# Lecture 2B

Wednesday, September 7, 2016 9:17 PM

Up to now we've been concerned with solving circuits in the "time domain". That is, given the circuit topology and component eqns., solve for how current and voltage behave over time. This led us to differential equations when the circuits contained L's and/or C's.

For circuits with many L's and/or C's, the order of the differential equations will be very high. Such higher order D.E. can be very difficult to solve analytically. Indeed, such high order systems are often solved by numerical integration.

However, there is a special class of signals for which there is a more straightforward and powerful approach to a solution. If the inputs (that is, the sources) of a circuit are sinusoids, then we can apply a different technique.

## 0. Preliminary Matters

Before we discuss the new method, note:

1) Consider a sinusoid:

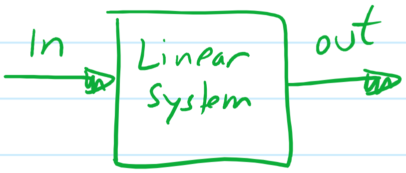
$$v(t) = V_0 \cos(\omega t + \phi)$$

Such a signal has three features:

$V_0$ The amplitude
$\omega$ The frequency
$\phi$ The phase

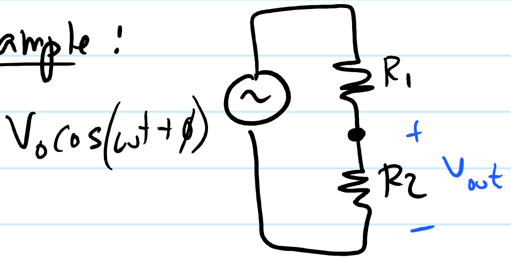
$\phi$ the phase.

2)



If a system is linear and we apply a sinusoidal input, all the currents and voltages in that circuit will be at the same frequency, $\omega$, as the input but the amplitudes and phases will change.

Simple example:



Note $V_{out} = \left(\frac{R_2}{R_1+R_2}\right) V_0 \cos(\omega t + \phi)$

(This applies to circuits with L, C (which are linear) because the derivative of a sinusoid at $\omega$ is still a Sinusoid at $\omega$.)

3) Any periodic signal can be represented as a sum of sinusoids of different frequencies, so if we can find a method that solves a circuit for an arbitrary sinusoidal signal, we can find it for any periodic signal.

## I. The Phasor Transform

$e^{j\phi} = \cos\phi + j\sin\phi$

Consider again $v(t) = V_0 \cos(\omega t + \phi)$

Note that by Euler's:

$$V_0 e^{j(\omega t+\phi)} = V_0\left[\cos(\omega t+\phi) + j\sin(\omega t+\phi)\right]$$

$$V_0 e^{j(\omega t+\phi)} = V_0\left[\cos(\omega t+\phi) + j\sin(\omega t+\phi)\right]$$

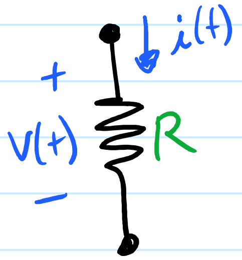
So that $v(t) = \mathrm{Re}\left\{\underbrace{V_0 e^{j\phi}}_{} e^{j\omega t}\right\}$

we know a (linear) circuit can change these two $(V_0, \phi)$

we know a (linear) circuit cannot change this

Let's call $V_0 e^{j\phi}$ a "phasor" and call it $\tilde{V}$.
Note the phasor encodes the amplitude and the phase of the sinusoid.

