EE16B - Fall'16 - Lecture 5A Notes[1]

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27 September 2016

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Stability of Linear State Models Continued

The Scalar Case

In the last lecture we saw that the solution of the scalar equation

$$x(t+1) = ax(t) + bu(t) \tag{1}$$

is:

$$\boxed{x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-1-k} bu(k) \quad t = 1, 2, 3, \ldots} \tag{2}$$

The first term $a^t x(0)$ represents the effect of the initial condition and the second term $\sum_{k=0}^{t-1} a^{t-1-k} bu(k)$ represents the effect of the input sequence $u(0), u(1), \ldots, u(t-1)$.

Definition. We say that a system is *stable* if its state $x(t)$ remains bounded for any initial condition and any bounded input sequence. Conversely, we say it is *unstable* if we can find an initial condition and a bounded input sequence such that $|x(t)| \to \infty$ as $t \to \infty$.

It follows from (2) that, if $|a| > 1$, then a nonzero initial condition $x(0) \neq 0$ in enough to drive $|x(t)|$ unbounded. This is because $|a|^t$ grows unbounded and, with $u(t) = 0$ for all t, we get $|x(t)| = |a^t x(0)| = |a|^t |x(0)| \to \infty$. Thus, (1) is unstable for $|a| > 1$.

Next, we show that (1) is stable when $|a| < 1$ is stable. In this case $a^t x(0)$ decays to zero, so we need only to show that the second term in (2) remains bounded for any bounded input sequence. A bounded input means we can find a constant M such that $|u(t)| \leq M$ for all t. Thus,

$$\left| \sum_{k=0}^{t-1} a^{t-1-k} bu(k) \right| \leq \sum_{k=0}^{t-1} |a|^{t-1-k} |b| |u(k)| \leq |b| M \sum_{k=0}^{t-1} |a|^{t-1-k}.$$

Defining the new index $s = t - 1 - k$ we rewrite the last expression as

$$|b| M \sum_{s=0}^{t-1} |a|^s,$$

and note that $\sum_{s=0}^{t-1} |a|^s$ is a geometric series that converges to $\frac{1}{1-|a|}$ since $|a| < 1$. Therefore, each term in (2) is bounded and we conclude stability for $|a| < 1$.

Summary: The scalar system (1) is stable when $|a| < 1$, and unstable when $|a| > 1$.

When a is a complex number, a perusal of the stability and instability arguments above show that the same conclusions hold if we interpret $|a|$ as the modulus of a, that is:

$$|a| = \sqrt{\text{Re}\{a\}^2 + \text{Im}\{a\}^2}.$$

What happens when $|a| = 1$? If we disallow inputs ($b = 0$), this case is referred to as "marginal stability" because $|a^t x(0)| = |x(0)|$, which neither grows nor decays. If we allow inputs ($b \neq 0$), however, we can find a bounded input to drive the second term in (2) unbounded. For example, when $a = 1$, the constant input $u(t) = 1$ yields:

$$\sum_{k=0}^{t-1} a^{t-1-k} b u(k) = \sum_{k=0}^{t-1} b = bt$$

which grows unbounded as $t \to \infty$. Therefore, $|a| = 1$ is a precarious case that must be avoided in designing systems.

The Vector Case

When $\vec{x}(t)$ is an n-dimensional vector governed by

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t), \tag{3}$$

recursive calculations lead to the expression

$$\boxed{\vec{x}(t) = A^t \vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} B u(k) \quad t = 1, 2, 3, \ldots} \tag{4}$$

This is similar to (2), except that the scalars a and b are now replaced with matrices A and B, and a^t is replaced with the matrix power $A^t = \underbrace{A \cdots A}_{t \text{ times}}$.

