This homework is due October 10, 2016, at Noon.

1. **Homework process and study group**

   (a) Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.)

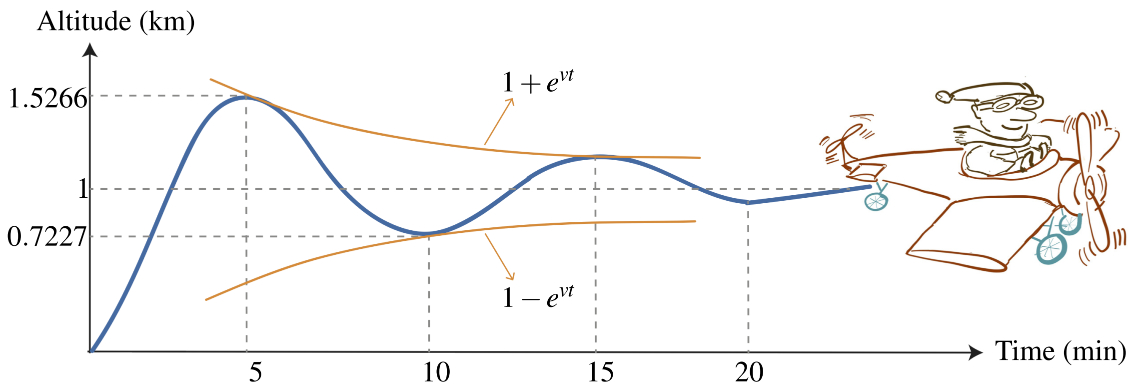
   (b) How long did you spend working on this homework? How did you approach it?

2. **Otto the Pilot**

   Otto has devised a control algorithm so that his plane climbs to the desired altitude by itself. However he is having oscillatory transients as shown in the figure. Prof. Arcak told him that if his system has complex eigenvalues

   $$\lambda_{1,2} = v \mp j\omega$$

   then his altitude would indeed oscillate with frequency ω about the steady state value, 1 km, and that the maxima and minima of these oscillations would lie on the curves $1+e^{vt}$ and $1-e^{vt}$, respectively.



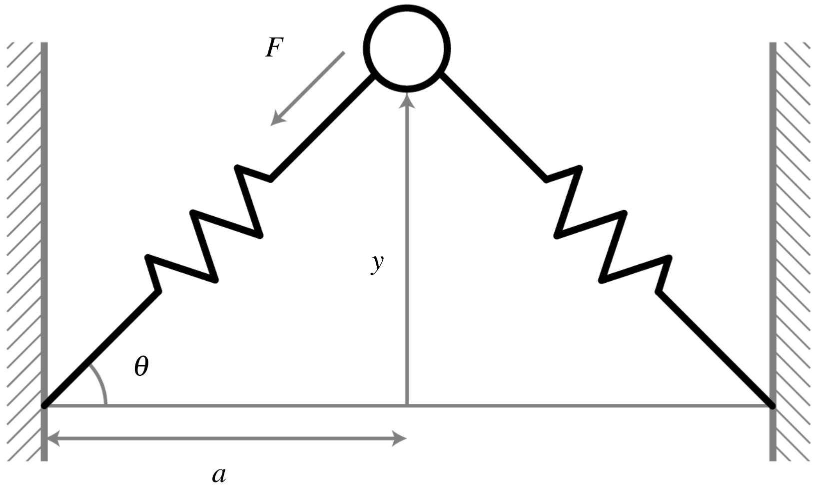
   (a) Find the real part v and the imaginary part ω from the altitude plot.

   (b) Let the dynamical model for the altitude be

   $$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

   where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find out what a_1 and a_2 are.

   (c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two negative, real eigenvalues at the same location.



3. Linearization of a spring mass system

In Discussion 5B we saw a spring-mass system and analyzed it in the (easier) case when the natural length of the spring X_0 is less than the shortest distance a between the spring base and mass. Now, we'll look at the same problem in the case that $X_0 > a$. Recall that in discussion we derived the following state space model of this system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0 \frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix}.$$

(a) Find the equilibria of the model assuming that $X_0 > a$. (Note: there are three.)

(b) Linearize your model about each of the three equilibria.

(c) Compute the eigenvalues for your model at each of the three equilibria. Which equilibria are stable/unstable/marginally stable (on the boundary between stable and unstable)?

