EE16B - Fall'16 - Lecture 14A Notes[1]

Murat Arcak

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LTI Systems and DFT

In Lecture 13A we started discussing LTI systems whose impulse response satisfy

$$h(t) = 0 \quad \text{when} \;\; t \notin \{0, 1, \ldots, M\} \tag{1}$$

for some integer M, so that only $h(0), \ldots, h(M)$ can be nonzero. Likewise we assumed a finite length input $u(t)$, $t = 0, 1, \ldots, L-1$, and showed that the output can be expressed as

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ y(M+L-1) \end{bmatrix}}_{\triangleq \vec{y}} = \begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \ddots & \vdots \\ \vdots & h(1) & \ddots & 0 \\ h(M) & \vdots & \ddots & h(0) \\ 0 & h(M) & & h(1) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h(M) \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(L-1) \end{bmatrix}. \tag{2}$$

Note that the first column represents the impulse response and subsequent columns are obtained by shifting the first one downwards.

To match the size of the input and output vectors we augment the input vector with M zeros,

$$\vec{u} \triangleq \begin{bmatrix} u(0) \; u(1) \; \cdots \; u(L-1) \;\; \underbrace{0 \; \cdots \; 0}_{M \text{ zeros}} \end{bmatrix}^T, \tag{3}$$

and complete the $(M+L) \times L$ matrix above into a square matrix by adding M columns. Although the choice of these columns is arbitrary (they get multiplied by the zero entries of $\vec{u}$) we preserve the structure and continue shifting the columns downwards and wrap around the entries from the bottom back to the top:

$$\vec{y} = \underbrace{\begin{bmatrix} h(0) & 0 & \cdots & 0 & h(M) & \cdots & h(1) \\ h(1) & h(0) & \ddots & \vdots & 0 & \ddots & \vdots \\ \vdots & h(1) & \ddots & 0 & \vdots & \ddots & h(M) \\ h(M) & \vdots & \ddots & h(0) & 0 & \vdots & 0 \\ 0 & h(M) & & h(1) & h(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & h(M) & h(M-1) & \cdots & h(0) \end{bmatrix}}_{\triangleq H} \vec{u}. \quad (4)$$

The resulting $(M+L) \times (M+L)$ square matrix H belongs to a class of matrices called "circulant."

Circulant Matrices and Diagonalization with DFT Basis

A circulant matrix has the general form

$$C = \begin{bmatrix} c_0 & c_{N-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{N-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{N-2} & \vdots & \ddots & \ddots & c_{N-1} \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{bmatrix} \quad (5)$$

which encompasses H in (4) as a special case. Circulant matrices have the following useful property proven at the end of these notes:

The DFT basis diagonalizes all circulant matrices. Specifically,

$$F^*CF = \sqrt{N} \begin{bmatrix} C(0) & & & \\ & C(1) & & \\ & & \ddots & \\ & & & C(N-1) \end{bmatrix} \quad (6)$$

where $F = [\vec{\Phi}_0 \cdots \vec{\Phi}_{N-1}]$ is the Fourier matrix defined in Lecture 12B and $C(0), C(1), \ldots, C(N-1)$ are the DFT coefficients of the sequence $c_0, c_1, \ldots, c_{N-1}$ obtained from the first column.

We now use this property to derive a relation between the input and output of a LTI system in the DFT basis. Multiplying both sides of (4) from the left with F^* we write:

$$F^*\vec{y} = F^*H\vec{u} = F^*HFF^*\vec{u}$$

where the second equality follows because $FF^* = I$. Since $F^*\vec{y}$ and $F^*\vec{u}$ give the vector of DFT coefficients $\vec{Y}$ and $\vec{U}$, this means

$$\vec{Y} = F^* H F \vec{U}.$$

Finally, adapting (6) to the matrix H in (4), we get

$$\begin{bmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(M+L-1) \end{bmatrix} = \sqrt{M+L} \begin{bmatrix} H(0) & & & \\ & H(1) & & \\ & & \ddots & \\ & & & H(M+L-1) \end{bmatrix} \begin{bmatrix} U(0) \\ U(1) \\ \vdots \\ U(M+L-1) \end{bmatrix}$$

where $H(0), H(1), \ldots, H(M+L-1)$ are the DFT coefficients of the length $M+L$ sequence obtained from the first column of H:

$$h(0)\ h(1)\ \cdots\ h(M)\ \underbrace{0\ \cdots\ 0}_{L-1 \text{ zeros}}.$$

Likewise, $U(0), U(1), \ldots, U(M+L-1)$ represent the DFT for the input in (3) which has been "padded" with M zeros so that it has the same length as the output and, thus, has as many DFT coefficients.

