative of $\vec{x}$ we can obtain second derivatives. The second order differential equation can be written as:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\vec{x}(t) \tag{17}$$

where the first row of matrix A is given by definition, and the second row is exactly given by the second order differential equation.

(b) What are the eigenvalues and eigenvectors of A?

Solution:

First, we write out the characteristic equation:

$$det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0 \tag{18}$$

Therefore, the eigenvalues are $\lambda_0 = 3$, $\lambda_1 = -1$. Now let's find the eigenvectors for each case. For $\lambda_0 = 3$, define the vector $\begin{bmatrix} u \\ v \end{bmatrix}$,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ 3u+2v \end{bmatrix} = 3 \begin{bmatrix} u \\ v \end{bmatrix} \tag{19}$$

Therefore we have $v = 3u$, in other words, $\vec{v}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is the corresponding eigenvector. Similarly, for $\lambda_1 = -1$, we have:

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ 3u+2v \end{bmatrix} = - \begin{bmatrix} u \\ v \end{bmatrix} \tag{20}$$

Therefore we have $v = -u$, in other words, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the corresponding eigenvector.

(c) Based on the above results, what is the solution to the second-order equation?

Solution:

The general solution to the first order differential equation in the matrix form, as discussed in the lecture, is:

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{21}$$

where c_1 and c_2 are constants that need to be determined. Using the initial conditions that $\vec{x}(0) = \begin{bmatrix} y(0) \\ \frac{d}{dt}y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have

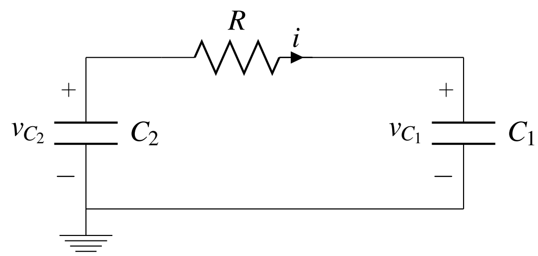
$$c_1 + c_2 = 0$$
$$-c_1 + 3c_2 = 1$$

where we know that $c_1 = -\frac{1}{4}, c_2 = \frac{1}{4}$. Therefore, we have $\vec{x}(t) = -\frac{1}{4}e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{4}e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. From the above, we can see that the solution for the second order differential equation is:

$$y(t) = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t}. \tag{22}$$

5. Charge sharing, revisited

Consider the following circuit which features two capacitors connected together with a resistor. With the exception of the resistor in between, you should recognize this as the idea of charge sharing from EE 16A which you used in your capacitive touchscreen lab. In this problem, we will take a more in-depth look at this charge sharing circuit to make it more rigorous and intuitive on a circuit level.



We will analyze this circuit for $t \geq 0$. The initial voltage across C_2 is given as V_2, and the initial voltage across C_1 is given as V_1. In other words:

$$v_{C_1}(t \leq 0) = V_1$$
$$v_{C_2}(t \leq 0) = V_2$$

(a) Determine a formula for $v_{C_1}(t)$ for $t \geq 0$.

Solution: Here we provide two different approaches to solve (a) and (b).

From KCL and the capacitor equations:

$$i(t) = C_1 \frac{d}{dt} v_{C_1}(t) = -C_2 \frac{d}{dt} v_{C_2}(t) \tag{23}$$

From KVL and Ohm's law:

$$v_{C_2} = i(t)R + v_{C_1}$$

Isolating $i(t)$:

$$i(t) = \frac{1}{R}(-v_{C_1} + v_{C_2}) \tag{24}$$

Combining equations 23 and 24:

$$\frac{d}{dt} v_{C_1}(t) = \frac{1}{RC_1}(-v_{C_1} + v_{C_2})$$
$$\frac{d}{dt} v_{C_2}(t) = \frac{1}{RC_2}(v_{C_1} - v_{C_2})$$

Rewrite in matrix form:

$$\begin{bmatrix} \frac{d}{dt} v_{C_1}(t) \\ \frac{d}{dt} v_{C_2}(t) \end{bmatrix} = \frac{1}{R} \begin{bmatrix} -\frac{1}{C_1} & \frac{1}{C_1} \\ \frac{1}{C_2} & -\frac{1}{C_2} \end{bmatrix} \begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix}$$