## a. Resistor



- In the time domain, $i(t) = \dfrac{v(t)}{R}$
- We know both $i(t)$ and $v(t)$ will have the same frequency, $\omega$, since

$$i(t) = \frac{V_0}{R}\cos(\omega t+\phi)$$

Now, if $i(t) = \mathrm{Re}\left\{I_0 e^{j\phi_1} e^{j\omega t}\right\}$

and $v(t) = \mathrm{Re}\left\{V_0 e^{j\phi} e^{j\omega t}\right\}$

then by ohm's law

$$\mathrm{Re}\left\{I_0 e^{j\phi_1} e^{j\omega t}\right\} = \frac{\mathrm{Re}\left\{V_0 e^{j\phi} e^{j\omega t}\right\}}{R}$$

$$\mathrm{Re}\left\{\tilde{I} e^{j\omega t}\right\} = \frac{\mathrm{Re}\left\{\tilde{V} e^{j\omega t}\right\}}{R}$$

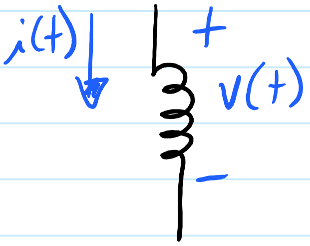
$$\text{Re}\{I\, e \quad\} = \frac{\text{Re}\{V\, e \quad\}}{R}$$

$$\tilde{I} = \frac{\tilde{V}}{R} \quad \text{if } R \text{ is real}$$

$$\boxed{\frac{\tilde{V}}{\tilde{I}} = R}$$

let's remember this. Looks like Ohm's Law.

## 2. Inductor



$$v(t) = L\frac{di(t)}{dt}$$

$$\text{Re}\{V_0 e^{j\phi} e^{j\omega t}\} = L\frac{d}{dt}\left(\text{Re}\{I_0 e^{j\phi_1} e^{j\omega t}\}\right)$$

$$\text{Re}\{V_0 e^{j\phi} e^{j\omega t}\} = j\omega L\, \text{Re}\{I_0 e^{j\phi_1} e^{j\omega t}\}$$

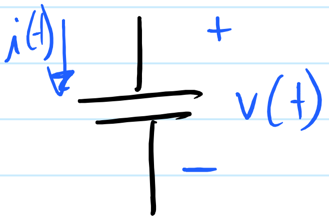
$$\text{Re}\{\tilde{V} e^{j\omega t}\} = j\omega L\, \text{Re}\{\hat{I} e^{j\omega t}\}$$

$$\tilde{V} = j\omega L \tilde{I}$$

$$\boxed{\frac{\tilde{V}}{\tilde{I}} = j\omega L}$$

this sort of looks like Ohm's Law but the value is imaginary!

## 3. Capacitor



$$i(t) = C\frac{dv(t)}{dt}$$

$$\mathrm{Re}\{I_0 e^{j\phi_1} e^{j\omega t}\} = C\frac{d}{dt}\left(\mathrm{Re}\{V_0 e^{j\phi} e^{j\omega t}\}\right)$$

$$\mathrm{Re}\{I_0 e^{j\phi_1} e^{j\omega t}\} = j\omega C\left(\mathrm{Re}\{V_0 e^{j\phi} e^{j\omega t}\}\right)$$

$$\mathrm{Re}\{\tilde{I} e^{j\omega t}\} = j\omega C \, \mathrm{Re}\{\tilde{V} e^{j\omega t}\}$$

$$\boxed{\frac{1}{j\omega C} = \frac{-j}{\omega C}}$$

$$\tilde{I} = j\omega C \tilde{V}$$

$$\boxed{\frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}}$$

Again, looks like ohm's Law.

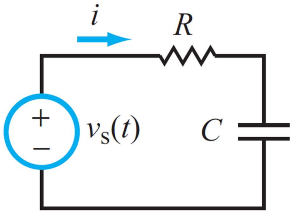
Why did we just do this exercise above?

If we transform all voltages and currents from time domain $[v(t), i(t)]$ into the phasor domain $[\tilde{V}, \tilde{I}]$ we now have eqns.

for R, C, L. Maybe we can solve a circuit entirely in the Phasor domain, then convert back. If we do this, there won't be any D.E.'s to solve!!!