Because the matrix H in (4) is now diagonalized we simply multiply the DFT coefficients of the input and the DFT coefficients of the impulse response to obtain the DFT coefficients of the output:

$$\boxed{Y(k) = \sqrt{M+L}\, H(k) U(k) \quad k = 0, 1, \ldots, L+M-1.} \tag{7}$$

We can then reconstruct $y(t)$ from its DFT $Y(k)$. This method is preferable to the convolution sum performed in (4) since DFT is computed efficiently with FFT algorithms.

Summary: In the DFT basis a LTI system simply scales the DFT coefficients of the input. Each frequency component k is scaled by a factor $H(k)$ that comes from the DFT of the impulse response.

You can compare $H(k)$ to the transfer functions $H(\omega)$ studied in Lecture 3A for circuits. Unlike Lecture 3A, however, the derivation above is for discrete time systems and $H(k)$ is also discrete.

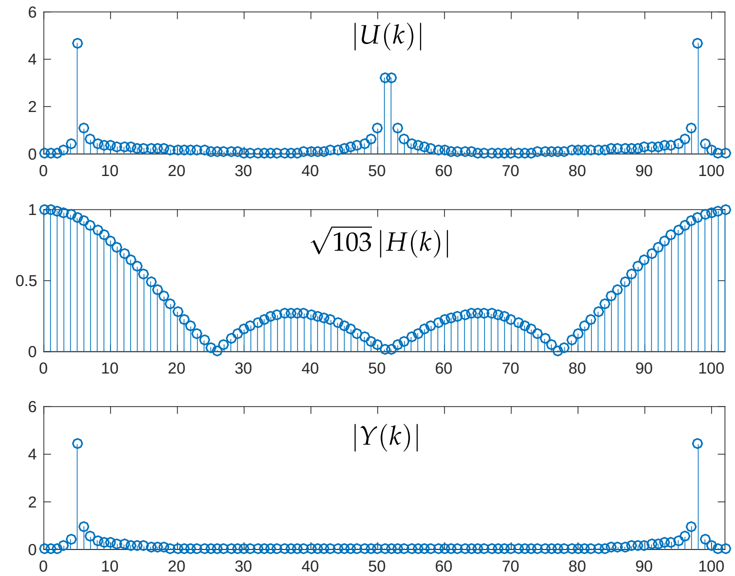
Example: Suppose we apply 100 samples of the signal

$$u(t) = \cos(0.1\pi t) + 0.5\cos(\pi t) \quad t = 0, 1, \ldots, 99$$

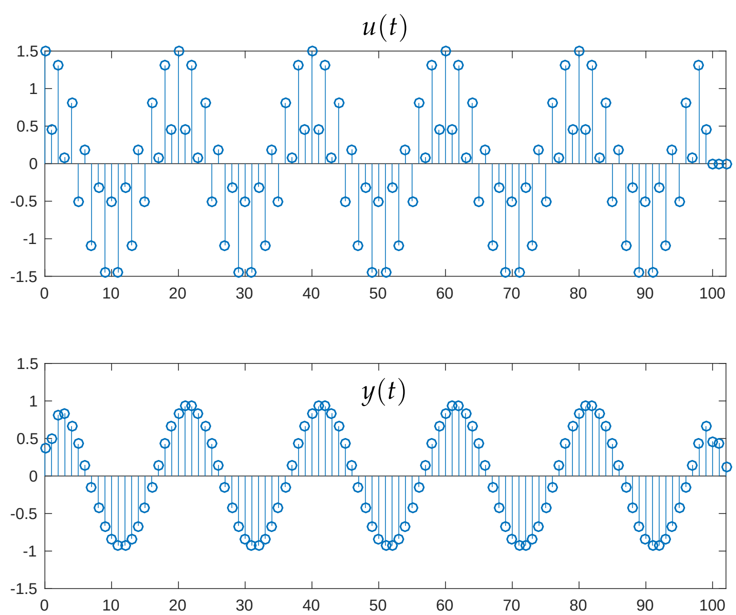
to a LTI system whose impulse response is

$$h(t) = \begin{cases} \frac{1}{4} & t = 0, 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

