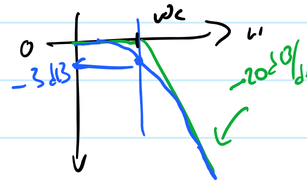
Lecture 3B

Wednesday, September 14, 2016 1:46 PM

We've seen the basics of how to plot transfer functions using the Bode plot method.

Key points:

1) Bode plots are straight line approximations. That is, for each zero and pole, we can quickly deduce the behavior as $\omega \to 0$ and $\omega \to \infty$ and indicate that with straight lines. The real frequency response is not "straight" but changes smoothly from one asymptote to the other. An example of this was provided in Lecture 3A notes.

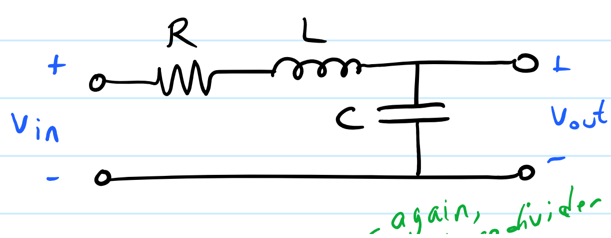


2) We can write the transfer function H as products and fractions of zero's and poles. If these are written in canonical form, constructing the Bode plot is easy: find the Bode plot of each zero and pole and construct the result by addition.

Example of canonical form:
If you see $1 + j\omega RC$, call $\omega_c = \frac{1}{RC}$
such that $\left(1 + \frac{j\omega}{\omega_c}\right)$ ← canonical form

I. "2nd order" filters

What about this circuit:



again, impedance divider

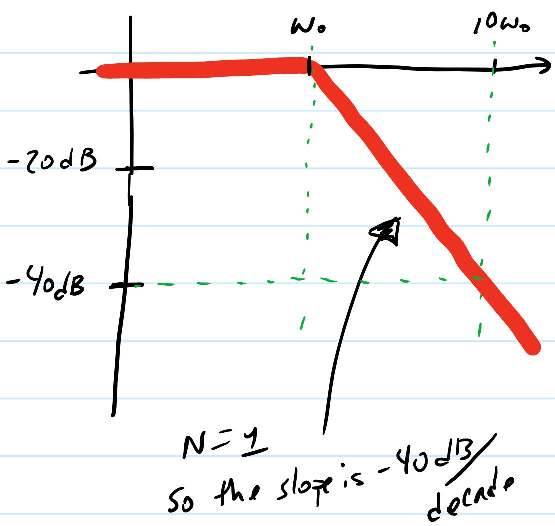
$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} \cdot \frac{j\omega C}{j\omega C}$$

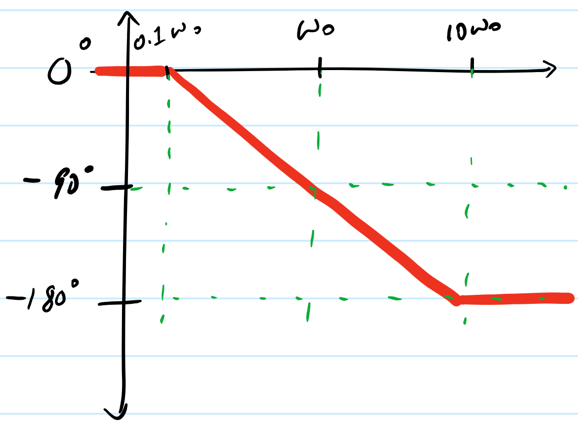
$$= \frac{1}{1 + j\omega RC + (j\omega C)(j\omega L)}$$

$$= \frac{1}{1 + j2\xi\frac{\omega}{\omega_0} + \left(\frac{j\omega}{\omega_0}\right)^2}$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$, $\xi = \frac{R}{2L}$

This is a quadratic pole, as per the table. So, the Bode plots look like this:

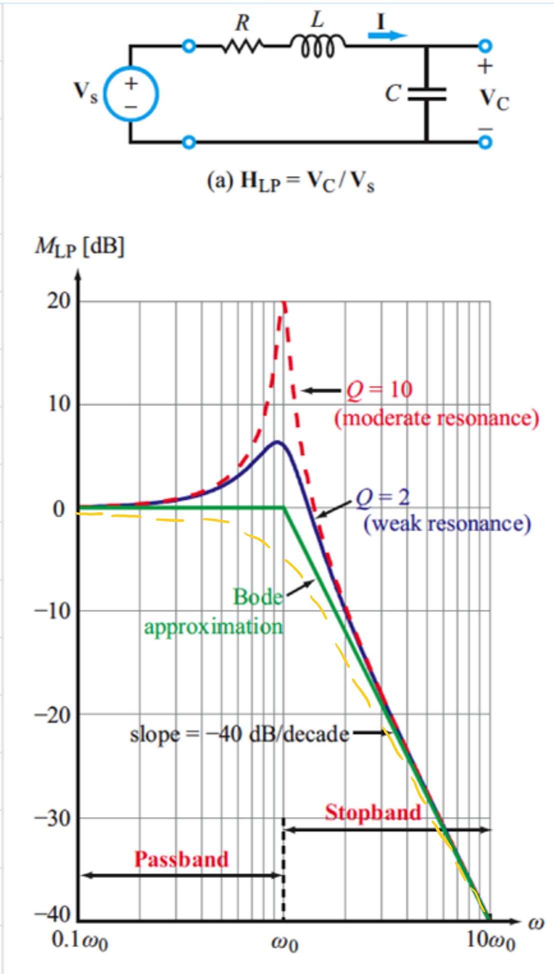




But is this the whole story? What about the ξ term in the expression. Also, the form of the pole seems to bear a striking similarity to the 2nd order ODE for the cap voltage from a few lectures ago. Notice that $R/2L$ and $1/\sqrt{LC}$ showed up again.