## Ⅱ) The Method (followed by an example):

**Step 1**
Adopt Cosine Reference
(Time Domain)



$v_s(t) = 12 \sin(\omega t - 45°)$ (V)

if needed convert sin to cos

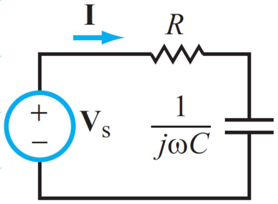
transform

**Step 2**
Transfer to Phasor Domain

$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$

(boldface is the same as ~)

$\mathbf{V} = \tilde{V}$



$\mathbf{V}_s = 12e^{-j135°}$ (V)

**Step 3**
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

KVL, KCL, Nodal, etc. still apply!

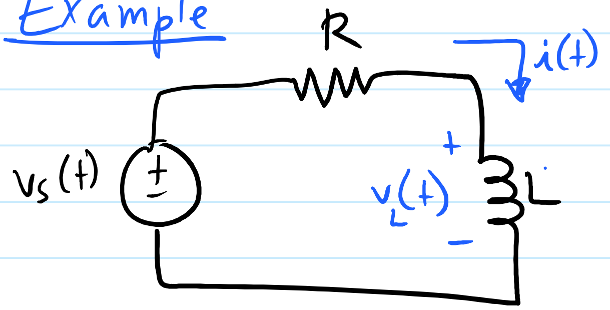
**Step 4**
Solve for Unknown Variable
(Phasor Domain)

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

**Step 5**
Transform Solution
Back to Time Domain

$$i(t) = \Re\mathfrak{e}[\mathbf{I}e^{j\omega t}]$$
$$= 6 \cos(\omega t - 105°)$$
(mA)

## II. Example



$R = 3\Omega$
$L = 0.1$ mH

Find $v_L(t)$ if $v_s(t) = 15 \sin(\omega t + \phi)$

$\omega = 4 \times 10^4$ rad/s
$\phi = -30°$

Step 1: Adopt cosine reference

$$v_s(t) = 15\sin(\omega t + \phi) = 15\cos(\omega t + \phi - 90^\circ)$$

$$= 15\cos(\omega t - 120^\circ)$$

## Step 2: Transform circuit

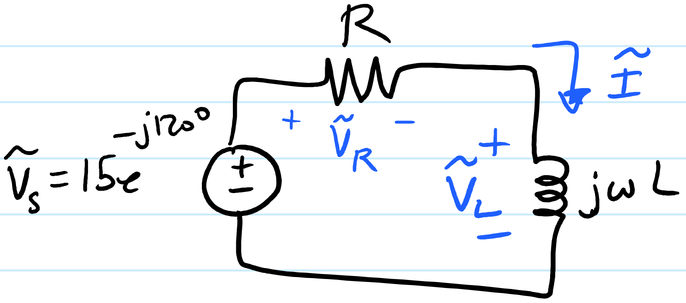
$$v_s(t) \longrightarrow \tilde{V}_s = 15e^{-j120^\circ}$$

you can also write

$15\angle{-120^\circ}$

$$L \longrightarrow j\omega L$$

$$R \longrightarrow R$$



Use KVL:

$$-\tilde{V}_s + \tilde{V}_R + \tilde{V}_L = 0$$

$$\tilde{I}R + \tilde{V}_L = \tilde{V}_s$$

$$\tilde{I}R + (j\omega L)\tilde{I} = \tilde{V}_s$$

$$\tilde{I}(R + j\omega L) = \hat{V}_s$$

$$\ddot{I} - \frac{V_s}{R + j\omega L} = \frac{15e^{-j120^\circ}}{\text{[illegible]})(0.1\times10^{-3})}$$