The next few steps might get a little hairy. To prevent that we'll simplify to get rid of the fractions inside the matrix by multiplying and dividing by $-C_1C_2$:

$$\begin{bmatrix} \frac{d}{dt} v_{C_1}(t) \\ \frac{d}{dt} v_{C_2}(t) \end{bmatrix} = -\frac{1}{RC_1C_2} \overbrace{\begin{bmatrix} C_2 & -C_2 \\ -C_1 & C_1 \end{bmatrix}}^{A} \begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix}$$

The next step is to diagonalize the matrix. Again, we will try to simplify the math, so we'll diagonalize the matrix A and multiply the eigenvalues by $-\frac{1}{RC_1C_2}$ later.

$$0 = \det(A - \lambda I) = (C_2 - \lambda)(C_1 - \lambda) - C_1C_2 = C_2C_1 - \lambda C_1 - \lambda C_2 + \lambda^2 - C_1C_2 = \lambda(\lambda - (C_1 + C_2))$$

Therefore the eigenvalues of A are 0 and $C_1 + C_2$

$$A - 0I = \begin{bmatrix} C_2 & -C_2 \\ -C_1 & C_1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 * C_2 + R_0 * C_1} \begin{bmatrix} C_2 & -C_2 \\ 0 & 0 \end{bmatrix}$$

We'll select $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ as the eigenvector for 0.

$$A - (C_1 + C_2)I = \begin{bmatrix} C_2 - (C_1 + C_2) & -C_2 \\ -C_1 & C_1 - (C_1 + C_2) \end{bmatrix} = \begin{bmatrix} -C_1 & -C_2 \\ -C_1 & -C_2 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_0} \begin{bmatrix} -C_1 & -C_2 \\ 0 & 0 \end{bmatrix}$$

We could select $\begin{bmatrix} -\frac{C_2}{C_1} & 1 \end{bmatrix}^T$ as the eigenvector for $C_1 + C_2$. But the next steps would be simpler if we choose a simpler eigenvector, such as $\begin{bmatrix} C_2 & -C_1 \end{bmatrix}^T$. We still need to multiply the eigenvectors by $-\frac{1}{RC_1C_2}$ so we get:

$$\lambda_1 = 0 \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -\frac{C_1 + C_2}{RC_1C_2} \quad v_2 = \begin{bmatrix} C_2 \\ -C_1 \end{bmatrix}$$

The solution to the differential equation is, as we've seen in class:

$$\begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix} = \begin{bmatrix} 1 & C_2 \\ 1 & -C_1 \end{bmatrix} \begin{bmatrix} K_1 e^{\lambda_1 t} \\ K_2 e^{\lambda_2 t} \end{bmatrix} \tag{25}$$

Substituting the values for $t = 0$:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & C_2 \\ 1 & -C_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

We'll solve for K_1, K_2 by Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & C_2 & V_1 \\ 1 & -C_1 & V_2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 C_2 + R_0 C_1} \left[\begin{array}{cc|c} 1 & C_2 & V_1 \\ C_2 + C_1 & 0 & V_2 C_2 + V_1 C_1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{C_1 + C_2} R_1} \left[\begin{array}{cc|c} 1 & C_2 & V_1 \\ 1 & 0 & \frac{V_2C_2 + V_1C_1}{C_2 + C_1} \end{array}\right]$$

$$\xrightarrow{R_0 \leftarrow R_0 - R_1} \left[\begin{array}{cc|c} 0 & C_2 & V_1 - \frac{V_2C_2 + V_1C_1}{C_2 + C_1} \\ 1 & 0 & \frac{V_2C_2 + V_1C_1}{C_2 + C_1} \end{array}\right] = \left[\begin{array}{cc|c} 0 & C_2 & \frac{V_1C_2 + V_1C_1 - V_2C_2 - V_1C_1}{C_2 + C_1} \\ 1 & 0 & \frac{V_2C_2 + V_1C_1}{C_2 + C_1} \end{array}\right] = \left[\begin{array}{cc|c} 0 & C_2 & \frac{V_1C_2 - V_2C_2}{C_2 + C_1} \\ 1 & 0 & \frac{V_2C_2 + V_1C_1}{C_2 + C_1} \end{array}\right]$$

$$\xrightarrow{R_0 \leftarrow \frac{1}{C_2} R_0} \left[\begin{array}{cc|c} 0 & 1 & \frac{V_1 - V_2}{C_2 + C_1} \\ 1 & 0 & \frac{V_2C_2 + V_1C_1}{C_2 + C_1} \end{array}\right]$$

