EE16B - Fall'16 - Lecture 11A Notes[1]

Murat Arcak

8 November 2016

Interpolation with Basis Functions

Recall that in this method we assign to each x_i a function $\phi_i(x)$ that satisfies:

$$\phi_i(x_i) = 1 \quad \text{and} \quad \phi_i(x_j) = 0 \text{ when } j \neq i \tag{1}$$

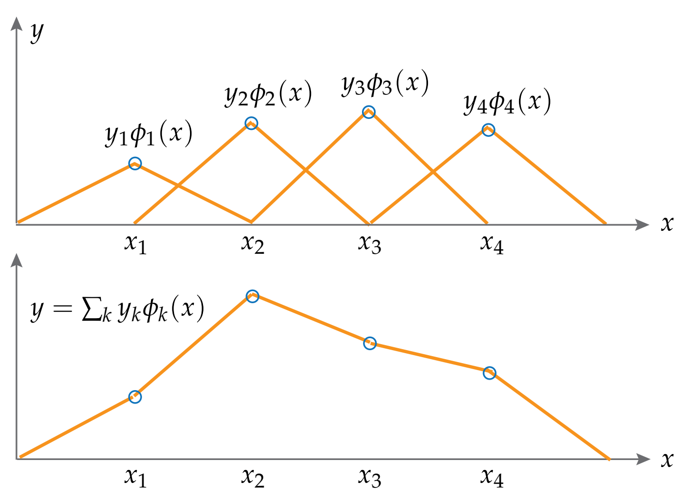
and interpolate between the data points (x_i, y_i) with the function:

$$y = \sum_k y_k \phi_k(x). \tag{2}$$

When $x = x_i$ this expression yields $y = y_i$ as desired, because $\phi_k(x_i) = 0$ except when $k = i$.

We refer to $\phi_i(x)$ as "basis functions" since our interpolation is obtained from a linear combination of these functions. Basis functions restrict the behavior of the interpolation between the data points, thus avoiding the erratic results of polynomial interpolation.

The figure below uses triangular basis functions which lead to a linear interpolation.



Below we discuss three commonly used basis functions. For simplicity we assume x_i are in increasing order and evenly spaced:

$$x_{i+1} - x_i = \Delta \quad \text{for all } i.$$

This allows us to obtain basis functions $\phi_i(x)$ by shifting a single function $\phi(x)$:

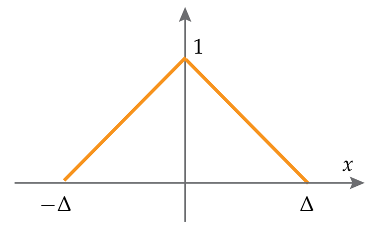
$$\phi_i(x) = \phi(x - x_i). \tag{3}$$

Note that (1) holds if we define $\phi(x)$ such that

$$\phi(0) = 1 \quad \text{and} \quad \phi(k\Delta) = 0 \text{ when } k \neq 0. \tag{4}$$

Linear Interpolation

When $\phi(x)$ is as depicted on the right, that is

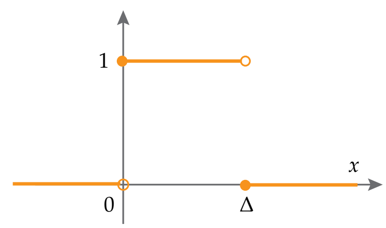


$$\phi(x) = \begin{cases} 1 - \frac{|x|}{\Delta} & |x| \leq \Delta \\ 0 & \text{otherwise,} \end{cases}$$

and the basis functions are obtained from (3), then the interpolation (2) connects the data points with straight lines. (See figure above.)

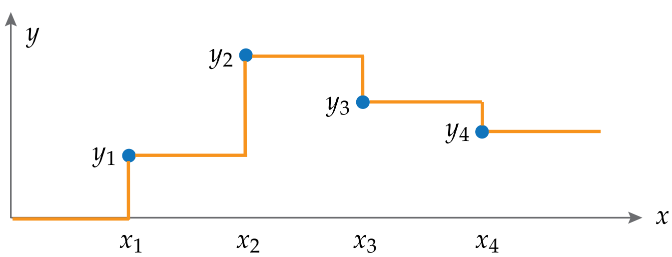
Zero Order Hold Interpolation

When



$$\phi(x) = \begin{cases} 1 & x \in [0, \Delta) \\ 0 & \text{otherwise} \end{cases}$$

as depicted on the right, the interpolation (2) keeps y constant between the data points:

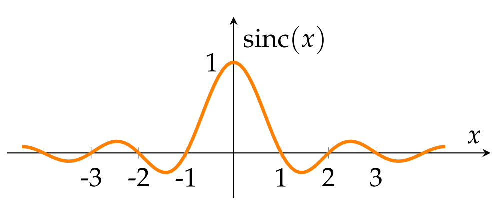


Sinc Interpolation

The *sinc* function is defined as

$$\text{sinc}(x) \triangleq \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

