This homework is due October 24, 2016, at Noon.

1. **Homework process and study group**

   (a) Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.)

   (b) How long did you spend working on this homework? How did you approach it?

2. **Design for controllability and observability I**

   We are given a system

$$\dot{\vec{x}}(t) = \begin{bmatrix} -2 & 1 \\ \gamma & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{x}(t)$$

   with tuneable parameter γ.

   (a) How should we tune γ to make the system controllable but not observable?

   (b) How should we tune γ to make the system observable but not controllable?

3. **Design for controllability and observability II**

   We are given a new system

$$\dot{\vec{x}}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}(t).$$

   along with only one sensor and one actuator to control and observe the system.

   (a) Which state should we control with the actuator to make the system controllable?

   (b) Which state should we measure with the sensor to make the system observable?

4. **LLR circuit**

   Consider the circuit below driven by a voltage source with voltage $u(t)$. The output $y(t)$ is the current through the resistor and the state variables are the inductor currents as marked in the circuit diagram.

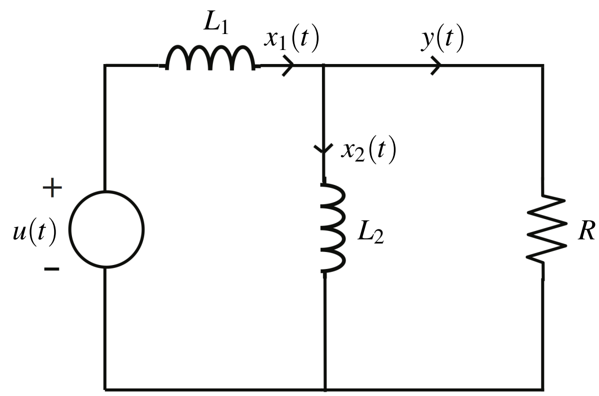
   (a) Write a state model for this circuit.

   (b) Find all equilibrium points when $u(t) = 0$ for all t.

   (c) Determine if the system is controllable.

   (d) Determine if the system is observable.

   (e) If your answer to part (c) or (d) is no, explain the physical reason for lack or controllability or observability, whichever is applicable.



5. **Force feedback for an accelerometer**

Many MEMS accelerometers use force feedback to improve the linearity of the sensors. One implementation of this is shown in figure 1. When the the accelerometer moves, the frame moves while the proof mass stays stationary because of its large inertia. As a result, the proof mass looks like it is accelerating with respect to the accelerometer frame. This displacement is sensed by an electrode that acts as a variable capacitor. The displacement signal is then used to drive an actuation electrode that will force the proof mass' position back to the equilibrium point. Ideally, the feedback system keeps the accelerometer still, and the amount of force required to keep it still can be used to calculate the acceleration. The equations below describe some important quantities:

