EE16B - Fall'16 - Lecture 12A Notes[1]

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Discrete Fourier Transform (DFT)

The DFT represents a length-N discrete-time sequence $x(t)$, $t = 0, 1, \ldots, N-1$, as a linear combination of the "basis" sequences:

$$\boxed{\Phi_k(t) \triangleq \frac{1}{\sqrt{N}} e^{jk\omega t}, \quad k = 0, 1, \ldots, N-1 \quad \text{where} \quad \omega = \frac{2\pi}{N}.} \tag{1}$$

Note from the identity $e^{j\theta} = \cos\theta + j\sin\theta$ that

$$\Phi_k(t) = \frac{1}{\sqrt{N}} \cos(k\omega t) + j\frac{1}{\sqrt{N}} \sin(k\omega t)$$

which has frequency $k\omega$. The figure below shows $\Phi_0(t), \ldots, \Phi_3(t)$ when $N = 4$, that is $\omega = \frac{\pi}{2}$.

Finding the DFT of $x(t)$ means finding coefficients $X(k)$, $k = 0, 1, \ldots, N-1$, such that

$$\boxed{x(t) = \sum_{k=0}^{N-1} X(k)\Phi_k(t).} \tag{2}$$

This amounts to a change of basis where $x(t)$, $t = 0, 1, \ldots, N-1$, is replaced with $X(k)$, $k = 0, 1, \ldots, N-1$.

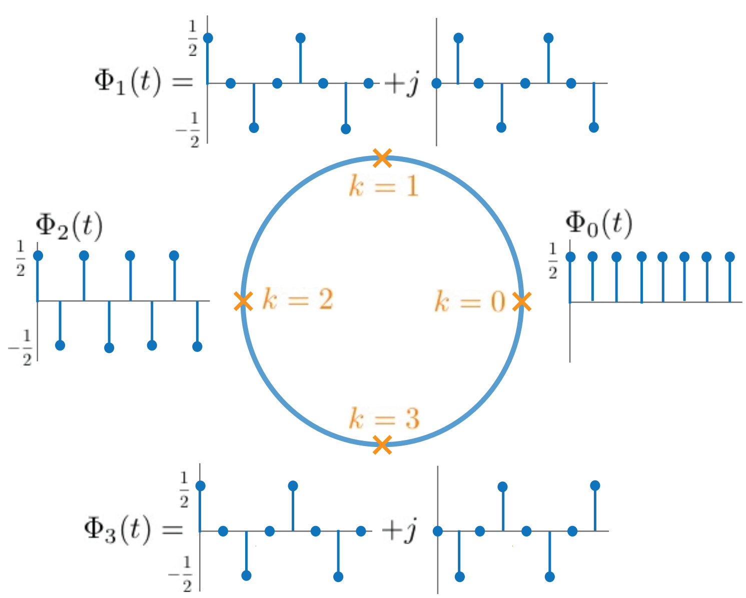

Figure 1: Basis functions for $N = 4$. Here $\omega = \frac{\pi}{2}$ and $e^{jk\omega}$ is marked on the unit circle for $k = 0, 1, 2, 3$. Note from (1) that $\Phi_k(t) = \frac{1}{2}(e^{jk\omega})^t$. The functions Φ_0 and Φ_2 are real-valued while Φ_1 and Φ_3 are complex.

The advantage of the new basis is that, instead of the values of $x(t)$ at each time t, we represent the sequence with coefficients $X(k)$ of its frequency components. This allows, for example, compression algorithms that allocate more bits to accurately store the coefficients of frequency components that matter more to the quality of sound than other frequencies.

Example ($N = 2$): Consider the length-two signal where

$$x(0) = 2, \quad x(1) = 3.$$

Since $N = 2$, we have $\omega = \pi$,

$$\Phi_0(t) = \frac{1}{\sqrt{2}} e^{j0t} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \Phi_1(t) = \frac{1}{\sqrt{2}} e^{j\pi t} = \frac{1}{\sqrt{2}} (-1)^t.$$

We view $x(t)$, $\Phi_0(t)$, $\Phi_1(t)$ as length-two vectors whose entries are the values that each sequence takes at $t = 0, 1$:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{\Phi}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{\Phi}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then DFT becomes a change of basis that we can perform using standard linear algebra:

$$\vec{x} = X(0)\vec{\Phi}_0 + X(1)\vec{\Phi}_1.$$

If we multiply both sides from the left with $\vec{\Phi}_0^T$ and note that $\vec{\Phi}_0$ and $\vec{\Phi}_1$ are orthonormal, that is $\vec{\Phi}_0^T\vec{\Phi}_1 = 0$ and $\vec{\Phi}_0^T\vec{\Phi}_0 = \vec{\Phi}_1^T\vec{\Phi}_1 = 1$, we get

$$\vec{\Phi}_0^T \vec{x} = X(0)\, \vec{\Phi}_0^T \vec{\Phi}_0 \quad \Rightarrow \quad X(0) = \vec{\Phi}_0^T \vec{x} = \frac{5}{\sqrt{2}}.$$

