EE16B - Fall'16 - Lecture 7A Notes[1]

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11 October 2016

State Feedback Control Continued

Although we discussed only discrete-time systems in the last lecture, the idea of state feedback is identical for the continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t), \quad u(t) \in \mathbb{R}. \tag{1}$$

To bring $\vec{x}(t)$ to the equilibrium $\vec{x} = 0$ we apply

$$u(t) = K\vec{x}(t) \tag{2}$$

and obtain the closed-loop system

$$\frac{d}{dt}\vec{x}(t) = (A + BK)\vec{x}(t). \tag{3}$$

The only difference from discrete-time is the stability criterion: we must choose K such that $\mathrm{Re}\{\lambda_i(A + BK)\} < 0$ for each eigenvalue λ_i.

Today we will argue that controllability allows us to arbitrarily assign the eigenvalues of $A + BK$ with the choice of K. We will again default to discrete-time models, but the arguments are unchanged for continuous-time.

Eigenvalue Assignment: The General Case

In the previous lecture we studied the example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t) \tag{4}$$

and showed that the eigenvalues of $A + BK$ are the roots of

$$\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1)$$

which can be assigned to desired values λ_1 and λ_2 by matching the polynomial above to

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

We now generalize the special structure of A and B in (4) to $n > 2$:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

The first benefit of this structure is the simple form of the polynomial

$$\det(\lambda I - A) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_2\lambda - a_1$$

whose roots constitute the eigenvalues of A. The second benefit is that $A + BK$ preserves the structure of A, except that the entry a_i is replaced by $a_i + k_i$, $i = 1, \ldots, n$. Therefore,

$$\det(\lambda I - (A + BK)) = \lambda^n - (a_n + k_n)\lambda^{n-1} \cdots - (a_2 + k_2)\lambda - (a_1 + k_1)$$

and assigning the closed-loop eigenvalues to desired values $\lambda_1, \ldots, \lambda_n$ amounts to matching the coefficients of this polynomial to those of

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Thus eigenvalue assignment is straightforward when A and B have the special form above, known as the *controller canonical form*.

Example: Suppose we want all eigenvalues of $A + BK$ to be 0 for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This means that we must match the polynomial

$$\det(\lambda I - (A + BK)) = \lambda^3 - (3 + k_3)\lambda^2 - (2 + k_2)\lambda - (1 + k_1)$$

to λ^3 which has three roots at 0. This is accomplished with the choice $k_1 = -1$, $k_2 = -2$, $k_3 = -3$. □

It turns out that we can bring any controllable, single-input system

$$\vec{x}(t+1) = \tilde{A}\vec{x}(t) + \tilde{B}u(t) \tag{6}$$

to the controller canonical form with a change of variables $\vec{z} = T\vec{x}$; that is, there exists T such that

$$T\tilde{A}T^{-1} = A \quad \text{and} \quad T\tilde{B} = B \tag{7}$$

where A and B are as in (5).

This means that we can design a state feedback $u = K\vec{z}$ to assign the eigenvalues of $A + BK$ using the procedure above for the controller canonical form. Since $\vec{z} = T\vec{x}$, $u = K\vec{z}$ is identical to $u = \tilde{K}\vec{x}$ where

$$\tilde{K} = KT \tag{8}$$

and, since $T(\tilde{A} + \tilde{B}\tilde{K})T^{-1} = A + BK$, the eigenvalues of $\tilde{A} + \tilde{B}\tilde{K}$ are the same as those of $A + BK$.

Conclusion: If the system (6) is controllable, then we can arbitrarily assign the eigenvalues of $\tilde{A} + \tilde{B}\tilde{K}$ with an appropriate choice of $\tilde{K}$.

How can we find a matrix T that satisfies (7)? Recall the matrix we use for checking controllability:

$$R_n = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix}. \tag{9}$$

If we substitue (7), we see that

$$\begin{aligned} R_n &= \begin{bmatrix} (T\tilde{A}^{n-1}T^{-1})(T\tilde{B}) & \cdots & (T\tilde{A}T^{-1})(T\tilde{B}) & (T\tilde{B}) \end{bmatrix} \\ &= T \underbrace{\begin{bmatrix} \tilde{A}^{n-1}\tilde{B} & \cdots & \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix}}_{=\tilde{R}_n} \end{aligned} \tag{10}$$

which suggests the choice $T = R_n \tilde{R}_n^{-1}$.

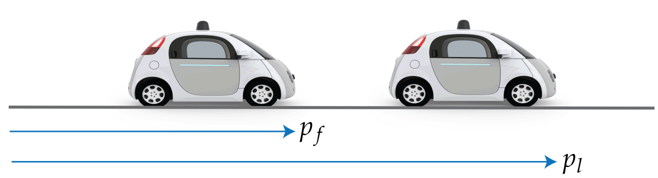
$\tilde{R}_n$ is full rank, thus invertible, because (6) is controllable. Likewise, R_n has the lower diagonal form

$$R_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \star & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \star & \cdots & \star & 1 \end{bmatrix}$$

that has rank n regardless of the values of the entries marked as '$\star$'. Then $T = R_n \tilde{R}_n^{-1}$ is itself invertible, thus a viable choice for (7).

Case Study: Cooperative Adaptive Cruise Control (CACC)

Consider a vehicle following another as depicted below.



We denote by p_l the position of the leader and by p_f the position of the follower, and write the continuous-time models

$$\begin{aligned} \frac{d}{dt}p_l(t) &= v_l(t) \\ \frac{d}{dt}v_l(t) &= u_l(t) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \frac{d}{dt}p_r(t) &= v_r(t) \\ \frac{d}{dt}v_r(t) &= u_r(t). \end{aligned} \tag{12}$$

To maintain a constant distance δ between p_l and p_f, we define

$$x_1 \triangleq p_l - p_f - \delta \quad x_2 \triangleq v_l - v_f \quad u \triangleq u_l - u_f, \tag{13}$$

and obtain from (11) and (12) the following model that describes the relative motion of the two vehicles:

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_2(t) \\ \frac{d}{dt}x_2(t) &= u(t). \end{aligned} \tag{14}$$

Then the task is to stabilize the equilibrium $x_1 = 0$, $x_2 = 0$ which means $p_l - p_f = \delta$ and $v_l = v_f$. This is accomplished with

$$u(t) = k_1 x_1(t) + k_2 x_2(t)$$

where k_1 and k_2 are selected such that the eigenvalues of

$$A + BK = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$$

have negative real parts.

Recall from (13) that $u = u_l - u_f$. Thus, if the lead vehicle broadcasts its acceleration $u_l(t)$ to the follower via vehicle-to-vehicle wireless communication[2], then the follower can implement the controller

[2] hence the term *cooperative* adaptive cruise control

$$\begin{aligned} u_f(t) &= u_l(t) - u(t) \\ &= u_l(t) - k_1 x_1(t) - k_2 x_2(t) \\ &= u_l(t) - k_1(p_l(t) - p_f(t) - \delta) - k_2(v_l(t) - v_f(t)) \end{aligned} \tag{15}$$

using range sensors to obtain $p_l(t) - p_f(t)$ and $v_l(t) - v_f(t)$. In this implementation the lead vehicle chooses its own input $u_l(t)$ without regard to the follower, and the follower applies the controller (15) to automatically follow the leader with constant relative distance δ.