# *EE16B - Fall'16 - Lecture 4B Notes*[1]

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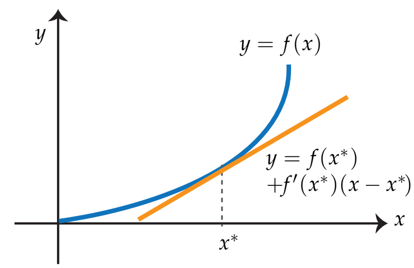
## *Linearization*

Linear models are advantageous because their solutions, stability properties, and stabilizing controllers can be studied using linear algebra. The methods applicable to nonlinear models are limited; therefore it is common practice to approximate a nonlinear model with a linear one that is valid around a desired operating point.




Recall that the Taylor approximation of a differentiable function $f$ around a point $x^*$ is:

$$f(x) \approx f(x^*) + \nabla f(x)|_{x=x^*}(x - x^*),$$

as illustrated on the right for a scalar-valued function of a single variable. When $x$ and $f(x)$ are $n$-vectors as in our state models, $\nabla f(x)$ must be interpreted as the $n \times n$ matrix of partial derivatives:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f_1(x_1,\ldots,x_n)}{\partial x_1} & \frac{\partial f_1(x_1,\ldots,x_n)}{\partial x_2} & \cdots & \frac{\partial f_1(x_1,\ldots,x_n)}{\partial x_n} \\ \frac{\partial f_2(x_1,\ldots,x_n)}{\partial x_1} & \frac{\partial f_2(x_1,\ldots,x_n)}{\partial x_2} & \cdots & \frac{\partial f_2(x_1,\ldots,x_n)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(x_1,\ldots,x_n)}{\partial x_1} & \frac{\partial f_n(x_1,\ldots,x_n)}{\partial x_2} & \cdots & \frac{\partial f_n(x_1,\ldots,x_n)}{\partial x_n} \end{bmatrix}.$$

We linearize nonlinear state models by applying this approximation around an *equilibrium* point. For the continuous-time system

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t)), \tag{1}$$

$\vec{x}^*$ is called an *equilbrium* when $f(\vec{x}^*) = 0$ because, if the initial condition is $\vec{x}^*$, then $\frac{d}{dt}\vec{x}(t) = 0$ and $\vec{x}(t)$ remains at $\vec{x}^*$. If we define the deviation of $\vec{x}$ from $\vec{x}^*$ as:

$$\boxed{\tilde{x}(t) \triangleq \vec{x}(t) - \vec{x}^*} \tag{2}$$

then we see that

$$\frac{d}{dt}\tilde{x}(t) = f(\vec{x}(t)) \approx f(\vec{x}^*) + \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*}\tilde{x}(t).$$

Substituting $f(\vec{x}^*) = 0$ and defining

$$\boxed{A \triangleq \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*}} \tag{3}$$

we obtain the linearization of (1) around the equilibrium $\vec{x}^*$:

$$\boxed{\frac{d}{dt}\tilde{x}(t) \approx A\tilde{x}(t).}$$

In the discrete-time case

$$\vec{x}(t+1) = f(\vec{x}(t)),$$

$\vec{x}^*$ is an equilbrium point if $f(\vec{x}^*) = \vec{x}^*$. The vector $\tilde{x}(t)$ defined in (2) satisfies:

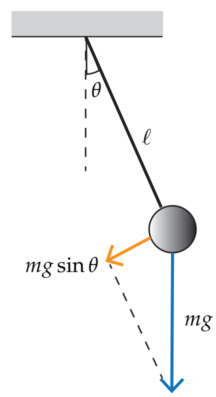
$$\tilde{x}(t+1) = \vec{x}(t+1) - \vec{x}^* = f(\vec{x}(t)) - \vec{x}^* \approx f(\vec{x}^*) - \vec{x}^* + \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*}\,\tilde{x}(t).$$

Substituting $f(\vec{x}^*) - \vec{x}^* = 0$ and defining $A$ as in (3), we get

$$\boxed{\tilde{x}(t+1) \approx A\tilde{x}(t).}$$

<u>Example:</u> Recall the pendulum model derived in Lecture 4A:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t) \end{aligned} \tag{4}$$




where

$$x_1(t) \triangleq \theta(t) \quad \text{and} \quad x_2(t) \triangleq \frac{d\theta(t)}{dt}.$$

