# *EE16B - Fall'16 - Lecture 6A Notes*[1]

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## *Controllability*

As we saw in Lecture 5A the solution of the discrete-time state model

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t), \tag{1}$$

where $\vec{x}(t)$ is an $n$-dimensional vector, is given by

$$\vec{x}(t) = A^t\vec{x}(0) + A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + ABu(t-2) + Bu(t-1)$$

or, equivalently,

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{\triangleq\, R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-2) \\ u(t-1) \end{bmatrix}. \tag{2}$$

Can we find an input sequence $u(0), u(1), \ldots, u(t-1)$ that brings the state from $\vec{x}(0)$ to a desired value $\vec{x}(t) = \vec{x}_{\text{target}}$? The answer is yes if $\vec{x}_{\text{target}} - A^t\vec{x}(0)$ lies in the range space of $R_t$ because, then we can find appropriate values for $u(0), u(1), \ldots, u(t-1)$, *i.e.* an appropriate linear combination of the columns of $R_t$, so that the right hand side of (2) is $\vec{x}_{\text{target}} - A^t\vec{x}(0)$ and, thus, $\vec{x}(t) = \vec{x}_{\text{target}}$.

Now, if we want to be able to reach any $\vec{x}_{\text{target}}$ from any initial $\vec{x}(0)$ – a property we henceforth refer to as *controllability* – then the range of $R_t$ must be the whole $n$-dimensional space, that is $R_t$ must have $n$ linearly independent columns for some $t$.

Since increasing $t$ amounts to adding more columns to $R_t$, it may appear that we may eventually have $n$ independent columns. However, this is not so: *if we don't have $n$ independent columns in $R_t$ at $t = n$, we never will for $t > n$.* This is a consequence of a result[2] in linear algebra which states that, if $A$ is $n \times n$, then $A^n$ can be written as a linear combination of $A^{n-1}, \ldots, A, I$:

[2] This result is known as the Cayley-Hamilton Theorem and its details are beyond the scope of this course. You can consult the Wikipedia article if you are interested.

$$A^n = \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \quad \text{for some} \quad \alpha_{n-1}, \ldots, \alpha_1, \alpha_0.$$

Multiplying both sides from the right by $B$ we see that $A^nB$ is itself a linear combination of $A^{n-1}B, \ldots, AB, B$. This means that the new columns in $R_{n+1}$ are merely linear combinations of the columns of $R_n$, and the same argument extends to $R_{n+2}, R_{n+3}, \cdots$.

Thus, for controllability, we need $R_n$ to have $n$ linearly independent columns, which means rank $= n$:

$$\text{Controllability} \quad \Leftrightarrow \quad \text{rank} \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix} = n.$$

Example: The system

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)$$

is uncontrollable because the matrix

$$\begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has rank $= 1$ rather than $n = 2$. The reason for uncontrollability becomes clear if we write the equation for $x_2(t)$ explicitly:

$$x_2(t+1) = 2x_2(t).$$

The right hand side doesn't depend on $u(t)$ or $x_1(t)$, which means that $x_2(t)$ evolves independently and can be influenced neither directly by input $u(t)$, nor indirectly through the other state $x_1(t)$.

Example: Consider a vehicle moving in a lane with speed $v(t)$. We are able to change the acceleration, $u(t)$, every $T$ seconds and it remains constant for the next $T$ seconds. This suggests a discrete-time model which we obtain from the relationship

$$\frac{d}{dt}v(t) = u(t)$$

by integrating both sides from $t$ to $t+T$ and keeping in mind that $u(t)$ is constant in this interval, that is $u(t+\tau) = u(t)$ for $\tau \in [0, T)$:

$$v(t+T) - v(t) = \int_0^T u(t+\tau)d\tau = Tu(t). \qquad (3)$$

Next, we let $p(t)$ denote the position and note that

$$\frac{d}{dt}p(t) = v(t).$$

Integrating from $t$ to $t+T$ and substituting $v(t+\tau) = v(t) + \tau u(t)$ we get

$$p(t+T) - p(t) = \int_0^T (v(t) + \tau u(t))d\tau = Tv(t) + \frac{1}{2}T^2 u(t). \qquad (4)$$

Finally we combine (3) and (4) into the state model:

$$\begin{bmatrix} p(t+T) \\ v(t+T) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}}_{B} u(t)$$

which is of the form (1) if we take the unit time to be $T$ seconds.

The controllability condition above holds: the rank of

$$\begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} \frac{3}{2}T^2 & \frac{1}{2}T^2 \\ T & T \end{bmatrix}$$

is indeed $n = 2$. This confirms that we can move the vehicle to an arbitrary target location $p_{\text{target}}$ and stop there ($v_{\text{target}} = 0$) by applying an appropriate input sequence obtained from (2).

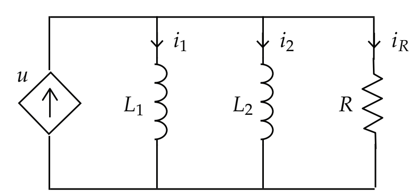
## *Controllability in Continuous-Time*

The controllability condition for the continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t) \tag{5}$$

is exactly the same: $\text{rank}(R_n) = n$ where $R_n$ is as defined above. We omit the derivation for this case but illustrate the result with a circuit example.

Example: For the circuit depicted on the right we treat the current source as the control $u(t)$, and the inductor currents $i_1(t)$ and $i_2(t)$ as the state variables.




Since the voltage across each capacitor is the same as the voltage across the resistor, we have

$$\begin{aligned} L_1 \frac{di_1(t)}{dt} &= Ri_R(t) \\ L_2 \frac{di_2(t)}{dt} &= Ri_R(t). \end{aligned} \tag{6}$$

Substituting $i_R = u - i_1 - i_2$ from KCL and dividing the equations by $L_1$ and $L_2$ respectively, we get

$$\begin{bmatrix} \frac{di_1(t)}{dt} \\ \frac{di_2(t)}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix}}_{A} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix}}_{B} u(t).$$

Note that

$$AB = \begin{bmatrix} -\frac{R}{L_1}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \\ -\frac{R}{L_2}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \end{bmatrix} = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right) B$$

which means that $AB$ and $B$ are linearly dependent. Thus

$$\text{rank}\begin{bmatrix} AB & B \end{bmatrix} = 1,$$

and the model is *not* controllable.

To see the physical obstacle to controllability note that the two inductors in parallel share the same voltage:

$$L_1 \frac{di_1(t)}{dt} = L_2 \frac{di_2(t)}{dt}.$$

Thus,

$$\frac{d}{dt}\left(L_1 i_1(t) - L_2 i_2(t)\right) = 0$$

which means that the difference between the two inductor fluxes, $L_1 i_1 - L_2 i_2$, remains constant no matter what $u$ we apply.

Because of this constraint we can't control $i_1$ and $i_2$ independently. We can, however, control the total current $i_L = i_1 + i_2$ which obeys, from (6),

$$\frac{di_L(t)}{dt} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right) R i_R(t) = \frac{R}{L}(-i_L(t) + u(t))$$

where $L \triangleq \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$. Note that this is the governing equation for the circuit on the right where the two inductors are lumped into one.