EE16B - Fall'16 - Lecture 6B Notes[1]

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6 October 2016

State Feedback Control

Suppose we are given a single-input control system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t), \quad u(t) \in \mathbb{R}, \tag{1}$$

and we wish to bring the solution $\vec{x}(t)$ to the equilibrium $\vec{x} = 0$ from any initial condition $\vec{x}(0)$.

To achieve this goal we will study a "control policy" of the form

$$u(t) = k_1 x_1(t) + k_2 x_2(t) + \cdots + k_n x_n(t) \tag{2}$$

where $k_1, k_2, \ldots, k_n$ are to be determined. Rewriting (2) as

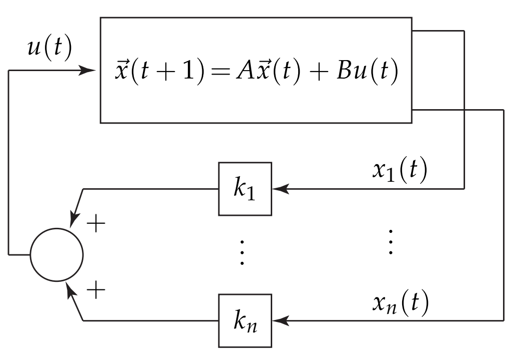
$$\boxed{u(t) = K\vec{x}(t)} \tag{3}$$

with row vector $K = [k_1\, k_2\, \cdots\, k_n]$, and substituting in (1), we get

$$\boxed{\vec{x}(t+1) = (A + BK)\vec{x}(t).} \tag{4}$$

Thus, if we can choose K such that all eigenvalues of $A + BK$ satisfy the stability condition $|\lambda_i(A + BK)| < 1$, then $\vec{x}(t) \to 0$ from any $\vec{x}(0)$.

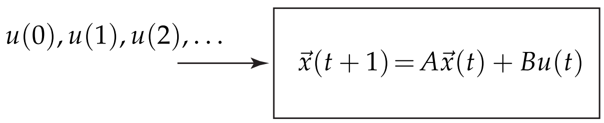
We will see in the next lecture that if the system (1) is controllable, then we can arbitrarily assign the eigenvalues of $A + BK$ with the choice of K. Thus, in addition to bringing the eigenvalues inside the unit disk for stability, we can place them in favorable locations to shape the transients, *e.g.*, to achieve a well damped convergence.



We refer to (4) as the "closed-loop" system since the control policy (2) generates a feedback loop as depicted in the block diagram. The state variables are measured at every time step t and the input $u(t)$ is synthesized as a linear combination of these measurements.

Comparison to Open Loop Control

Recall from the last lecture that controllability allows us to calculate an input sequence $u(0), u(1), u(2), \ldots$ that drives the state from $\vec{x}(0)$ to any $\vec{x}_{\text{target}}$. Thus, an alternative to the feedback control (2) is to select $\vec{x}_{\text{target}} = 0$, calculate an input sequence based on $\vec{x}(0)$, and to apply this sequence in an "open-loop" fashion without using further state measurements as depicted below.



The trouble with this open-loop approach is that it is sensitive to uncertainties in A and B, and does not make provisions against disturbances that may act on the system.

By contrast, feedback offers a degree of robustness: if our design of K brings the eigenvalues of $A + BK$ to well within the unit disk, then small perturbations in A and B would not move these eigenvalues outside the disk. Thus, despite the uncertainty, solutions converge to $\vec{x} = 0$ in the absence of disturbances and remain bounded in the presence of bounded disturbances.

Example (Cruise Control): Consider again a vehicle moving in a lane (Lecture 6A) where the state equation for the velocity $v(t)$ is

$$v(t+1) = v(t) + Tu(t). \tag{5}$$

Suppose we want to stabilize the velocity to a desired value v^*. Define $\tilde{v}(t) = v(t) - v^*$, and subtract v^* from both sides of the equation:

$$\tilde{v}(t+1) = \tilde{v}(t) + Tu(t). \tag{6}$$

Then the controller

$$u(t) = k\tilde{v}(t) = k(v(t) - v^*)$$

results in the closed-loop system

$$\tilde{v}(t+1) = (1 + kT)\tilde{v}(t)$$

which is stable if we choose the coefficient k such that $|1 + kT| < 1$.

Eigenvalue Assignment: Second Order Examples

Example: Consider the second order system

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t) \tag{7}$$

and note that the eigenvalues of A are the roots of the polynomial[2]

$$\lambda^2 - a_2\lambda - a_1.$$

[2] obtained from $\det(\lambda I - A)$

If we substitute the control

$$u(t) = K\vec{x}(t) = k_1 x_1(t) + k_2 x_2(t)$$

the closed-loop system becomes

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}}_{A+BK} \vec{x}(t) \tag{8}$$

and, since $A + BK$ has the same structure as A with a_1, a_2 replaced by $a_1 + k_1$, $a_2 + k_2$, the eigenvalues of $A + BK$ are the roots of

$$\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1).$$

Now if we want to assign the eigenvalues of $A + BK$ to desired values λ_1 and λ_2, we must match the polynomial above to

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2,$$

that is,

$$a_2 + k_2 = \lambda_1 + \lambda_2 \quad \text{and} \quad a_1 + k_1 = -\lambda_1\lambda_2.$$

This is indeed accomplished with the choice $k_1 = -a_1 - \lambda_1\lambda_2$ and $k_2 = -a_2 + \lambda_1 + \lambda_2$, which means that we can assign the closed-loop eigenvalues as we wish.

Example: Let's apply the eigenvalue assignment procedure above to

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t).$$

Now we have

$$A + BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 + k_1 & 1 + k_2 \\ 0 & 2 \end{bmatrix}$$

and, because this matrix is upper diagonal, its eigenvalues are the diagonal entries:

$$\lambda_1 = 1 + k_1 \quad \text{and} \quad \lambda_2 = 2.$$

Note that we can move λ_1 with the choice of k_1, but we have no control over λ_2. In fact, since $|\lambda_2| > 1$, the closed-loop system remains unstable no matter what control we apply.

This is a consequence of the uncontrollability of this example which was shown in the previous lecture. The second state equation

$$x_2(t+1) = 2x_2(t)$$

can't be influenced by $u(t)$, and $x_2(t) = 2^t x_2(0)$ grows exponentially.

By contrast the previous example was controllable[3]. In the next section we argue that controllability allows us to arbitrarily assign the eigenvalues of $A + BK$ with the choice of K.

[3] Show this.