$x + jy \Leftrightarrow M\angle\phi^\circ$

$$= \frac{15e^{-j120^\circ}}{3 + j4}$$

$x+jy \Leftrightarrow M\angle\phi^\circ$

$M = \sqrt{x^2+y^2}$

$\phi = \tan^{-1}(y/x)$

$$= \frac{15e^{-j}}{3+j4}$$

$$= \frac{15\,e^{-j120^\circ}}{5\,e^{j53.1^\circ}}$$

$$= \frac{15}{5}\,e^{j(-120^\circ-53.1^\circ)}$$

$$= 3e^{-j173.1^\circ}$$

btw, this means $i(t) = \mathrm{Re}\left\{\tilde{I}e^{j\omega t}\right\}$

$$= \mathrm{Re}\left\{3e^{-j173.1^\circ}e^{j\omega t}\right\}$$

$$= 3\cos(\omega t - 173.1^\circ)$$

But we want $v_L(t)$. Well,

$$\tilde{V}_L = j\omega L\hat{I}$$

$$\hat{V}_L = j\omega L\left(3e^{-j173.1^\circ}\right)$$

$$= j(4\times10^4)(0.1\times10^{-3})\,3e^{-j173.1^\circ}$$

$$= j12e^{-173.1^\circ}$$

$$= 12e^{-j173.1^\circ}\cdot e^{j90^\circ}$$

$e^{j90} = \cos 90 + j\sin 90$

$= j$

$$= 12\, e^{-j173.1} \cdot e^{j90}$$

$$= 12\, e^{-j83.1°}$$

$$e = \cos 90 + j\sin 90$$
$$= j$$
so $j = e^{j90°}$

$$v_L(t) = \text{Re}\left\{12\, e^{-j83.1°} e^{j\omega t}\right\} = \boxed{12\cos(\omega t - 83.1°)}$$

Done!

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The beauty of this method is:

1) No D.E.'s
2) We can solve circuits with any number of L's and C's
3) We can extend it to make fundamental observations about circuits (next lecture!)

**BUT** it only works for sinusoids!

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## IV. Impedance

The phasor I-V relationship leads us to a more general concept, impedance, Z:

$$Z = \underset{\substack{\uparrow \\ \text{resistance}}}{R} + j\underset{\substack{\nwarrow \\ \text{reactance}}}{X}$$

Resistance you already know. It is real and comes from dissipative elements.

Reactance arises only from energy storage components

$Z_R = R$ (no reactance)

$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$ ($X_C = -\frac{1}{\omega C}$ => caps have negative reactance)

$Z_L = j\omega L$ ($X_L = \omega L$ => inductors have positive reactance)

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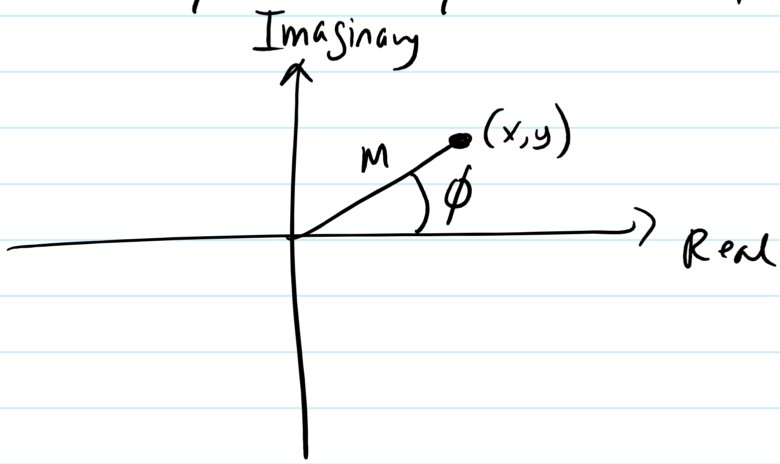
Remember $x + jy \iff M\angle\phi$ or $Me^{j\phi}$

rectangular polar

$x = M\cos\phi$ $M = \sqrt{x^2 + y^2}$

$y = M\sin\phi$ $\phi = \tan^{-1} y/x$



Also,

$$\frac{M_1 e^{j\phi_1}}{M_2 e^{j\phi_2}} = \frac{M_1}{M_2} e^{j(\phi_1 - \phi_2)}$$