EE16B - Fall'16 - Lecture 12B Notes[1]

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Discrete Fourier Transform (DFT) Continued

The DFT represents a length-N discrete-time sequence $x(t)$, $t = 0, 1, \ldots, N-1$, as a linear combination of the "basis" sequences:

$$\boxed{\Phi_k(t) \triangleq \frac{1}{\sqrt{N}} e^{jk\omega t}, \quad k = 0, 1, \ldots, N-1 \quad \text{where} \quad \omega = \tfrac{2\pi}{N}.} \tag{1}$$

To simplify the notation we define

$$W_k \triangleq e^{jk\frac{2\pi}{N}} \tag{2}$$

which lies on the unit circle, and rewrite the basis functions (1) as

$$\Phi_k(t) \triangleq \frac{1}{\sqrt{N}} W_k^t, \quad k = 0, 1, \ldots, N-1,$$

or as vectors whose entries are the values of the sequence $\Phi_k(t)$ at $t = 0, 1, \ldots, N-1$:

$$\vec{\Phi}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ W_k \\ W_k{}^2 \\ \vdots \\ W_k^{(N-1)} \end{bmatrix}. \tag{3}$$

To see that these vectors form an orthonormal basis, note that

$$\vec{\Phi}_k^* \vec{\Phi}_l = \frac{1}{N} \sum_{t=0}^{N-1} (W_k^t)^* W_l^t = \frac{1}{N} \sum_{t=0}^{N-1} e^{j(l-k)\frac{2\pi}{N}t} = \frac{1}{N} \sum_{t=0}^{N-1} W_{l-k}^t$$

where the second equality follows from $(W_k^t)^* = (e^{jk\frac{2\pi}{N}t})^* = e^{-jk\frac{2\pi}{N}t}$. Now, if $l = k$, $e^{j(l-k)\frac{2\pi}{N}t} = 1$ and the summation gives N. If $l \neq k$ the summation gives zero because the numbers W_{l-k}^t, $t = 0, \ldots, N-1$ are spread evenly around the unit circle and add up to zero. Thus,

$$\vec{\Phi}_k^* \vec{\Phi}_l = \begin{cases} 0 & k \neq l \\ 1 & k = l. \end{cases} \tag{4}$$

DFT Analysis and Synthesis Equations

Recall that the DFT representation of the sequence $x(t)$ is

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = X(0)\vec{\Phi}_0 + X(1)\vec{\Phi}_1 + \cdots + X(N-1)\vec{\Phi}_{N-1}. \quad (5)$$

To determine $X(k)$ we multiply both sides of this equation by $\vec{\Phi}_k^*$ and use the orthonormality property (4) to obtain:

$$\vec{\Phi}_k^* \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = X(k)\Phi_k^*\Phi_k = X(k).$$

Thus, stacking up $X(0), \ldots, X(N-1)$ in a vector gives the formula:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} \vec{\Phi}_0^* \\ \vec{\Phi}_1^* \\ \vec{\Phi}_2^* \\ \vdots \\ \vec{\Phi}_{N-1}^* \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \quad (6)$$

which we call the "analysis equation" since it amounts to analyzing the frequency components of the sequence $x(t)$.

Likewise, we rewrite the equation (5) compactly as:

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{\Phi}_0 & \vec{\Phi}_1 & \vec{\Phi}_2 & \cdots & \vec{\Phi}_{N-1} \end{bmatrix}}_{\triangleq F} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}$$

and call it the "synthesis equation" since it synthesizes the sequence $x(t)$ from its frequency components.

We refer to the $N \times N$ matrix F as the Fourier Matrix. It is apparent from (6) that

$$F^{-1} = F^*$$

which means that F is a *unitary* matrix.

Some Properties of DFT

The first two properties below follow directly from (6).

1) Scaling:

If $x(t)$ has DFT coefficients $X(k)$ then, for any constant a, the sequence $ax(t)$ has DFT coefficients $aX(k)$.

2) Superposition:

If $x(t)$ and $y(t)$ have DFT coefficients $X(k)$ and $Y(k)$ respectively, then $x(t) + y(t)$ has DFT coefficients $X(k) + Y(k)$.

3) Parseval's relation: Given $x(t)$ and its DFT coefficients $X(k)$,

$$\sum_{t=0}^{N-1} |x(t)|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

which means $\|\vec{x}\|^2 = \|\vec{X}\|^2$. This follows because $\vec{x} = F\vec{X}$ where F is unitary ($F^*F = I$) as discussed above and, thus,

$$\|\vec{x}\|^2 = \vec{x}^*\vec{x} = \vec{X}^*F^*F\vec{X} = \vec{X}^*\vec{X} = \|\vec{X}\|^2.$$

4) Conjugate symmetry:

When $x(t)$ is real-valued, the DFT coefficients satisfy

$$\boxed{X(N-k) = X(k)^* \qquad k = 1, \ldots, N-1.} \tag{7}$$

For example, when $N = 4$, we get $X(3) = X(1)^*$ and $X(2) = X(2)^*$ which means $X(2)$ is real.

