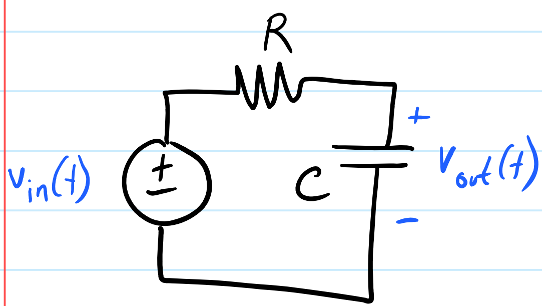
Lecture3A

Monday, September 12, 2016 5:05 PM

So now we know how to solve for the steady-state voltages and currents in linear circuits driven by sinusoidal inputs.

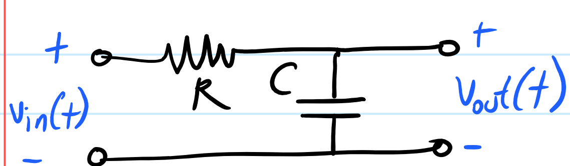
For example, if given this circuit:



If $v_{in}(t) = V_0 \cos(\omega t + \phi)$ we can convert the circuit to phasor domain, solve for $\tilde{V}_{out}$ and then convert back to obtain $v_{out}(t)$.

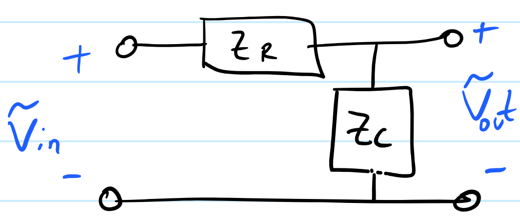
I Transfer functions

What if we re-write the circuit this way:



Before we do anything else, let's make some simple observations.

1.) The circuit, in the phasor domain, looks like



2) By voltage division, $\tilde{V}_{out} = \dfrac{Z_C}{Z_R + Z_C} \tilde{V}_{in}$

3) For $\omega \to 0$, $Z_C = \frac{1}{j\omega C} \to \infty$

Thus, $\boxed{\tilde{V}_{out} \to \tilde{V}_{in}}$ as $\boxed{\omega \to 0}$

4) For $\omega \to \infty$, $Z_C = \frac{1}{j\omega C} \to 0$

Thus, $\boxed{\tilde{V}_{out} \to 0}$ as $\boxed{\omega \to \infty}$

5) From 3) and 4), if $v_{in}(t)$ is at a low frequency (i.e. $\omega \to 0$) $v_{out}(t) \sim v_{in}(t)$ (i.e. the signal gets "through" the circuit. Conversely, if $v_{in}(t)$ is at a high frequency (i.e. $\omega \to \infty$), then $v_{out}(t) \sim 0$ (i.e. the input signal does not show up at the output.

6) Let's look at the behaviour in between $\omega = 0$ and $\omega \to \infty$:

$$\tilde{V}_{out} = \frac{Z_C}{Z_R + Z_C} \tilde{V}_{in}$$

$$\tilde{V}_{out} = \frac{(1/j\omega C)}{R + (1/j\omega C)} \tilde{V}_{in}$$

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \left(\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \right) \cdot \frac{j\omega C}{j\omega C} = \frac{1}{1 + j\omega RC}$$

$$\boxed{\tilde{H}_v(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j\omega RC}}$$

Notice we can solve $\frac{\tilde{V}_{out}}{}$ for a linear circuit.

Notice we can solve $\frac{\tilde{V}_{out}}{\tilde{V}_{in}}$ for a linear circuit.

we can also solve for $\frac{\tilde{I}_{out}}{\tilde{I}_{in}}$, $\frac{\tilde{V}_{out}}{\tilde{I}_{in}}$, $\frac{\tilde{I}_{out}}{\tilde{V}_{in}}$ where these can be any voltage or current in the circuit.

