EE16B - Fall'16 - Lecture 8A Notes[1]

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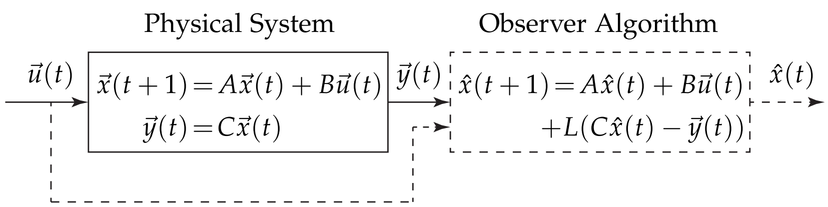
Observers Continued

In the last lecture we introduced an observer where the state estimate, denoted by $\hat{x}(t)$, is updated at each time according to

$$\hat{x}(t+1) = A\hat{x}(t) + L(C\hat{x}(t) - \vec{y}(t)) + B\vec{u}(t) \tag{1}$$

as shown in the block diagram below.



This observer incorporates the model of the physical system and augments it with the term $L(C\hat{x}(t) - \vec{y}(t))$. To see the role played by this term, define the estimation error

$$\vec{e}(t) \triangleq \hat{x}(t) - \vec{x}(t)$$

and note from (1) and the state model $\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$ that

$$\begin{aligned} \vec{e}(t+1) &= A(\hat{x}(t) - \vec{x}(t)) + L(C\hat{x}(t) - C\vec{x}(t)) \\ &= (A + LC)\vec{e}(t). \end{aligned} \tag{2}$$

Thus, if we design L such that all eigenvalues of $A + LC$ are within the unit disk then we guarantee $e(t) = \hat{x}(t) - \vec{x}(t) \to 0$; that is, the estimate $\hat{x}(t)$ approaches the state $\vec{x}(t)$ asymptotically.

It turns out that if the system is observable then we can assign the eigenvalues of $A + LC$ with an appropriate choice of L. We prove this by analogy to controllability. First note that taking the transpose of a matrix does not change its eigenvalues; therefore assigning the eigenvalues of $A + LC$ is equivalent to assigning the eigenvalues of

$$A^T + C^T L^T. \tag{3}$$

Now if we define $A_0 = A^T$, $B_0 = C^T$, $K_0 = L^T$, (3) takes the form $A_0 + B_0 K_0$ and we know that we can assign its eigenvalues by choice of K_0 if the pair (A_0, B_0) is controllable.

The next step, which we leave as an exercise, is to show that the columns of the controllability matrix for (A_0, B_0), when transposed, match the rows of the observability matrix for (A, C). Therefore they have identical ranks and observability of (A, C) implies controllability of (A_0, B_0). Then we can assign the eigenvalues of $A_0 + B_0 K_0$ with the design of K_0, and $L = K_0^T$ assigns the same eigenvalues to $A + LC$.

Example: Consider again the system

$$\begin{aligned} \vec{x}(t+1) &= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}(t) \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \vec{x}(t). \end{aligned} \tag{4}$$

If we take $\theta = \pi/2$, for which the system is observable, then

$$A + LC = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} l_1 & -1 \\ l_2 + 1 & 0 \end{bmatrix}$$

whose eigenvalues are the roots of

$$\det \begin{bmatrix} \lambda - l_1 & 1 \\ -l_2 - 1 & \lambda \end{bmatrix} = \lambda^2 - l_1\lambda + (l_2 + 1).$$

Note that, when $l_1 = l_2 = 0$, the eigenvalues are at $\pm j$ which are on the unit circle. To move these eigenvalues to desired values λ_1, λ_2 *inside* the unit disk we must match the coefficients of the polynomial above to those of

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

This means $l_1 = \lambda_1 + \lambda_2$ and $l_2 = \lambda_1\lambda_2 - 1$. For example, if we choose $\lambda_{1,2} = \pm 0.9j$ which are inside the unit disk, we get $l_1 = 0$, $l_2 = -0.19$.