And finally, we substitute the values of K_1 and K_2 into equation 25:

$$v_{C_1}(t) = \frac{V_2C_2 + V_1C_1}{C_2 + C_1} + C_2 \frac{V_1 - V_2}{C_2 + C_1} e^{-\frac{C_1 + C_2}{RC_1C_2} t}$$
$$v_{C_2}(t) = \frac{V_2C_2 + V_1C_1}{C_2 + C_1} + C_1 \frac{V_2 - V_1}{C_2 + C_1} e^{-\frac{C_1 + C_2}{RC_1C_2} t}$$

Solution: Alternatively, let us first begin with Ohm's law across the resistor R:

$$v_{C_2} - v_{C_1} = iR$$
$$i = \frac{v_{C_2} - v_{C_1}}{R}$$

We can relate this current to the capacitors using the capacitor equation $i_c = C\frac{dv}{dt}$:

$$i = C_1 \frac{dv_{C_1}}{dt}$$
$$-i = C_2 \frac{dv_{C_2}}{dt}$$

We'll use these equations as a starting point for analysis. Firstly, let us define $v \triangleq v_{C_2} - v_{C_1}$. Continuing:

$$i + (-i) = 0$$

$$C_1 \frac{dv_{C_1}}{dt} + C_2 \frac{dv_{C_2}}{dt} = 0$$

$$C_1 \frac{d(v_{C_2} - v)}{dt} + C_2 \frac{d(v + v_{C_1})}{dt} = 0$$

$$C_1 (\frac{dv_{C_2}}{dt} - \frac{dv}{dt}) + C_2 (\frac{dv}{dt} + \frac{dv_{C_1}}{dt}) = 0 \text{ (linearity of differential operator)}$$

$$C_1 \frac{dv_{C_2}}{dt} - C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_2 \frac{dv_{C_1}}{dt} = 0$$

$$C_1 \frac{dv_{C_2}}{dt} + C_2 \frac{dv_{C_1}}{dt} + C_2 \frac{dv}{dt} - C_1 \frac{dv}{dt} = 0$$

$$C_1 \frac{-i}{C_2} + C_2 \frac{i}{C_1} + C_2 \frac{dv}{dt} - C_1 \frac{dv}{dt} = 0 \text{ (capacitor equations)}$$

$$C_1 \frac{-v}{RC_2} + C_2 \frac{v}{RC_1} + C_2 \frac{dv}{dt} - C_1 \frac{dv}{dt} = 0 \text{ (Ohm's law)}$$

$$C_1 \frac{-v}{RC_2} + C_2 \frac{v}{RC_1} = C_1 \frac{dv}{dt} - C_2 \frac{dv}{dt}$$

$$C_1 \frac{-v}{RC_2} + C_2 \frac{v}{RC_1} = (C_1 - C_2) \frac{dv}{dt}$$

$$v(\frac{C_2}{RC_1} - \frac{C_1}{RC_2}) = (C_1 - C_2) \frac{dv}{dt}$$

$$v \frac{\frac{C_2}{RC_1} - \frac{C_1}{RC_2}}{C_1 - C_2} = \frac{dv}{dt}$$

$$v \frac{\frac{C_2}{RC_1}}{C_1 - C_2} - \frac{\frac{C_1}{RC_2}}{C_1 - C_2} = \frac{dv}{dt}$$

$$v \frac{C_2}{RC_1(C_1 - C_2)} - \frac{C_1}{RC_2(C_1 - C_2)} = \frac{dv}{dt}$$

$$v \frac{C_2^2}{RC_1C_2(C_1 - C_2)} - \frac{C_1^2}{RC_1C_2(C_1 - C_2)} = \frac{dv}{dt}$$