Also, notice $\tilde{H}$ is like <u>gain</u> (G) but it is:

a) frequency dependent

b) a complex number

$$\boxed{\tilde{H}(\omega) \text{ is called a } \underline{\text{transfer}}\ \underline{\text{function}}}$$

For our circuit, as expected,

$$H_v(\omega=0) = 1$$
$$H_v(\omega\to\infty) \to 0$$

How should we think about $\tilde{H}(\omega)$?

we could plot either $Re\{\tilde{H}(\omega)\}$ and $Im\{\tilde{H}(\omega)\}$

or $|\tilde{H}(\omega)|$ and $\phi(\tilde{H}(\omega))$

magnitude phase

Let's look at $|\tilde{H}|$ and ϕ.

$$|\tilde{H}_v(\omega)| = \frac{|Num|}{|Denom|} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}}$$

$$\frac{1}{1+j\omega RC}$$

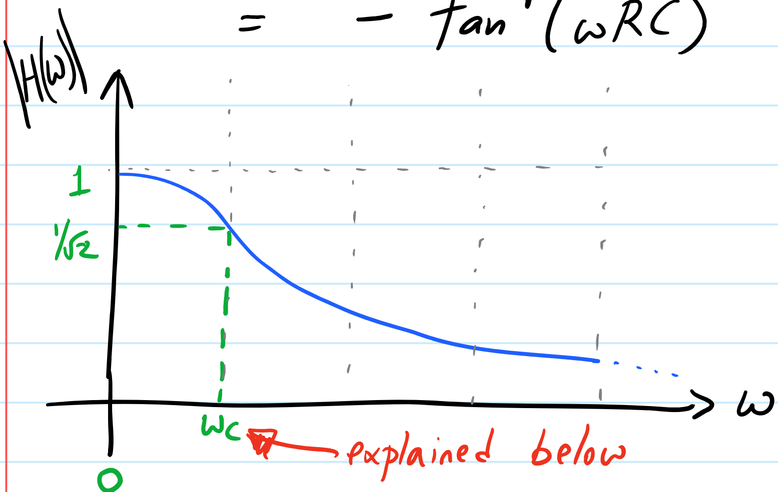
$x+jy$

$$\phi = \phi(Num) - \phi(Denom)$$

$x + jy$

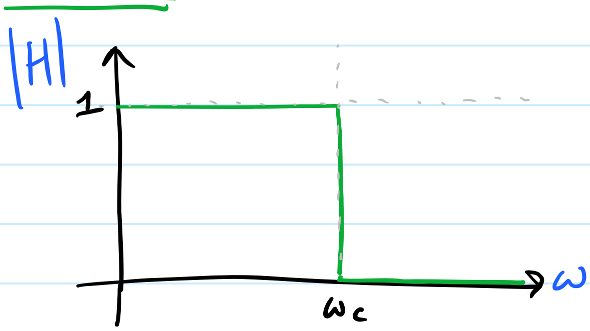
$z = \sqrt{x^2 + y^2}$

$$\phi' = \phi(\text{Num}) - \phi(\text{Denom})$$
$$= \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}(\omega RC)$$
$$= -\tan^{-1}(\omega RC)$$

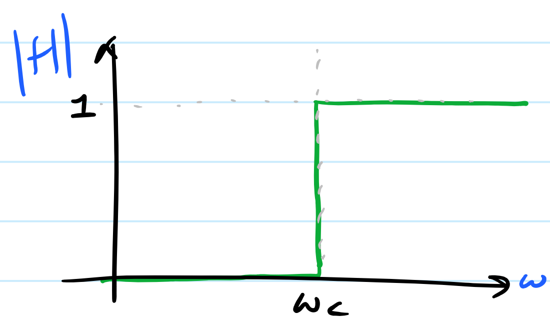


So, this circuit is a filter. It lets signals at some frequencies through but not others. It is not a very sharp filter, though. Because it has only one reactive element, it is a 1st order filter.