Now repeat this example for $\theta = \pi$, in which case

$$A + LC = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} l_1 - 1 & 0 \\ l_2 & -1 \end{bmatrix}$$

and the eigenvalues are the roots of

$$\det \begin{bmatrix} \lambda + 1 - l_1 & 0 \\ -l_2 & \lambda + 1 \end{bmatrix} = (\lambda + 1 - l_1)(\lambda + 1).$$

Note that an eigenvalue at $\lambda = -1$ persists regardless of the choice of l_1, l_2, which is a result of unobservability in the case where $\theta = \pi$.

Example: Navigation is the task of identifying a vehicle's position, attitude, velocity, etc. relative to an inertial frame by integrating multiple measurements which may be inaccurate individually. To relate this task to observers consider the model of a vehicle moving in a lane from Lecture 6A,

$$\begin{bmatrix} p(t+1) \\ v(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}}_{B} u(t),$$

where $p(t)$ is position, $v(t)$ is velocity, and $u(t)$ is acceleration.

Without further measurements, our observer would be a simple copy of the model above:

$$\begin{bmatrix} \hat{p}(t+1) \\ \hat{v}(t+1) \end{bmatrix} = A \begin{bmatrix} \hat{p}(t) \\ \hat{v}(t) \end{bmatrix} + Bu(t).$$

This primitive strategy is known as *dead reckoning* and leads to major errors over long distances because observer errors accumulate over time and are not dissipated due to the eigenvalues of A at 1.

Modern navigation systems use satellite-based measurements of position. However, since these measurements are noisy and intermittent, it is reasonable to combine them with the dead reckoning method above within the observer

$$\begin{bmatrix} \hat{p}(t+1) \\ \hat{v}(t+1) \end{bmatrix} = A \begin{bmatrix} \hat{p}(t) \\ \hat{v}(t) \end{bmatrix} + L(\hat{p}(t) - p(t)) + Bu(t)$$

where $p(t)$ is treated as the output; that is

$$C = [1 \quad 0].$$

The task is then to design L such that $A + LC$ has eigenvalues inside the unit disk, which is possible since the pair (C, A) is observable (show this).

With this architecture we make use of satellite measurements but do not rely exclusively on them; we also exploit the system model which can make accurate predictions in the short term.

A more elaborate form of the observer (1), where the matrix L is also updated at each time, is known as the Kalman Filter and is the industry standard in navigation. The Kalman Filter takes into account the statistical properties of the noise that corrupts measurements and minimizes the mean square error between $x(t)$ and $\hat{x}(t)$.

Figure 1: **Rudolf Kalman (1930-2016)** introduced the Kalman Filter as well as many of the state space concepts we studied, such as controllability and observability. He was awarded the National Medal of Science in 2009.

Primer for the Upcoming Control Lab

In a lab assignment you will work on a robot car with two wheels, each driven with a separate electric motor. Let $d_l(t)$ and $d_r(t)$ be the distance traveled by the left and right wheels, and let $u_l(t)$ and $u_r(t)$ denote the control input (duty cycle of pulse width modulated current) we apply to the respective motor.

An appropriate model relating these variables is

$$\begin{aligned} d_l(t+1) - d_l(t) &= \theta_l u_l(t) - \beta_l \\ d_r(t+1) - d_r(t) &= \theta_r u_r(t) - \beta_r \end{aligned} \tag{5}$$

where the right hand sides approximate the speed for each wheel. Experimental data show that the speed instantly settles to a steady state proportional to the input, possibly with a bias term, hence the expressions $\theta_l u_l(t) - \beta_l$ and $\theta_r u_r(t) - \beta_r$.

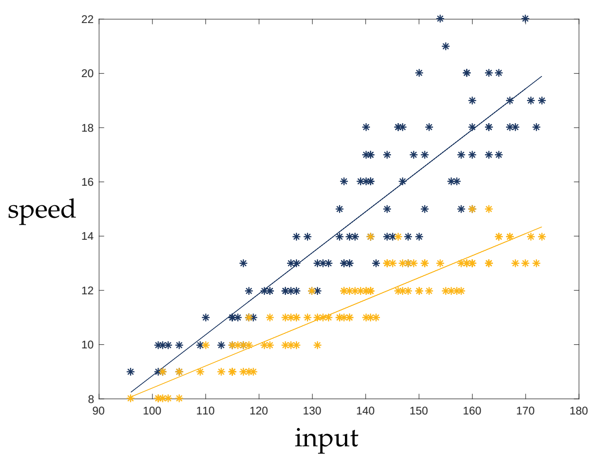


Figure 2: Speed vs. input data points from experiments. The left and right wheels are represented with different colors, and the lines show least squares estimates for each.