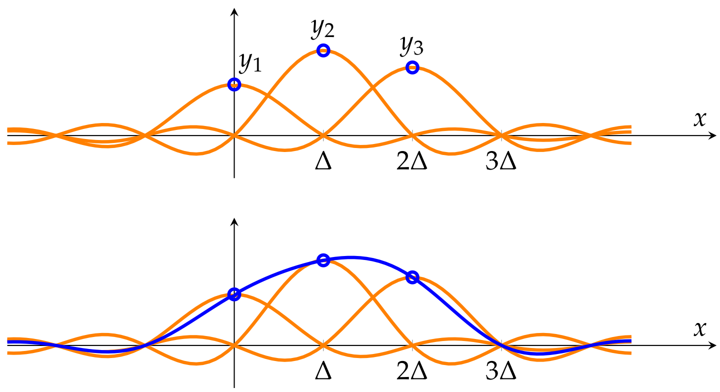
and depicted below. It is continuous since $\lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} = 1$, and vanishes whenever x is a nonzero integer.



In sinc interpolation we let

$$\phi(x) = \text{sinc}(x/\Delta)$$

and apply (2) with the shifted basis functions (3). To illustrate this, the first plot below superimposes $y_i\phi(x - x_i)$ for three data points $i = 1, 2, 3$. The second plot adds them up (blue curve) to complete the interpolation.



The interest in sinc interpolation is due to its smoothness – contrast the blue curve above with the kinks of linear interpolation and the discontinuities of zero order hold interpolation illustrated earlier.

To make this smoothness property more explicit we use the identity

$$\text{sinc}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(\omega x) d\omega \tag{5}$$

which you can verify by evaluating the integral. Viewing this integral as an infinite sum of cosine functions, we see that the fastest varying component has frequency $\omega = \pi$. Thus the sinc function can't exhibit variations faster than this component.

Functions that involve frequencies smaller than some constant are called "band-limited." This notion is made precise in EE 120 with continuous *Fourier Transforms*. For 16B it is sufficient to think of a band-limited signal as one that can be written as a sum or integral of sinusoidal components whose frequencies lie in a finite band, which is $[0, \pi]$ for the sinc function in (5).

Sampling Theorem

Sampling is the opposite of interpolation: given a function $f(x)$ we evaluate it at sample points x_i:

$$y_i = f(x_i) \quad i = 1, 2, 3, \ldots$$

Sampling is critical in digital signal processing where one uses samples of an analog sound or image. For example, digital audio is often recorded at 44.1 kHz which means that the analog audio is sampled 44,100 times per second; these samples are then used to reconstruct the audio when playing it back. Similarly, in digital images each pixel corresponds to a sample of an analog image.

An important problem in sampling is whether we can perfectly recover an analog signal from its samples. As we explain below, the answer is yes when the analog signal is band-limited and the interval between the samples is sufficiently short.

Suppose we sample the function $f : \mathbb{R} \to \mathbb{R}$ at evenly spaced points

$$x_i = \Delta i, \quad i = 1, 2, 3, \ldots$$

and obtain

$$y_i = f(\Delta i) \quad i = 1, 2, 3, \ldots$$

Then sinc interpolation between these data points gives:

$$\hat{f}(x) = \sum_i y_i \phi(x - \Delta i) \tag{6}$$

where

$$\phi(x) = \text{sinc}(x/\Delta),$$

which is band-limited by π/Δ from (5). This means that $\hat{f}(x)$ in (6) contains frequencies ranging from 0 to π/Δ.

Now if $f(x)$ involves frequencies smaller than π/Δ, then it is reasonable to expect that it can be recovered from (6) which varies as fast as $f(x)$. In fact the shifted sinc functions $\phi(x - \Delta i)$ form a basis for the space of functions[2] that are band-limited by π/Δ and the formula (6) is simply the representation of $f(x)$ with respect to this basis.

[2] for technical reasons this space is also restricted to square integrable functions

Sampling Theorem: If $f(x)$ is band-limited by frequency

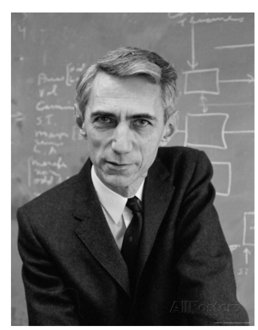
$$\omega_{\max} < \frac{\pi}{\Delta} \tag{7}$$

then the sinc interpolation (6) recovers $f(x)$, that is $\hat{f}(x) = f(x)$.

By defining the sampling frequency $\omega_s = 2\pi/\Delta$, we can restate the condition (7) as:

$$\omega_{\max} < \frac{\omega_s}{2}$$

which states that the function must be sampled faster than twice its maximum frequency. The Sampling Theorem was proven by Claude Shannon in the 1940s and was implicit in an earlier result by Harry Nyquist. Both were researchers at the Bell Labs.



Claude Shannon (1916-2001)

Harry Nyquist (1889-1976)