Ideally, we might want:

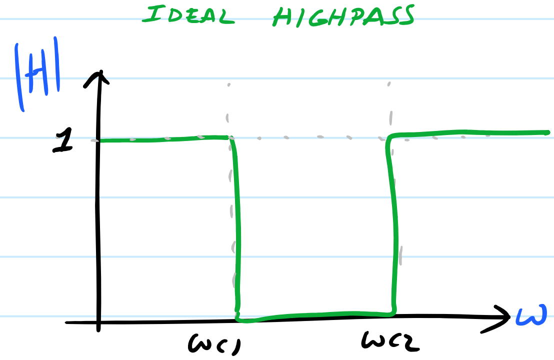


IDEAL LOWPASS



IDEAL HIGHPASS

IDEAL BANDPASS



IDEAL BANDSTOP

Note that our filter is a lowpass filter but it is far from ideal. Ideally we would want all frequencies below some cutoff to be passed ($|H|=1$) and all frequencies above to be stopped ($|H|=0$). The cutoff frequancy is labelled ω_c.

Cutoff frequency ω_c

How do we define ω_c? A long time ago it was decided that

$$\frac{P_{out}(\omega_c)}{P_{out,ref}} = \frac{1}{2}$$

In other words, when the power delivered to a load dropped to $\frac{1}{2}$ the value of that of a reference power (usually, the power in the passband). But how do we map this to voltages and currents?

Note:
$$\frac{P_{out}(\omega_c)}{P_{out,ref}} = \frac{\frac{V_{out}^2(\omega)}{R_L}}{\frac{V_{out,ref}^2}{R_L}} = \frac{1}{2}$$

$$\frac{V_{out}^2(\omega)}{V_{out,ref}^2} = \frac{1}{2} \quad \text{or} \quad V_{out}(\omega) = \frac{1}{\sqrt{2}} V_{out,ref}$$

So,
$$H_v(\omega_c) = \frac{H_v(\omega \to ref)}{\sqrt{2}}$$

By the same logic,
$P = I^2 R$ so
$$H_I(\omega_c) = \frac{H_I(\omega \to ref)}{\sqrt{2}}$$

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$P = I^2 R$ so
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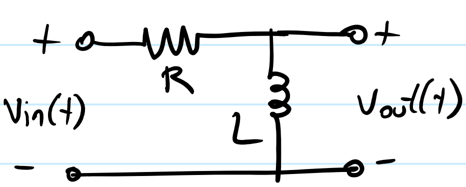
In our example above, the passband max value is $|H_v(0)| = 1$, so we'll use that as reference.

$$|H_v(\omega_c)| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

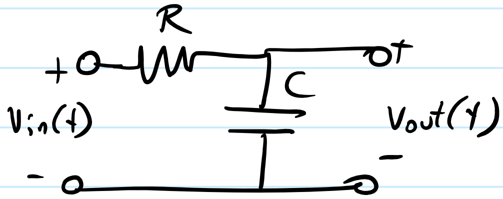
$$2 = 1 + \omega^2 R^2 C^2$$
$$\omega^2 R^2 C^2 = 1$$
$$\omega^2 = \frac{1}{R^2 C^2}$$
$$\boxed{\omega_c = \frac{1}{RC}}$$

In conclusion, we find ω_c by setting the expression we have for $|H(\omega)| = \frac{1}{\sqrt{2}} |H(\omega \to ref)|$ where $|H(\omega \to ref)|$ is the max value in the passband for our circuit ($= 1$ for the circuit above).