Unlike the scalar case (2), stability properties are not apparent from (4). However, when A is diagonalizable we can employ the change of variables $\vec{z} \triangleq T\vec{x}$ and select the matrix T such that

$$A_{\text{new}} = TAT^{-1}$$

is diagonal. A and A_{new} have the same eigenvalues and, since A_{new} is diagonal, the eigenvalues appear as its diagonal entries:

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

The state model for the new variables is

$$\vec{z}(t+1) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \vec{z}(t) + B_{\text{new}} u(t) \tag{5}$$

which nicely decouples into scalar equations:

$$z_i(t+1) = \lambda_i z_i(t) + b_i u(t), \quad i = 1, \ldots, n \tag{6}$$

where we denote by b_i the i-th entry of B_{new}. Then, the results of the previous section imply stability when $|\lambda_i| < 1$ and instability when $|\lambda_i| > 1$.

For the whole system to be stable each subsystem must be stable, therefore we need $|\lambda_i| < 1$ for each $i = 1, \ldots, n$ for stability. If there exists at least one eigenvalue λ_i with $|\lambda_i| > 1$ then we conclude instability because we can drive the corresponding state $z_i(t)$ unbounded.

Summary: The discrete-time system (3) is stable if $|\lambda_i| < 1$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of A, and unstable if $|\lambda_i| > 1$ for some eigenvalue λ_i.

Although we assumed diagonalizability of A above, the same stability and instability conditions hold when A is not diagonalizable. In that case a transformation exists that brings A_{new} to an upper-diagonal form with eigenvalues on the diagonal[2]. Thus, instead of (5) we have

[2] The details of this transformation are beyond the scope of this course.

$$\vec{z}(t+1) = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ & \ddots & \ddots & \vdots \\ & & \ddots & \star \\ & & & \lambda_n \end{bmatrix} \vec{z}(t) + B_{\text{new}} u(t) \tag{7}$$

where the entries marked with '$\star$' may be nonzero, but we don't need their explicit values for the argument that follows. Then it is not difficult to see that z_n obeys

$$z_n(t+1) = \lambda_n z_n(t) + b_n u(t) \tag{8}$$

which does not depend on other states, so we conclude $z_n(t)$ remains bounded for bounded inputs when $|\lambda_n| < 1$. The equation for z_{n-1} has the form

$$z_{n-1}(t+1) = \lambda_{n-1} z_{n-1}(t) + [\star\, z_n(t) + b_{n-1} u(t)] \tag{9}$$

where we can treat the last two terms in brackets as a bounded input since we have already shown that $z_n(t)$ is bounded. If $|\lambda_{n-1}| < 1$ we conclude $z_{n-1}(t)$ is itself bounded and proceed to the equation:

$$z_{n-2}(t+1) = \lambda_{n-2} z_{n-2}(t) + [\star\, z_{n-1}(t) + \star\, z_n(t) + b_{n-2} u(t)]. \quad (10)$$

Continuing this argument recursively we conclude stability when $|\lambda_i| < 1$ for each eigenvalue λ_i.

To conclude instability when $|\lambda_i| > 1$ for some eigenvalue, note that the ordering of the eigenvalues in (7) is arbitrary: we can put them in any order we want by properly selecting T. Therefore, we can assume without loss of generality that an eigenvalue with $|\lambda_i| > 1$ appears in the nth diagonal entry, that is $|\lambda_n| > 1$. Then, instability follows from the scalar equation (8).

Stability of Linear Continuous-Time State Models

We will state analogous results for the continuous-time case without detailed derivations. The first-order differential equation

$$\frac{dx(t)}{dt} = ax(t) + bu(t)$$

admits the solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-s)} bu(s) ds,$$

which is analogous to (2). It is stable if $a < 0$ and unstable if $a > 0$. If we allow a to be complex, these conditions become $\text{Re}\{a\} < 0$ and $\text{Re}\{a\} > 0$, respectively.

For the vector case

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \quad (11)$$

we use a change of variables that brings A to a diagonal form when it is diagonalizable, and to an upper-diagonal form otherwise. Then arguments similar to the discrete-time case lead to the following conclusion:

Summary: The continuous-time system (11) is stable if $\text{Re}\{\lambda_i\} < 0$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of A, and unstable if $\text{Re}\{\lambda_i\} > 0$ for some eigenvalue λ_i.