$$v \frac{C_2^2 - C_1^2}{RC_1C_2(C_1 - C_2)} = \frac{dv}{dt}$$

$$v(-\frac{C_2^2 - C_1^2}{RC_1C_2(C_2 - C_1)}) = \frac{dv}{dt}$$

$$v(-\frac{(C_2 - C_1)(C_2 + C_1)}{RC_1C_2(C_2 - C_1)}) = \frac{dv}{dt}$$

$$v(-\frac{C_2 + C_1}{RC_1C_2}) = \frac{dv}{dt}$$

$$-(\frac{1}{RC_1} + \frac{1}{RC_2})v = \frac{dv}{dt} \text{ (inspired from the simplification of } \frac{1}{x} + \frac{1}{y})$$

This is a first-order differential equation, so we will proceed to solve it for $[0, t]$.

Observe that $-(\frac{1}{RC_1} + \frac{1}{RC_2})v = \frac{dv}{dt}$ is in the form of $\frac{dv}{dt} = kv$. Hence, $v(t) = Ae^{kt}$ is an solution, since

Ae^{kt} is an eigenfunction of $\frac{d}{dt}$ as shown in lecture. Observe that the constant A must be $v(0)$, since $v(0) = Ae^0 = A$. [1]

$$v(t) = v(0)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$v_{C_2}(t) - v_{C_1}(t) = (v_{C_2}(0) - v_{C_1}(0))e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$v_{C_2}(t) - v_{C_1}(t) = (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

This is an equation for the voltage difference between the two capacitors. We can now relate this back to v_{C_1} independent of v_{C_2} by using Ohm's law and the capacitor equation for C_1 again:

$$v_{C_2}(t) - v_{C_1}(t) = (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$iR = (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$(C_1\frac{dv_{C_1}}{dt})R = (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$RC_1\frac{dv_{C_1}}{dt} = (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$\frac{dv_{C_1}}{dt} = \frac{(V_2 - V_1)}{RC_1}e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

This is a first-order differential equation in v_{C_1}, which we can solve for $[0,t]$. In fact, if we make a few simplifications, nothing beyond single variable calculus is required.

Firstly, let us make some definitions:

[1] Alternatively, this can be also solved using separation of variables:

$$-(\frac{1}{RC_1}+\frac{1}{RC_2})dt = \frac{1}{v}dv$$

$$\int_0^t -(\frac{1}{RC_1}+\frac{1}{RC_2})dt = \int_{v(0)}^{v(t)} \frac{1}{v}dv$$

$$-(\frac{1}{RC_1}+\frac{1}{RC_2})t = \ln\frac{v(t)}{v(0)}$$

$$e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t} = \frac{v(t)}{v(0)}$$

$$v(0)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t} = v(t)$$

$$B \triangleq \frac{(V_2 - V_1)}{RC_1}$$
$$k \triangleq (\frac{1}{RC_1} + \frac{1}{RC_2})$$

Now observe that v_{C_1} does not actually appear anywhere on the right hand side, which means that we can simply take an antiderivative:

$$\frac{dv_{C_1}}{dt} = Be^{-kt}$$
$$v_{C_1}(t) = B(\frac{-1}{k})e^{-kt} + C \text{ (where } C \text{ is an arbitrary constant)}$$

Finally, let us rewrite the constant C in terms of the initial condition $v_{C_1}(0)$:

$$v_{C_1}(0) = B(\frac{-1}{k})e^{0} + C$$
$$v_{C_1}(0) = \frac{-B}{k} + C$$
$$v_{C_1}(0) + \frac{B}{k} = C$$

This results in:

$$v_{C_1}(t) = B(\frac{-1}{k})e^{-kt} + \frac{B}{k} + v_{C_1}(0)$$
$$v_{C_1}(t) = \frac{B}{k}(1 - e^{-kt}) + v_{C_1}(0)$$