For practice, solve for $|H|$ and ϕ for:



(a)

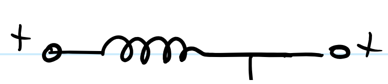


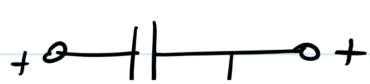
(b)

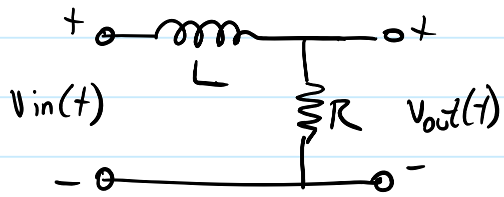
a) $H(\omega)$

b) $|H|, \phi$

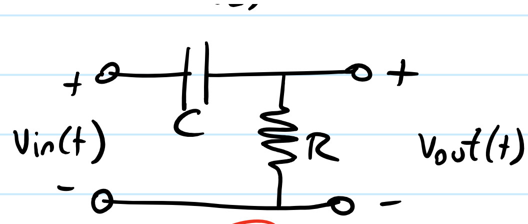
c) ω_c







(c)



(d)

II. Bode Plots

Plotting $|H|$ and ϕ on a linear scale is not a good idea. Why?

a) We often care about large spans of ω (e.g. from DC $\rightarrow$ 10 GHz)

b) The differences in $|H|$ value between the pass band and the stop band can be huge, like $> 10^9$! It is hard to see any detail on a y-axis that is linear.

What to do?

We could plot $\log_{10}|H|$ vs. $\log_{10}\omega$ and ϕ vs. $\log_{10}\omega$.

This is close, but not quite.

The decibel (dB)

- A decibel is a measure of power gain.

If $H_p = \dfrac{P_{out}}{P_{ref}}$ then

$$\boxed{H\,[dB] = 10\log_{10}|H_p|}$$

this is the definition of a dB

$$\boxed{H[dB] = 10 \log_{10} |H_p|}$$ of a dB

Once again, because $P_{out} = \frac{V_{out}^2}{R}$ and $P_{ref} = \frac{V_{ref}^2}{R}$,

we can also say that

$$H[dB] = 10 \log_{10} \left(\frac{V_{out}}{V_{ref}}\right)^2 = 20 \log_{10} \frac{V_{out}}{V_{ref}} = 20 \log_{10} H_v$$

- So, when dealing with voltage and current transfer functions, we use this:

$$\boxed{H[dB] = 20 \log_{10} |H|}$$ ← use this

- Note that if

1) $A = B \cdot C$ then $A[dB] = B[dB] + C[dB]$

2) $A = \frac{B}{C}$ then $A[dB] = B[dB] - C[dB]$

by the rules regarding logarithms.

- Try to develop some intuition for what dB values mean:

$\frac{P}{P_0}$	dB
10^N	$10N$ dB
10^3	30 dB
100	20 dB
10	10 dB
4	$\simeq 6$ dB
2	$\simeq 3$ dB
1	0 dB
0.5	$\simeq -3$ dB
0.25	$\simeq -6$ dB
0.1	-10 dB
10^{-N}	$-10N$ dB

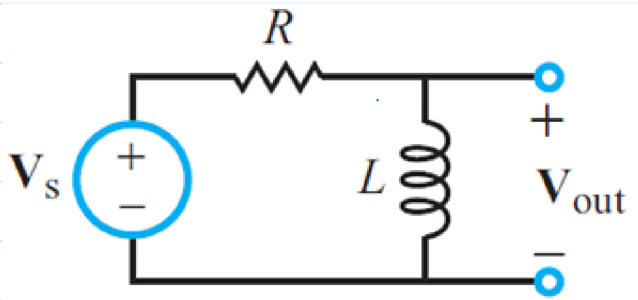
$\left\lvert\frac{\mathbf{V}}{\mathbf{V}_0}\right\rvert$ or $\left\lvert\frac{\mathbf{I}}{\mathbf{I}_0}\right\rvert$	dB
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	$\simeq 12$ dB
2	$\simeq 6$ dB
1	0 dB
0.5	$\simeq -6$ dB
0.25	$\simeq -12$ dB
0.1	-20 dB
10^{-N}	$-20N$ dB