(d) The state space model above is somewhat unrealistic because it doesn't model energy dissipation, so the mass will continue to oscillate forever. A more realistic model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0 \frac{x_1}{\sqrt{x_1^2+a^2}}\right) - \frac{c}{m}x_2 \end{bmatrix}$$

where c is a damping coefficient. How does the addition of this damping term change your answer for parts (a), (b) and (c) above? Where will the system come to rest in practice?

4. Linearization

Suppose we would like to control a spacecraft near the surface of the moon. The spacecraft has two thrusters located at opposites sides of the spacecraft that are capable of generating vertical thrust (relative to the spacecraft). For simplicity, we will only discuss the problem in two dimensions.

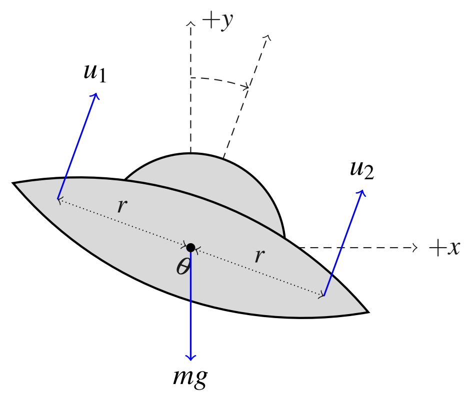


Figure 1: Spacecraft. The forces operating on the spacecraft are drawn in blue.

We use the following notation: the gravitation acceleration is g, the spacecraft has mass m, moment of inertia J, the distance between the thrusters and the center of mass is r, x is the position of the spacecraft along the x-axis, y is the position of the spacecraft along the y-axis, θ is the angle between the spacecraft and the y-axis, and u_1 and u_2 are the forces generated by the thrusters. The forces operating on the spacecraft can be seen in figure 1.

This problem uses some simple physics of the $F = ma$ variety. Please ask on Piazza if you don't know the simple physics involved, your fellow students will help you out. We'll start by modeling the behavior of the spacecraft with differential equations:

(a) Write a differential equation for Newton's second law in x-axis (relative to the moon's surface).

$$m\frac{d^2}{dt^2}x(t) = ?$$

(b) Write a differential equation for Newton's second law in the y-axis (relative to the moon).

$$m\frac{d^2}{dt^2}y(t) = ?$$

(c) Write the moment equation of the spacecraft around its center of mass.

$$J\frac{d^2}{dt^2}\theta(t) = ?$$

Next, we'll derive a state-space model for the system:

(d) Identify your state variables. How many do you have?

(e) Write the differential equations describing the system in standard form:

$$\frac{d}{dt}\vec{z}(t) = f(\vec{z}(t), \vec{u}(t)),$$

where $\vec{z}$ is the vector of state variables you have identified in the previous part.

The next step is to pick a point at which to linearize the system:

(f) Does the system have an operating point? That is, is there $\vec{z}$ such that $f(\vec{z}, \vec{0}) = \vec{0}$? What is the physical meaning of such a point?

(g) Choose some point $\vec{z}_0, \vec{u}_0$ where $\theta = 0$ such that the thursters counter the gravitational force exactly, and the spacecraft maintains the same position and rotation.

We now linearize the system and check if it is controllable.

(h) Linearize the system around the point $\vec{z}_0, \vec{u}_0$ by changing variables according to

$$\vec{\tilde{z}} = \vec{z} - \vec{z}_0$$
$$\vec{\tilde{u}} = \vec{u} - \vec{u}_0,$$

and writing the linearized system in standard form

$$\frac{d}{dt}\vec{\tilde{z}}(t) = A\vec{\tilde{z}} + B\vec{\tilde{u}}$$

(i) Is the spacecraft controllable around $\vec{z}_0$ using only these two thrustors?
Assume that the test for continuous-time controllability is the same as in discrete-time.

(j) Suppose that due to a mechanical problem, the right thruster stopped responding and is stuck at the nominal thrust setting for that thrustor. Is the system still controllable in the neighborhood around z_0, u_0?

5. Rotation matricies

Rotation matricies are convenient tools for rotating vectors. A 2x2 rotation matrix can be represented in the following form:

$$A = r\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

Where θ is the desired rotation angle and r is a scaling factor. A rotation operation can be applied to a vector by multiplying the vector by the rotation matrix:

$$\vec{x}_{\text{Rotated}} = r\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}\vec{x}$$

In this homework question, we will use iPython to visualize the use of these matricies.

(a) Complete the iPython notebook, "rotation_matricies.ipynb"

(b) How does the value of r affect the eigenvalues of the rotation matrix?

(c) How does the magnitude of the rotation matrix eigenvalues affect the behavior of the corresponding rotation operation?

Contributors:

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