To find the output $y(t)$, we pad three zeros to the input and take the DFT of the resulting length 103 sequence. Likewise we pad 99 zeros to the length four impulse response and calculate its DFT. The figure below shows $|U(k)|$ (top) and $\sqrt{103}\,|H(k)|$ (middle), as well as their product (bottom) which gives $|Y(k)|$ from (7).



We see that the spikes around $k = 51$ and 52 (corresponding to the high frequency π) are eliminated in the output because $H(k)$ is close to zero in that frequency range. This is also visible in the $y(t)$ plot below where the $\cos(\pi t)$ component of $u(t)$ has been filtered out.



Proof of (6)

We first rewrite (5) as the sum

$$C = c_0 I + c_1 S + c_2 S^2 + \cdots c_{N-1} S^{N-1} \tag{8}$$

where the matrix S is defined as

$$S \triangleq \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

As an illustration, (8) decomposes a 3×3 circulant matrix as

$$\begin{bmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{bmatrix} = c_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_1 \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{S} + c_2 \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{S^2}.$$

Next we claim that

$$\vec{\Phi}_k^* S = W_k^* \vec{\Phi}_k^* \tag{9}$$

where $W_k = e^{jk\frac{2\pi}{N}}$. To see this recall that

$$\vec{\Phi}_k^* = \frac{1}{\sqrt{N}} \left[1 \; W_k^* \cdots (W_k^*)^{N-1} \right]$$

and note that multiplication by S gives

$$\vec{\Phi}_k^* S = \frac{1}{\sqrt{N}} \left[W_k^* \cdots (W_k^*)^{N-1} \; 1 \right] = \frac{1}{\sqrt{N}} \left[W_k^* \cdots (W_k^*)^{N-1} \; (W_k^*)^N \right]$$

where we substituted[2] $(W_k^*)^N = 1$. Factoring out W_k^* from each entry we get the expression (9). With a recursive application of (9) we also conclude that

$$\vec{\Phi}_k^* S^t = (W_k^t)^* \vec{\Phi}_k^*.$$

Then, using (8),

$$\vec{\Phi}_k^* C = \sum_{t=0}^{N-1} c_t \vec{\Phi}_k^* S^t = \sum_{t=0}^{N-1} c_t (W_k^t)^* \vec{\Phi}_k^* = \sqrt{N} C(k) \vec{\Phi}_k^* \tag{10}$$

where the last equality follows because

$$\sum_{t=0}^{N-1} c_t (W_k^t)^* = \left[1 \; W_k^* \cdots (W_k^*)^{N-1} \right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \sqrt{N} \underbrace{\vec{\Phi}_k^* \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}}_{C(k)}.$$

[2] This follows because $W_k = e^{jk\frac{2\pi}{N}}$.

Finally note from (10) that

$$\underbrace{\begin{bmatrix} \vec{\Phi}_0^* \\ \vec{\Phi}_1^* \\ \vec{\Phi}_2^* \\ \vdots \\ \vec{\Phi}_{N-1}^* \end{bmatrix}}_{F^*} C = \sqrt{N} \begin{bmatrix} C(0) & & & \\ & C(1) & & \\ & & \ddots & \\ & & & C(N-1) \end{bmatrix} \underbrace{\begin{bmatrix} \vec{\Phi}_0^* \\ \vec{\Phi}_1^* \\ \vec{\Phi}_2^* \\ \vdots \\ \vec{\Phi}_{N-1}^* \end{bmatrix}}_{F^*}.$$

Multiplying both sides from the right by F and substituting $F^*F = I$ on the right hand side, we obtain (6).