EE16B - Fall'16 - Lecture 4A Notes[1]

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State Space Representation of Control Systems

State variables are a set of variables that fully represent the state of a dynamical system at a given time, *e.g.*, capacitor voltages and inductor currents in a circuit, or positions and velocities in a mechanical system. We denote by $\vec{x}(t)$ the vector of these state variables and refer to it as the *state vector*.

Continuous-Time Systems

In a continuous-time system with n state variables, $\vec{x}(t) \in \mathbb{R}^n$ evolves according to a differential equation of the form

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t)) \tag{1}$$

where $f(\vec{x}(t))$ is an n-vector that dictates the derivatives of each state variable according to the current value of the states. The form of f depends on the system we are modeling as we will see in examples.

If the system has input variables we can manipulate (*e.g.* voltage and current sources in a circuit, or force and torque delivered to a mechanical system), we represent the system as

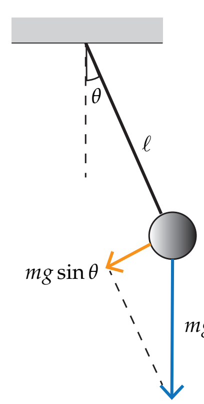
$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

where we refer to $\vec{u}(t)$ as the *control input*, since we can manipulate this input to influence the behavior of the system. Most of our examples will contain a single control input, but we write $\vec{u}(t)$ as a vector to allow for multiple control inputs.

Finally, we denote by $\vec{w}(t)$ other inputs that are not under our control, *e.g.* wind force in a flight control system, and add it to our model:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t), \vec{w}(t)).$$

The inputs contained in $\vec{w}(t)$ are often called *disturbances*.



Example 1: The motion of the pendulum depicted on the right is governed by the differential equation

$$m\ell\frac{d^2\theta(t)}{dt^2} = -k\ell\frac{d\theta(t)}{dt} - mg\sin\theta(t) \tag{2}$$

where the left hand side is mass×accelaration in the tangential direction and the right hand side is total force acting in that direction.

To bring this second order differential equation to state space form we define the state variables

$$x_1(t) \triangleq \theta(t) \qquad x_2(t) \triangleq \frac{d\theta(t)}{dt}$$

and note that they satisfy

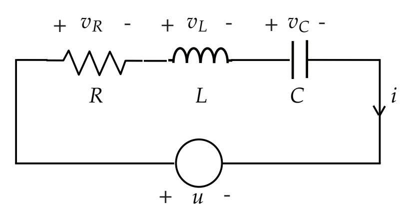
$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t). \end{aligned} \tag{3}$$

The first equation here follows from the definition of $x_2(t)$, and the second equation follows from (2). In this state representation we have two first order differential equations, one for each state variable, instead of the second order differential equation (2) for one variable.

Here we did not consider disturbances or control inputs that could be applied (say, to balance the pendulum in the upright position) so the equations (3) have the form (1) with

$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t) \end{bmatrix}.$$

Example 2: Consider the RLC circuit depicted on the right where u denotes the input voltage.



Since the capacitor and inductor satisfy the relations

$$\begin{aligned} C\frac{dv_c(t)}{dt} &= i(t) \\ L\frac{di(t)}{dt} &= v_L(t), \end{aligned} \tag{4}$$

we select $x_1 = v_C$ and $x_2 = i$ as the state variables, and eliminate $v_L(t)$ from the right hand side of (4) using KVL and Ohm's Law:

$$v_L = -v_C - v_R + u = -v_C - Ri + u.$$

Then the state model becomes

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \frac{1}{C}x_2(t) \\ \frac{dx_2(t)}{dt} &= \frac{1}{L}\left(-x_1(t) - Rx_2(t) + u(t)\right). \end{aligned} \tag{5}$$

Discrete-Time Systems

In a *discrete*-time system, $\vec{x}(t)$ evolves according to a *difference* equation rather than a differential equation:

$$\frac{d}{dt}\vec{x}(t+1) = f(\vec{x}(t), \vec{u}(t), \vec{w}(t)) \quad t = 0, 1, 2, \ldots$$

Here $f(\vec{x}(t), \vec{u}(t), \vec{w}(t))$ is a vector that dictates the value of the state vector at the next time instant based on the present values of the states and inputs.

Example 3: Let $s(t)$ denote the inventory of a manufacturer at the start of the t-th business day. The inventory at the start of the next day, $s(t+1)$, is the sum of $s(t)$ and the goods $g(t)$ manufactured, minus the goods $w(t)$ sold on day t. Assuming it takes a day to do the manufacturing, the amount of goods $g(t)$ manufactured is equal to the raw material available the previous day, $r(t-1)$. The raw material $r(t)$ is equal to the order placed the previous day, $u(t-1)$, assuming it takes a day for the order to arrive.

The state variables $s(t)$, $g(t)$, $r(t)$, thus evolve according to the model

$$\begin{aligned} s(t+1) &= s(t) + g(t) - w(t) \\ g(t+1) &= r(t) \\ r(t+1) &= u(t). \end{aligned} \tag{6}$$

Note that the order $u(t)$ is an input that the manufacturer can control, but the amount of goods sold, $w(t)$, depends on the customers.

Example 4: Let $p(t)$ be the number of EECS professors in a country in year t, and let $r(t)$ be the number of industry researchers with a PhD degree. A fraction, γ, of the PhDs become professors themselves and the rest become industry researchers. A fraction, δ, in each profession leaves the field every year due to retirement or other reasons.

Each professor graduates, on average, $u(t)$ PhD students per year. We treat this number as a control input because it can be manipulated by the government using research funding. This means there will be $p(t)u(t)$ new PhDs in year t, and $\gamma p(t)u(t)$ new professors. The state model is then

$$\begin{aligned} p(t+1) &= (1-\delta)p(t) + \gamma p(t)u(t) \\ r(t+1) &= (1-\delta)r(t) + (1-\gamma)p(t)u(t). \end{aligned} \tag{7}$$

Linear Systems

When $f(\vec{x},\vec{u},\vec{w}) \in \mathbb{R}^n$ is linear in $\vec{x} \in \mathbb{R}^n$, $\vec{u} \in \mathbb{R}^m$, $\vec{w} \in \mathbb{R}^d$, we can rewrite it in the form

$$f(\vec{x},\vec{u},\vec{w}) = A\vec{x} + B_u\vec{u} + B_w\vec{w}$$

where A is a $n \times n$ matrix, B_u is $n \times m$, and B_w is $n \times d$. The state equations then take the form

$$\begin{aligned} \frac{d}{dt}\vec{x}(t) &= A\vec{x}(t) + B_u\vec{u}(t) + B_w\vec{w}(t) \\ \vec{x}(t+1) &= A\vec{x}(t) + B_u\vec{u}(t) + B_w\vec{w}(t) \end{aligned}$$

for a continuous- and discrete-time system, respectively. When we don't need to differentiate between control and disturbance inputs, we drop the subscripts u and w from B.

Note that the models in Examples 2 and 3 above are linear[2]. In particular we can write (6) in the matrix form:

$$\underbrace{\begin{bmatrix} s(t+1) \\ g(t+1) \\ r(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s(t) \\ g(t) \\ r(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_u} u(t) + \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{B_w} w(t).$$

Likewise, we rewrite (5) as:

$$\underbrace{\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix}}_{\frac{d}{dt}\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{B} u(t).$$

[2] Why are Examples 1 and 4 nonlinear?