Similarly, multiplying both sides from the left with $\vec{\Phi}_1^T$ gives

$$\vec{\Phi}_1^T \vec{x} = X(1)\, \vec{\Phi}_1^T \vec{\Phi}_1 \quad \Rightarrow \quad X(1) = \vec{\Phi}_1^T \vec{x} = -\frac{1}{\sqrt{2}}.$$

Complex Inner Products

In the example above $\vec{\Phi}_0$ and $\vec{\Phi}_1$ were real-valued and we used the usual definition of inner product, $\vec{\Phi}_0^T\vec{\Phi}_1$, to show their orthogonality. For complex-valued vectors $\vec{x}$ and $\vec{y}$ the appropriate inner product is

$$\vec{x}^* \vec{y}$$

where $\vec{x}^*$ is the *conjugate transpose* which means that, in addition to transposing, we take the complex conjugate. As an illustration,

$$\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \quad \Rightarrow \quad \vec{x}^T = \begin{bmatrix} 1 & j \end{bmatrix} \quad \vec{x}^* = \begin{bmatrix} 1 & -j \end{bmatrix}. \tag{3}$$

For a real-valued $\vec{x}$ there is no difference between $\vec{x}^*$ and $\vec{x}^T$, as the complex conjugate of a real number is itself. Note that

$$\vec{x}^*\vec{x} = \|\vec{x}\|^2$$

which follows because $\vec{x}^*\vec{x} = \sum_i x(i)^* x(i) = \sum_i |x(i)|^2$. For the example in (3), $\vec{x}^*\vec{x} = 1 - j^2 = 2$ which means $\|\vec{x}\| = \sqrt{2}$. By contrast, $\vec{x}^T\vec{x} = 1 + j^2 = 0$ which shows the necessity of conjugation when defining complex inner products.

Example ($N = 4$): Consider now a length-four sequence. We have $\omega = \frac{2\pi}{N} = \frac{\pi}{2}$ and $\Phi_k(t)$ are as shown in Figure 1 above, specifically:

$$\begin{aligned}
\Phi_0(t) &= \frac{1}{2}e^{j0t} = \frac{1}{2} \\
\Phi_1(t) &= \frac{1}{2}e^{j\frac{\pi}{2}t} = \frac{1}{2}(j)^t \\
\Phi_2(t) &= \frac{1}{2}e^{j2\frac{\pi}{2}t} = \frac{1}{2}(-1)^t \\
\Phi_3(t) &= \frac{1}{2}e^{j3\frac{\pi}{2}t} = \frac{1}{2}(-j)^t.
\end{aligned}$$

We view $\Phi_0(t), \ldots, \Phi_3(t)$ as length-four vectors whose entries are the values that each sequence takes at $t = 0, 1, 2, 3$:

$$\vec{\Phi}_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{\Phi}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} \quad \vec{\Phi}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \vec{\Phi}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix}.$$

To find $X(k)$, $k = 0, 1, 2, 3$, such that

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = X(0)\vec{\Phi}_0 + X(1)\vec{\Phi}_1 + X(2)\vec{\Phi}_2 + X(3)\vec{\Phi}_3 \tag{4}$$

we will again use the orthonormality of the basis vectors $\vec{\Phi}_0, \ldots, \vec{\Phi}_3$. You can indeed show that

$$\vec{\Phi}_k^*\vec{\Phi}_l = \begin{cases} 0 & k \neq l \\ 1 & k = l. \end{cases}$$

Then, if we multiply both sides of (4) from the left by $\vec{\Phi}_k^*$, we get

$$\vec{\Phi}_k^* \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = X(k)\vec{\Phi}_k^*\vec{\Phi}_k = X(k).$$

Combining these equations for $k = 0, 1, 2, 3$ we get

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} \vec{\Phi}_0^* \\ \vec{\Phi}_1^* \\ \vec{\Phi}_2^* \\ \vec{\Phi}_3^* \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}.$$

As an illustration, for the sequence $x(0) = 1$, $x(1) = x(2) = x(3) = 0$, we get

$$X(0) = X(1) = X(2) = X(3) = \frac{1}{2}.$$

The summation of $\Phi_0(t), \ldots, \Phi_3(t)$ with these weights indeed recovers $x(t)$ as shown in the figure below. The shaded region demarcates the time interval of interest: $t = 0, 1, 2, 3$.