Substituting back the definitions of B and k and continuing with the analysis:

$$v_{C_1}(t) = \frac{(V_2 - V_1)}{RC_1} \frac{1}{\frac{1}{RC_1} + \frac{1}{RC_2}} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t}) + v_{C_1}(0)$$

$$RC_1(v_{C_1}(t) - v_{C_1}(0)) = (V_2 - V_1) \frac{1}{\frac{1}{R}(\frac{1}{C_1} + \frac{1}{C_2})} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t})$$

$$RC_1(v_{C_1}(t) - v_{C_1}(0)) = (V_2 - V_1) \frac{R}{(\frac{1}{C_1} + \frac{1}{C_2})} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t})$$

$$v_{C_1}(t) - v_{C_1}(0) = (V_2 - V_1) \frac{1}{C_1(\frac{1}{C_1} + \frac{1}{C_2})} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t})$$

$$v_{C_1}(t) = v_{C_1}(0) + (V_2 - V_1) \frac{1}{1 + \frac{C_1}{C_2}} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t})$$

$$v_{C_1}(t) = V_1 + (V_2 - V_1) \frac{1}{1 + \frac{C_1}{C_2}} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t})$$

$$v_{C_1}(t) = V_1 + (V_2 - V_1) \frac{1}{1 + \frac{C_1}{C_2}} (1 - e^{-(\frac{C_1 + C_2}{RC_1 C_2})t}) \text{ (equally)}$$

(b) Determine a formula for $v_{C_2}(t)$ for $t \geq 0$. (Hint: use the previous part of this problem.)

Solution:

From the previous part, we have the following set of equations:

$$v_{C_2}(t) - v_{C_1}(t) = (V_2 - V_1) e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t}$$

$$v_{C_1}(t) = V_1 + (V_2 - V_1) \frac{1}{1 + \frac{C_1}{C_2}} (1 - e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})t})$$

Combining them, we obtain,

$$v_{C_2}(t) - v_{C_1}(t) = (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$v_{C_2}(t) = v_{C_1}(t) + (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$v_{C_2}(t) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}(1 - e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}) + (V_2 - V_1)e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

$$v_{C_2}(t) = V_1 + (V_2 - V_1)(\frac{1}{1+\frac{C_1}{C_2}} + e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}(1 - \frac{1}{1+\frac{C_1}{C_2}}))$$

$$v_{C_2}(t) = V_1 + (V_2 - V_1)(\frac{1}{1+\frac{C_1}{C_2}} + (1 - \frac{1}{1+\frac{C_1}{C_2}})(e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}))$$

(c) Determine a formula for the current $i(t)$ through the resistor for $t \geq 0$.

Solution:

Since $i(t) = \frac{v(t)}{R}$, we use results from the previous parts of the problem to obtain:

$$i(t) = \frac{v(t)}{R}$$

$$i(t) = \frac{v_{C_2}(t) - v_{C_1}}{R}$$

$$i(t) = \frac{(V_2 - V_1)}{R}e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})t}$$

(d) Check your work by showing that at $t = 0$, the initial conditions given at the start of this problem are satisfied. (In other words, show that $v_{C_1}(t = 0) = V_1$ and that $v_{C_2}(t = 0) = V_2$.

Solution:

Substituting into derivations from the previous parts:

$$v_{C_1}(t = 0) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}(1 - e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})0})$$

$$v_{C_1}(t = 0) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}(1 - 1)$$

$$v_{C_1}(t = 0) = V_1$$

$$v_{C_2}(t=0) = V_1 + (V_2 - V_1)(\frac{1}{1+\frac{C_1}{C_2}} + (1 - \frac{1}{1+\frac{C_1}{C_2}})(e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})0}))$$
$$v_{C_2}(t=0) = V_1 + (V_2 - V_1)(\frac{1}{1+\frac{C_1}{C_2}} + 1 - \frac{1}{1+\frac{C_1}{C_2}})$$
$$v_{C_2}(t=0) = V_1 + (V_2 - V_1)(1)$$
$$v_{C_2}(t=0) = V_1 + V_2 - V_1$$
$$v_{C_2}(t=0) = V_2$$

(e) At steady-state ($t \to \infty$), what happens to the voltages across the two capacitors? Is this consistent with what you learned in EE 16A?