To see why (7) holds first note from (2) that

$$W_{N-k} = e^{j(N-k)\frac{2\pi}{N}} = e^{j2\pi}e^{-jk\frac{2\pi}{N}} = e^{-jk\frac{2\pi}{N}} = W_k^*$$

and, similarly, $W_{N-k}^t = (W_k^t)^*$, $t = 2, 3, \ldots$ It follows from (6) that

$$X(k) = \Phi_k^*\vec{x} = \sum_{t=0}^{N-1} (W_k^t)^* x(t) = \sum_{t=0}^{N-1} (W_{N-k}^t) x(t).$$

Since $x(t)$ is real, we have $x(t)^* = x(t)$; thus taking the conjugate of both sides gives

$$X(k)^* = \sum_{t=0}^{N-1} (W_{N-k}^t)^* x(t) = \Phi_{N-k}^*\vec{x} = X(N-k).$$

Example 1: Let $x(t) = e^{j\omega_0 t}$, $t = 0, 1, \ldots, N-1$. If

$$\omega_0 = k_0\frac{2\pi}{N} \quad \text{for some integer } k_0 \in \{0, 1, \ldots, N-1\} \tag{8}$$

then $x(t)$ can be recovered from the k_0-th basis function alone:

$$x(t) = \sqrt{N}\Phi_{k_o}(t).$$

This means that

$$X(k) = \begin{cases} \sqrt{N} & \text{if } k = k_0 \\ 0 & \text{otherwise.} \end{cases}$$

As a concrete example, if we have the length $N = 100$ sequence $x(t) = e^{j0.2\pi t}$, $t = 0, 1, \ldots, 99$, then (8) holds with $k_0 = 10$. Thus $X(10) = \sqrt{100} = 10$ and all other $X(k)$ values are zero. Note that $X(90) \neq X(10)^*$ because the conjugate symmetry property (7) does not necessarily hold when $x(t)$ is complex-valued.

The simplifying assumption (8) implied that $x(t)$, interpreted as a vector $\vec{x}$, is exactly aligned with the k_0-th basis vector $\vec{\Phi}_{k_0}$. When ω_0 is *not* exactly equal but close to $k_0\frac{2\pi}{N}$, we can expect that $\vec{x}$ will be approximately aligned with $\vec{\Phi}_{k_0}$ which means that $X(k)$ peaks around k_0 and is smaller away from k_0.

Example 2: Now let $x(t) = \cos(\omega_0 t)$ where ω_0 satisfies (8). Rewriting

$$\begin{aligned} x(t) &= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \\ &= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{(2\pi-\omega_0)t} \\ &= \frac{1}{2}e^{jk_0\frac{2\pi}{N}t} + \frac{1}{2}e^{(N-k_0)\frac{2\pi}{N}t} \\ &= \frac{\sqrt{N}}{2}\Phi_{k_0}(t) + \frac{\sqrt{N}}{2}\Phi_{N-k_0}(t) \end{aligned}$$

we conclude

$$X(k) = \begin{cases} \frac{\sqrt{N}}{2} & \text{if } k = k_0 \text{ or } k = N - k_0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

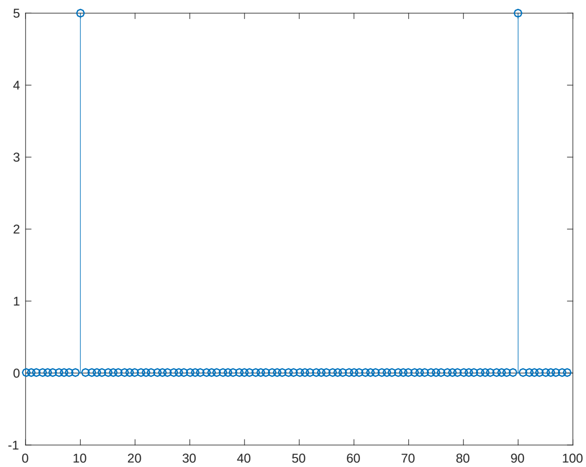
As an illustration, for

$$x(t) = \cos(0.2\pi t), \quad t = 0, 1, \ldots, N-1 \tag{10}$$

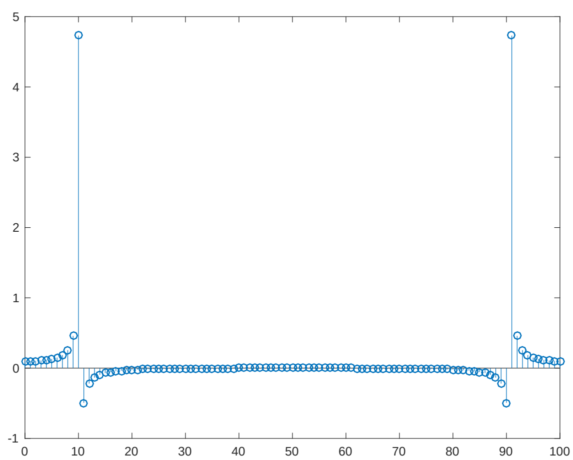
with $N = 100$ we get $X(10) = X(90) = 5$ and $X(k) = 0$ for all other k, as depicted below. The conjugate symmetry property (7) is now satisfied because $x(t)$ is real-valued.

To illustrate what happens when (8) does not hold, we next take $N = 101$ and plot the real and imaginary parts of the resulting DFT. As expected, $X(k)$ peaks near $k = 10$ and $k = 90$.

$X(k)$ for $N = 100$



Real part of $X(k)$ for $N = 101$



Imaginary part of $X(k)$ for $N = 101$