Bode Plot example

When constructing Bode plots of magnitude and phase:

Magnitude Bode plot	Phase Bode Plot
y-axis: $H[\text{dB}] = 20\log_{10}\lvert H\rvert$	y-axis: $\phi(\omega)$
x-axis: $\log_{10}\omega$	x-axis: $\log_{10}\omega$

Let's consider a different circuit.



$$\mathbf{V}_{\text{out}} = \frac{j\omega L\mathbf{V}_s}{R + j\omega L},$$

which leads to

you do not need to
"solve" for we before
...ing. Just factor
...sin into this

$$V_{out} = \frac{\quad}{R + j\omega L},$$

which leads to

$$\mathbf{H} = \frac{\mathbf{V}_{out}}{\mathbf{V}_s} = \frac{j\omega L}{R + j\omega L} = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)},$$

you "solve" Jus factoring. the expression in form.

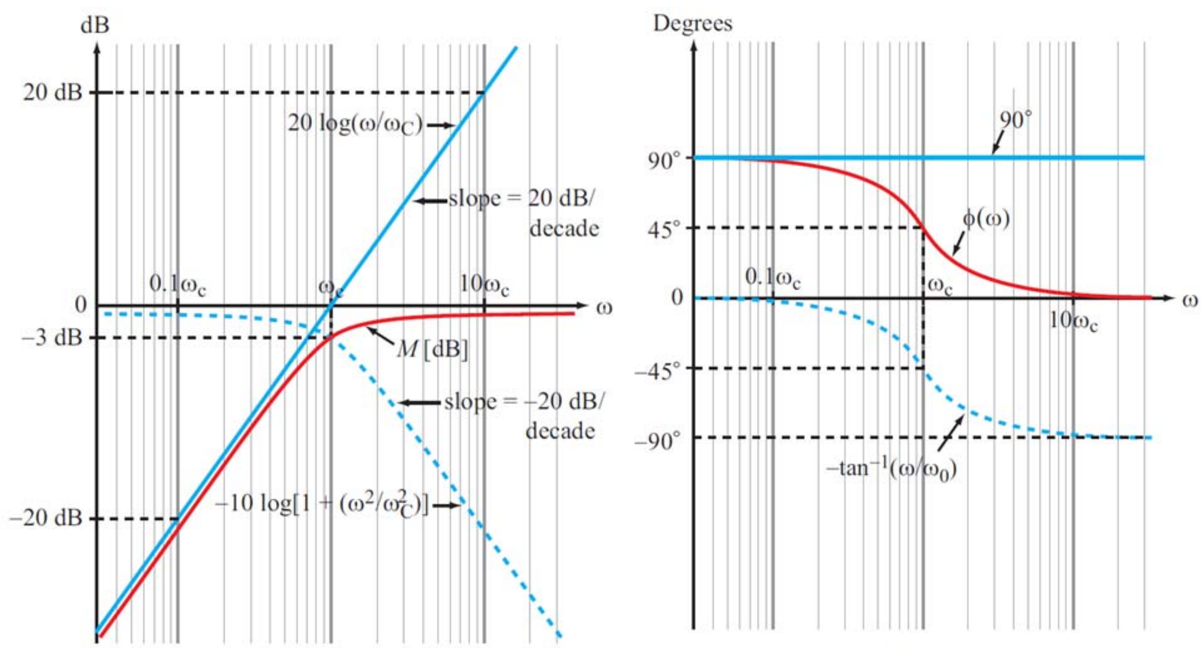
with $\omega_c = R/L$.