The parameters θ_l, θ_r, β_l, β_r, are estimated experimentally using least squares. We apply a time-varying input sequence $u_l(0)$, $u_l(1)$, $\ldots$, $u_l(k)$ to the left motor, collect the resulting data $d_l(1)$, $d_l(2)$, $\ldots$, $d_l(k+1)$, and set up the overdetermined equation

$$\underbrace{\begin{bmatrix} u_l(0) & -1 \\ u_l(1) & -1 \\ \vdots & \vdots \\ u_l(k) & -1 \end{bmatrix}}_{\triangleq S} \begin{bmatrix} \theta_l \\ \beta_l \end{bmatrix} = \begin{bmatrix} d_l(1) - d_l(0) \\ d_l(2) - d_l(1) \\ \vdots \\ d_l(k+1) - d_l(k) \end{bmatrix}.$$

Then the least squares estimate is

$$\begin{bmatrix} \hat{\theta}_l \\ \hat{\beta}_l \end{bmatrix} = (S^T S)^{-1} S^T \begin{bmatrix} d_l(1) - d_l(0) \\ d_l(2) - d_l(1) \\ \vdots \\ d_l(k+1) - d_l(k) \end{bmatrix}.$$

Parameter estimates $\hat{\theta}_r$, $\hat{\beta}_r$ for the right wheel are obtained similarly. However, the parameters for the two wheels may be significantly different. Thus, applying an identical input to both wheels would lead to nonidentical speeds, and the car would go in circles. We straighten the trajectory of the car in two steps:

Step 1: Apply the constant inputs

$$u_l = \frac{v^* + \hat{\beta}_l}{\hat{\theta}_l} \qquad u_r = \frac{v^* + \hat{\beta}_r}{\hat{\theta}_r} \tag{6}$$

which aim to set each speed to the same value v^*. Indeed, if our parameter estimates were exact, we would obtain v^* on the right hand side of each equation in (5).

While this step should improve the trajectory it may not completely straighten it, as the parameter estimates are unlikely to be perfectly accurate. In the next step we augment (6) with a feedback term.

Step 2: Modify (6) as

$$\begin{aligned} u_l(t) &= \frac{v^* + \hat{\beta}_l}{\hat{\theta}_l} + \frac{k_l}{\hat{\theta}_l}(d_l(t) - d_r(t)) \\ u_r(t) &= \frac{v^* + \hat{\beta}_r}{\hat{\theta}_r} + \frac{k_r}{\hat{\theta}_r}(d_l(t) - d_r(t)) \end{aligned} \tag{7}$$

where the feedback gains k_l and k_r are to be designed.

Assume for now that $\hat{\theta}_l = \theta_l$, $\hat{\beta}_l = \beta_l$, $\hat{\theta}_r = \theta_r$, $\hat{\beta}_r = \beta_r$, and substitute (7) in (5) to get

$$\begin{aligned} d_l(t+1) - d_l(t) &= v^* + k_l(d_l(t) - d_r(t)) \\ d_r(t+1) - d_r(t) &= v^* + k_r(d_l(t) - d_r(t)). \end{aligned} \tag{8}$$

Next, define $\delta(t) \triangleq d_l(t) - d_r(t)$ and note from (8) that it satisfies

$$\delta(t+1) - \delta(t) = (k_l - k_r)\delta(t)$$

or, equivalently,

$$\delta(t+1) = (1 + k_l - k_r)\delta(t).$$

Thus, to ensure $\delta(t) \to 0$, we need to select k_l and k_r such that

$$|1 + k_l - k_r| < 1.$$

Without the feedback terms in (7), that is $k_l = k_r = 0$, we get

$$\delta(t+1) = \delta(t)$$

which means that the error accumulated in $\delta(t)$ persists and is in fact likely to grow when we incorporate a disturbance term to account for the mismatch between the parameters and their estimates $\hat{\theta}_l, \hat{\theta}_r, \hat{\beta}_l, \hat{\beta}_r$. The feedback in (7) is thus essential to dissipate the error $\delta(t)$ and to keep it bounded in the presence of parameter mismatch.