Solution:

Substituting into derivations from the previous parts:

$$v_{C_1}(t\to\infty) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}(1 - e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})\infty})$$
$$v_{C_1}(t\to\infty) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}(1)$$
$$v_{C_1}(t\to\infty) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}$$
$$v_{C_1}(t\to\infty) = V_1 + (V_2 - V_1)\frac{C_2}{C_2+C_1} \quad \text{(alternatively)}$$

$$v_{C_2}(t\to\infty) = V_1 + (V_2 - V_1)(\frac{1}{1+\frac{C_1}{C_2}} + (1 - \frac{1}{1+\frac{C_1}{C_2}})(e^{-(\frac{1}{RC_1}+\frac{1}{RC_2})\infty}))$$
$$v_{C_2}(t\to\infty) = V_1 + (V_2 - V_1)(\frac{1}{1+\frac{C_1}{C_2}} + (1 - \frac{1}{1+\frac{C_1}{C_2}})(0))$$
$$v_{C_2}(t\to\infty) = V_1 + (V_2 - V_1)\frac{1}{1+\frac{C_1}{C_2}}$$
$$v_{C_2}(t\to\infty) = V_1 + (V_2 - V_1)\frac{C_2}{C_2+C_1} \quad \text{(alternatively)}$$

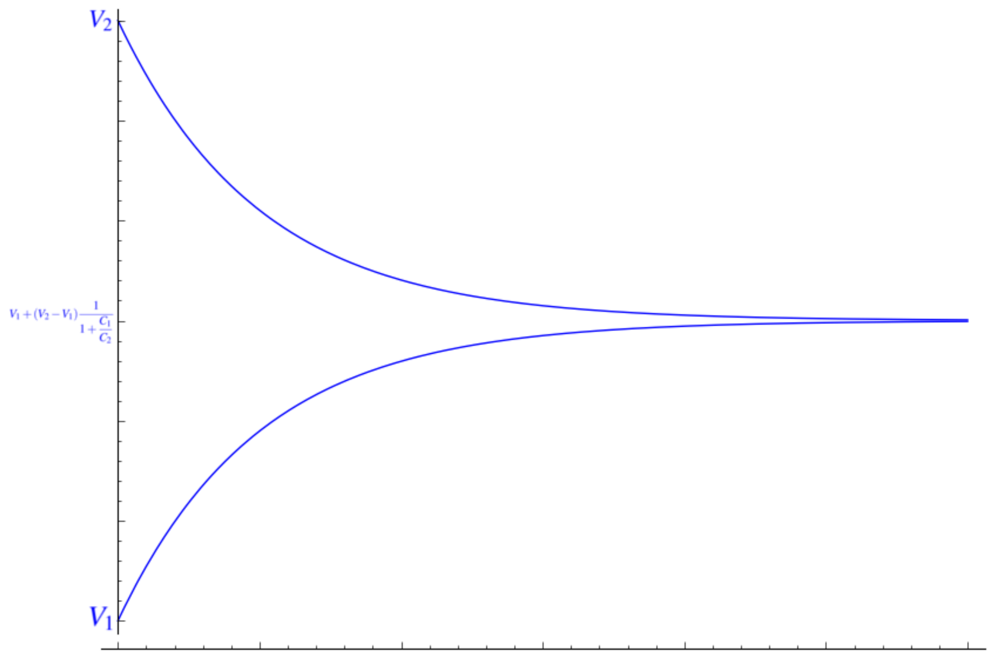
Observe that they both converge to the same voltage in the end, and that voltage is determined by the ratio of C_1 to C_2.

If $C_1 \gg C_2$, then the final voltage settles near V_1. (Verify this for yourself using the equation.) This is consistent with EE16A - if we dump charge from a small bucket into a much larger bucket, the voltage will not rise by much.

If $C_1 \ll C_2$, then the final voltage settles near V_2. This is also consistent with EE16A - if we dump charge from a large bucket into a smaller bucket, the voltage will rise by a lot.

(f) Sketch the two voltages over time on the same chart.

Solution: A representative sketch might look like the following:



(g) Is there any current flowing at steady-state? Why or why not?

Solution:

No, no current will flow at steady-state. From derivations in the previous parts:

$$i(t \to \infty) = \frac{(V_2 - V_1)}{R} e^{-(\frac{1}{RC_1} + \frac{1}{RC_2})\infty}$$
$$i(t \to \infty) = \frac{(V_2 - V_1)}{R} 0$$
$$i(t \to \infty) = 0$$

This also makes intuitive sense if you just think about it - if the voltage difference across a resistor is zero, will any current flow across it?

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