$$M = |\mathbf{H}| = \frac{(\omega/\omega_c)}{|1 + j(\omega/\omega_c)|} = \frac{(\omega/\omega_c)}{\sqrt{1 + (\omega/\omega_c)^2}}.$$

$$\phi(\omega) = 90° - \tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

Since H is a voltage ratio, the appropriate dB scaling factor is 20, so

$$\begin{aligned} M\ [\text{dB}] &= 20 \log M \\ &= 20 \log(\omega/\omega_c) - 20 \log[1 + (\omega/\omega_c)^2]^{1/2} \\ &= 20 \log(\omega/\omega_c) - 10 \log[1 + (\omega/\omega_c)^2]. \qquad (9.35) \end{aligned}$$



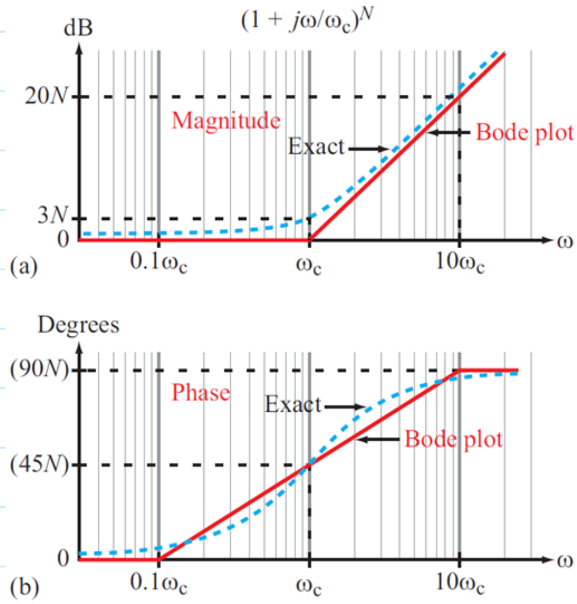
(b) Magnitude plot

(c) Phase plot

Notice that the asymptotes of the expressions are easy to see. This is the "trick" behind Bode plots: just draw the straightline approximation of the frequency response:

dB $(1 + j\omega/\omega_c)^N$

just draw the straightline approximation of the frequency response:

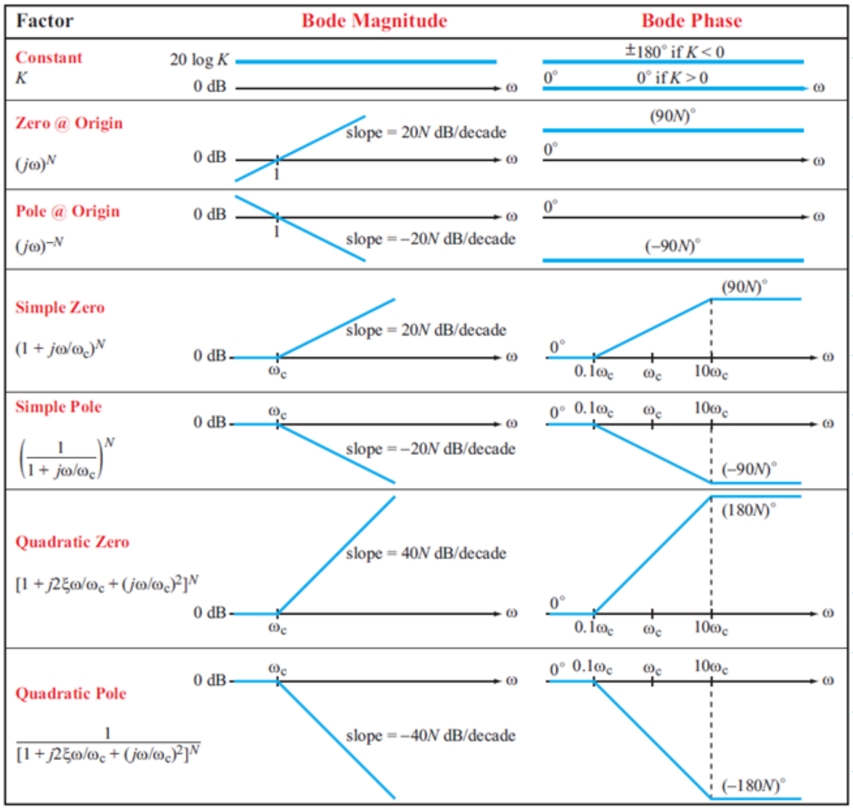


The technique:

1) Solve for $H(\omega)$
2) Solve for $|H|$ and ϕ
3) Factor $|H|$ into canonical form (see below)
4) Draw the Bode plot by simple geometrical construction.