To find the equilibrium points note that

$$f(\vec{x}) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} = 0$$

when $x_2 = 0$ and $\sin x_1 = 0$. Thus the two distinct equilibrium points are the downward position:

$$x_1 = 0, \quad x_2 = 0, \tag{5}$$

and the upright position:

$$x_1 = \pi, \quad x_2 = 0. \tag{6}$$

Since the entries of $f(\vec{x})$ are $f_1(\vec{x}) = x_2$ and $f_2(\vec{x}) = -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1$, we have

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & \frac{-k}{m} \end{bmatrix}.$$

By evaluating this matrix at (5) and (6), we obtain the linearization around the respective equilibrium point:

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & \frac{-k}{m} \end{bmatrix} \qquad A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-k}{m} \end{bmatrix}. \tag{7}$$

## *Changing State Variables*

It is important to note that the choice of state variables is not unique. Given the state vector $\vec{x} \in \mathbb{R}^n$ any transformation of the form

$$\vec{z} \triangleq T\vec{x}, \tag{8}$$

where $T$ is a $n \times n$ invertible matrix, defines new variables $z_i$, $i = 1, \ldots, n$, as a linear combination of the original variables $x_1, \ldots, x_n$.

To see how this change of variables affects the state equation

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t),$$

note that

$$\vec{z}(t+1) = T\vec{x}(t+1) = TA\vec{x}(t) + TB\vec{u}(t)$$

and substitute $\vec{x} = T^{-1}\vec{z}$ in the right hand side to obtain:

$$\vec{z}(t+1) = TAT^{-1}\vec{z}(t) + TB\vec{u}(t).$$

Thus the original $A$ and $B$ matrices are replaced with:

$$A_{\text{new}} = TAT^{-1}, \qquad B_{\text{new}} = TB.$$

The same change of variables brings the *continuous-time* system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

to the form

$$\frac{d}{dt}\vec{z}(t) = A_{\text{new}}\vec{z}(t) + B_{\text{new}}\vec{u}(t)$$

where $A_{\text{new}}$ and $B_{\text{new}}$ are as defined above.

In subsequent lectures we will use particular choices of $T$ to bring $A_{\text{new}}$ and $B_{\text{new}}$ to special forms that will make it easy to analyze properties such as *stability* and *controllability*.

## *Stability of Linear State Models*

### *The Scalar Case*

We first study a system with a single state variable $x(t)$ that obeys

$$x(t+1) = ax(t) + bu(t) \tag{9}$$

where $a$ and $b$ are constants. If we start with the initial condition $x(0)$, then we get by recursion

$$
\begin{aligned}
x(1) &= ax(0) + bu(0) \\
x(2) &= ax(1) + bu(1) = a^2x(0) + abu(0) + bu(1) \\
x(3) &= ax(2) + bu(2) = a^3x(0) + a^2bu(0) + abu(1) + bu(2) \\
&\vdots \\
x(t) &= a^t x(0) + a^{t-1}bu(0) + a^{t-2}bu(1) + \cdots + abu(t-2) + bu(t-1),
\end{aligned}
$$

rewritten compactly as:

$$
\boxed{x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-1-k} bu(k) \quad t = 1, 2, 3, \ldots} \qquad (10)
$$

The first term $a^t x(0)$ represents the effect of the initial condition and the second term $\sum_{k=0}^{t-1} a^{t-1-k} bu(k)$ represents the effect of the input sequence $u(0), u(1), \ldots, u(t-1)$.

**Definition.** We say that a system is *stable* if its state $x(t)$ remains bounded for any initial condition and any bounded input sequence. Conversely, we say it is *unstable* if we can find an initial condition and a bounded input sequence such that $|x(t)| \to \infty$ as $t \to \infty$.

It follows from (10) that, if $|a| > 1$, then a nonzero initial condition $x(0) \neq 0$ is enough to drive $|x(t)|$ unbounded. This is because $|a|^t$ grows unbounded and, with $u(t) = 0$ for all $t$, we get $|x(t)| = |a^t x(0)| = |a|^t |x(0)| \to \infty$. Thus, (9) is unstable for $|a| > 1$.

In the next lecture we will show that (9) is stable when $|a| < 1$ .