$$F_{fb} = \frac{1}{2}\varepsilon_0 (V_p - V_f)^2 \frac{A}{(d+x)^2}$$

$$C_a = C_s = \frac{\varepsilon_0 A}{(d-x)}$$

$$C_f = 10\mu F$$

$$d = 2\mu m$$

$$A = 1000\mu m^2$$

$$\varepsilon_0 = 8.854 * 10^{-12} \frac{F}{m}$$

$$m_p = 2.5 * 10^{-8} kg$$

$$k = 0.01 \frac{N}{m}$$

$$V_p = 150V$$

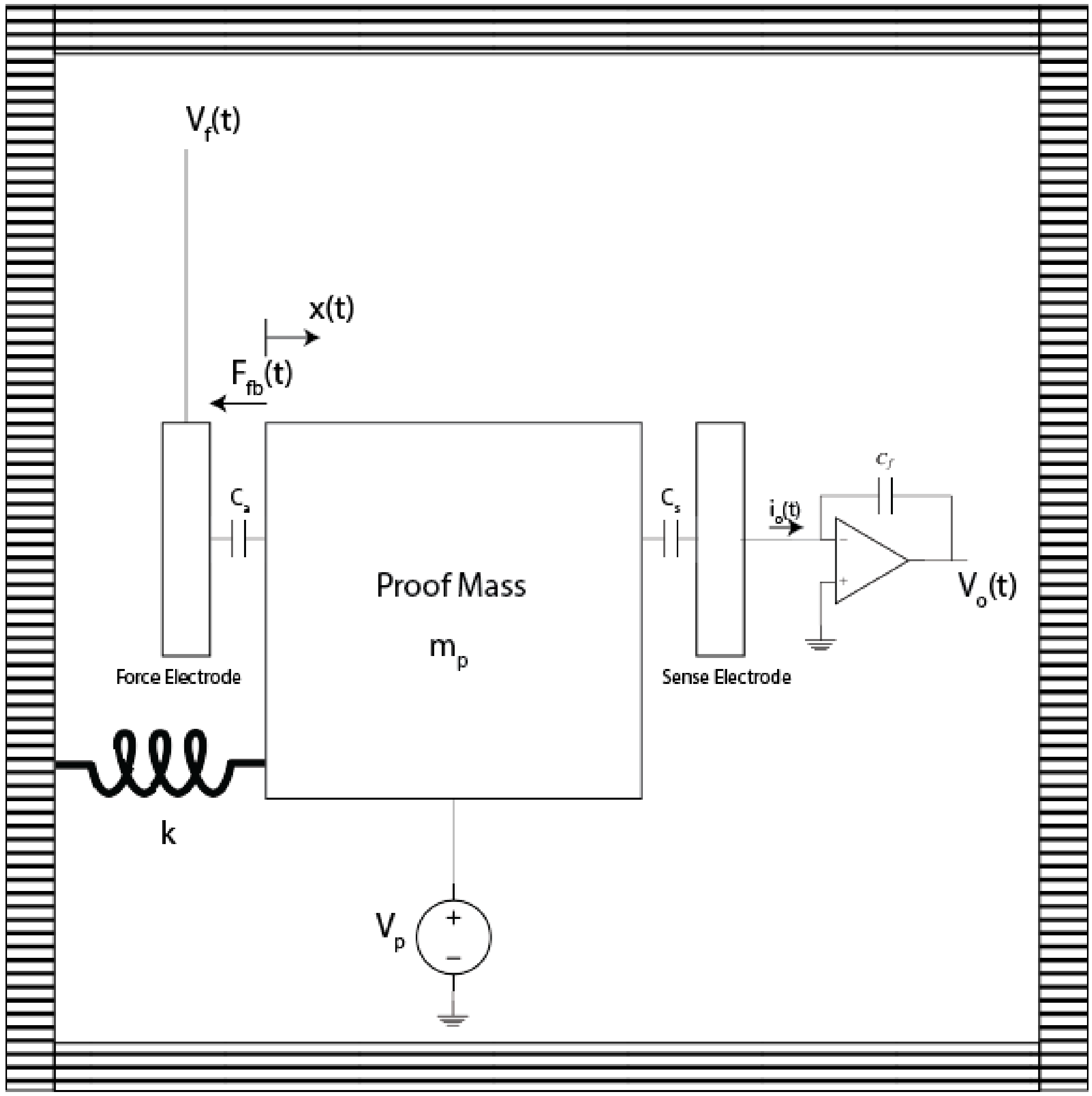


Figure 1: Open-Loop Force Feedback Accelerometer

(a) Assuming V_p is constant, find an expression for i_o in terms of $\frac{dx}{dt}$. (Hint: the full expression for current in a capacitor is $i_c = C\frac{dV_c}{dt} + V\frac{dC}{dt}$. We usually neglect the second term because our capacitors are normally constant, in this case, they aren't.)

(b) Find an expression for $V_o(t)$ as a function of $i_o(t)$ for $t > 0$. Assume the capacitor is discharged at $t = 0$ and $x(0) = 0$. Your expression should be in the time domain, not the phasor domain.

(c) Write a state space model for this system with input $u(t) = V_f(t)$, output $y(t) = V_o(t)$ and states $x(t)$ and $\dot{x}(t)$.

(d) For $u(t) = 2V_p$, what is the equilibrium point of this system?

(e) Linearize this state space model about the equilibrium point. What are the A, B and C matrices?

(f) Design an output feedback controller $u(t) = Ky(t)$ to make the closed-loop system stay at the equilibrium point.

$$\begin{bmatrix} \frac{dx}{dt}(t) \\ \frac{d\dot{x}}{dt}(t) \end{bmatrix} = A + BKC \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

6. **Continuous-time analog observer design: ship autopilots**

Modern ships use autopilots for steering. The main task of the autopilot is to maintain constant heading. A common system model used for ship steering controllers is the *Nomoto* first-order model. It is described using the following differential equation:

$$T\ddot{\psi} + \dot{\psi} = K\delta,$$

where ψ is the ship heading, δ is the rudder angle, and K and T are constants that are empirically estimated during sea trials. The "dot" notation used here is the physics convention (Newton's notation) that is very convenient for problems where nothing more that a second derivative is needed.

The only sensor is a gyrocompass, which reports the ship's current heading $y(t) = \psi(t)$. We would also like to provide a good estimate of an additional important parameter, the rate of turn — the derivative of the ship's current heading.

The input of the ship model is the rudder angle δ, and the output is the heading ψ, as measured by the gyrocompass.

(Note for the curious: undoubtably, some of you are wondering why we don't just take the derivative of the measurement and be done with it. The reason is that although we are describing everything without any noise, in the real-world, all measurements are noisy. Taking the derivative of noise is a very bad idea because it is in the nature of noise to shake a lot and so the derivative gets swamped by the shaking of the noise.)

In this problem you'll construct an analog continous-time observer, and then analyze its behaviour.

(a) Choose your state variables so that you have a two-dimensional state.

(b) Write down the system as a state-space model with a two-dimensional state.

(c) Is the system observable?

(d) Write down a model for the observer in matrix form using $\vec{\ell}$ to represent how you weigh the difference between the observed output $y(t)$ and the estimated output $\hat{y}(t)$ coming from within your observer.

(e) Find $\vec{\ell} = \begin{bmatrix} l_0 \\ l_1 \end{bmatrix}$ to place both the eigenvalues of the estimation error evolution at -2.

Now that we have designed the output-feedback and placed the eigenvalues of the estimation error. We'll design a circuit implementing the observer.

We will represent the state variables as voltages. Each input, output, and state variable will be implemented as a node in our circuit. The output of the original systems (the gyrocompass) would be an input of this system, and so would the rudder angle.

Recall that in EE16A and previously in EE16B, you have seen how to implement the following operations using simpler circuit elements (mainly resistors, capacitors and op-amps): differentiation, integration, scaling, addition and negation. This will be enough to implement the observer.

(f) Design a circuit whose output is the integral of its input with respect to time.

(g) Design a circuit whose output is a scaled version by a constant a_0 of its input.

(h) Design a circuit whose output is the negation of its input.

(i) Design a circuit whose output is the sum of its two inputs.

Now that we have the basic circuit elements. We'll implement the observer as a circuit.

(j) Use the circuits you designed above to construct the observer as a circuit driven by the output of the gyrocompass.

Contributors:

- John Maidens.
- Murat Arcak.
- Brian Kilberg.
- Baruch Sterin.