EE16B - Fall'16 - Lecture 13A Notes[1]

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Linear Time Invariant (LTI) Systems

In signal processing nomenclature a *system* is a block that takes in an input signal $u(t)$ and returns an output $y(t)$ as in the figure below.

A *linear* system is one that satisfies the following two conditions:

1. Scaling: If an input $u(t)$ generates[2] the output $y(t)$, then

$$au(t) \to ay(t) \text{ for any constant } a. \tag{1}$$

[2] we will use the notation $u(t) \to y(t)$ as a shorthand for "$u(t)$ generates $y(t)$"

2. Superposition: If $u_1(t) \to y_1(t)$ and $u_2(t) \to y_2(t)$, then

$$u_1(t) + u_2(t) \to y_1(t) + y_2(t). \tag{2}$$

If the input to a linear system is 0 then the output must be 0 as well. This follows by choosing $a = 0$ in (1).

A system is called *time invariant* if a time shift in the input results is an identical time shift in the output:

$$u(t-T) \to y(t-T). \tag{3}$$

This means that the system reacts the same way to an input applied later as it would have reacted if the input was applied now.

Example: Consider the discrete-time state space model

$$\begin{aligned} \vec{x}(t+1) &= A\vec{x}(t) + Bu(t) \\ y(t) &= C\vec{x}(t) \end{aligned} \tag{4}$$

and recall that the solution $\vec{x}(t)$ for $t = 1, 2, 3, \ldots$ is

$$\vec{x}(t) = A^t\vec{x}(0) + A^{t-1}Bu(0) + \cdots + ABu(t-2) + Bu(t-1).$$

Multiplying both sides from the left by C we get

$$y(t) = CA^t\vec{x}(0) + CA^{t-1}Bu(0) + \cdots + CABu(t-2) + CBu(t-1)$$

and, if the initial condition $\vec{x}(0)$ is zero, we conclude

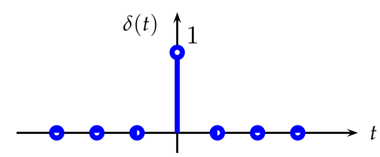
$$y(t) = CA^{t-1}Bu(0) + \cdots + CABu(t-2) + CBu(t-1). \tag{5}$$

This relationship satisfies the scaling and superposition properties; therefore the system is linear. It is also time invariant because A, B, C do not depend on time: if we apply the same input starting at time $t = T$ instead of $t = 0$, we get the same solution shifted in time by T.

Impulse Response of LTI Systems

Systems that are both linear and time invariant possess many useful properties. One such property is that, if we know the system's response to the *unit impulse* sequence

$$\delta(t) \triangleq \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$



depicted on the right, we can predict its response to any other input.

We denote the impulse response by $h(t)$, that is $\delta(t) \rightarrow h(t)$. To see how we can use $h(t)$ to predict the system's response to another input $u(t)$, rewrite $u(t)$ as

$$\begin{aligned} u(t) &= \ldots + u(0)\delta(t) + u(1)\delta(t-1) + u(2)\delta(t-2) + \ldots \\ &= \sum_k u(k)\delta(t-k). \end{aligned}$$

Since $\delta(t) \rightarrow h(t)$, by time-invariance we have $\delta(t-k) \rightarrow h(t-k)$. Then, by linearity, $\sum_k u(k)\delta(t-k) \rightarrow \sum_k u(k)h(t-k)$, therefore

$$\boxed{y(t) = \sum_k u(k)h(t-k).} \tag{6}$$

This expression is known as the *convolution sum* and shows how we can predict the output for any input using the impulse response $h(t)$.

Example: For the state space example above, note from (5) that

$$h(t) = CA^{t-1}B\delta(0) + \cdots + CAB\delta(t-2) + CB\delta(t-1) = CA^{t-1}B$$

for $t \geq 1$, and $h(0) = C\vec{x}(0) = 0$ since we assumed $\vec{x}(0) = 0$ to ensure linearity. Thus,

$$h(0) = 0,\ h(1) = CB,\ h(2) = CAB,\ h(3) = CA^2B,\ \ldots$$

and (5) can be interpreted as a convolution sum.

Example: For a LTI system whose output is

$$y(t) = \frac{u(t) + u(t-1)}{2}$$

the impulse response is zero except at $t = 0, 1$ where $h(0) = h(1) = \frac{1}{2}$.

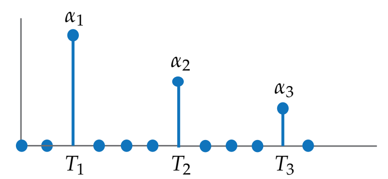
Example: Radio waves from a wireless transmitter may be deflected by obstacles, creating echoes of the transmitted signal that reach the receiver with different delays. For example, if there are three paths, the received signal can be written as

$$y(t) = \alpha_1 u(t - T_1) + \alpha_2 u(t - T_2) + \alpha_3 u(t - T_3)$$

which includes three copies of the transmitted signal $u(t)$, each with delay T_i and gain α_i, $i = 1, 2, 3$. In a discrete-time model T_i should be interpreted as the nearest number of sample periods. This input/output relationship is linear; it is also time invariant if we assume that α_i and T_i, $i = 1, 2, 3$ do not change over time. The impulse response is obtained by substituting the unit impulse for the input,

$$h(t) = \alpha_1 \delta(t - T_1) + \alpha_2 \delta(t - T_2) + \alpha_3 \delta(t - T_3),$$

and is depicted on the right.



Finite Impulse Response (FIR) Systems

For the rest of this discussion we restrict our attention to LTI systems whose impulse response is of finite length. Specifically we assume

$$h(t) = 0 \quad \text{when} \quad t \notin \{0, 1, \ldots, M\} \tag{7}$$

for some integer M, so that only $h(0), \ldots, h(M)$ can be nonzero. Likewise we assume a finite length input $u(t)$, $t = 0, 1, \ldots, L - 1$.

It then follows from (6) that

$$y(t) = \sum_{k=0}^{L-1} u(k) h(t - k) \tag{8}$$

and it can be shown from (7) that

$$y(t) = 0 \quad \text{when } t \notin \{0, 1, \ldots, M + L - 1\}.$$

For $t \in \{0, 1, \ldots, M + L - 1\}$ we write (8) in matrix form as

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ y(M+L-1) \end{bmatrix}}_{\triangleq \vec{y}} = \begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \ddots & \vdots \\ \vdots & h(1) & \ddots & 0 \\ h(M) & \vdots & \ddots & h(0) \\ 0 & h(M) & & h(1) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h(M) \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(L-1) \end{bmatrix} \tag{9}$$

where each row in the matrix corresponds to a particular t in (8).

Note that if we apply the unit impulse as the input, then

$$\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(L-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and multiplication with the matrix in (9) gives

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ y(M+L-1) \end{bmatrix} = \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which, as expected, is the impulse response.