Zeros and Poles

For all of the circuits in this class, you can factor $H(\omega)$ into a fraction where the numerator and denominator are products of one or more of the following basic expressions.



If the expression is in the numerator, it is called a zero

If in the denominator, it is called a pole.

use table

Example:

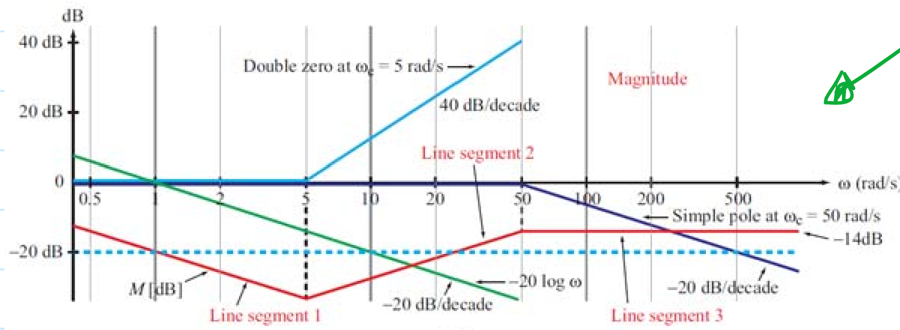
$$\mathbf{H}(\omega) = \frac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)}$$

$$H(\omega) = \frac{400(1 + j\omega/5)^2}{j4000\,\omega(1 + j\omega/50)} = \frac{\overset{K}{0.1}\,\overset{\text{Simple zero},\ \omega_c = 5,\ N = 2}{(1 + j\omega/5)^2}}{\underset{\text{pole@ origin}}{j(\omega)}\ \underset{\text{simple pole } \omega_c = 50,\ N = 1}{(1 + j\omega/50)}}$$

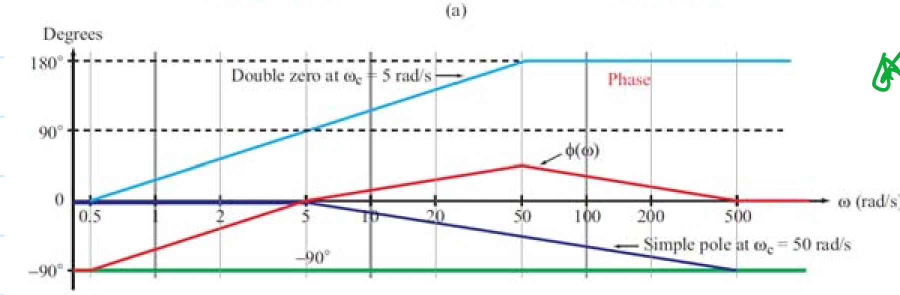
$$M\ [\text{dB}] = 20\log|\mathbf{H}|$$
$$= 20\log 0.1 + 40\log|1 + j\omega/5|$$
$$- 20\log\omega - 20\log|1 + j\omega/50|$$
$$= -20\text{ dB} + 40\log|1 + j\omega/5|$$
$$- 20\log\omega - 20\log|1 + j\omega/50|.$$

$$\phi = -90^\circ + 2\tan^{-1}\frac{\omega}{5} - \tan^{-1}\frac{\omega}{50}$$

each of these four terms is plotted below. The combination is the Bode plot for H. (in red)



(a)



(b)