Let's plot the actual frequency response magnitude:

$$|H(\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2}}$$



In green is the Bode plot. But notice that as ξ lessen ξ relative to ω_0, a peak appears right at ω_0!! For $\xi \ll \omega_0$, this peak can be thousands of times higher than the low frequency "passband."

Also, recall that $\xi < \omega_0$ is the underdamped condition when looking at this circuit in the time domain.

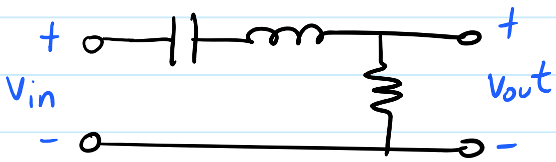
Lastly, if $\xi = 0$ (i.e. $R=0$), then $|H(\omega=\omega_0)| = \infty$!! In other words, for an LC with no R, the frequency response is a delta at ω_0.

So, it seems that the more underdamped the circuit is in the time domain, the sharper this peak is in the frequency domain. Also, the peak doesn't show up <u>at all</u> in Bode plot approximations.

These peaks might be great as bandpass filters, no? Let's explore this more.

If there is a peak, how wide is it?

Let's answer these questions but vary it a little for practice. Consider:



$$H(\omega) = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

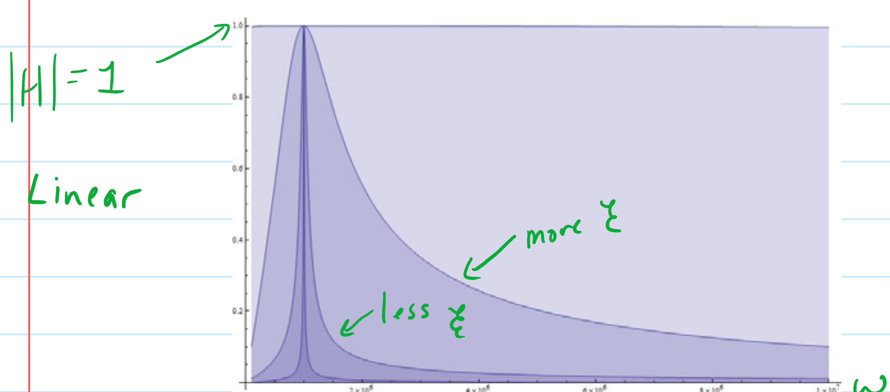
$$= \frac{j\omega RC}{1 + j\omega RC + (j\omega L)(j\omega C)}$$

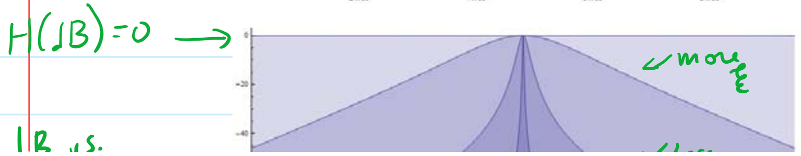
$$= \frac{j\omega RC}{1 + j2\xi\frac{\omega}{\omega_0} + j\left(\frac{\omega}{\omega_0}\right)^2}$$

zero

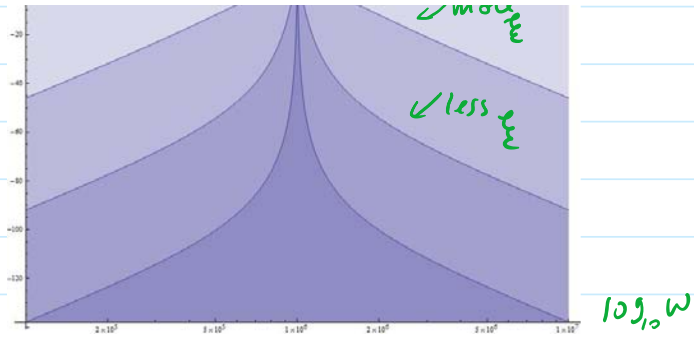
← same quadratic pole as before

$|H|$ is plotted below in linear and in dB scales. Note that as $\xi \to 0$, the peak narrows. Also notice this is a bandpass filter. How do we find the width of the passband?





dB vs. log ω

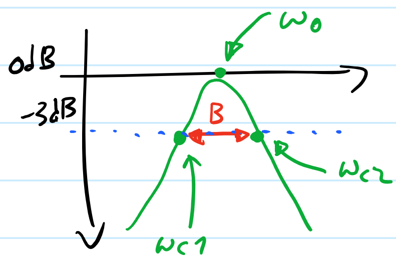


$$|H| = \frac{1}{\sqrt{2}}$$ (just like before)

$$\frac{1}{\sqrt{2}} = \left| \frac{j\omega RC}{1 + j\omega RC - \omega^2 LC} \right|$$ solve for ω

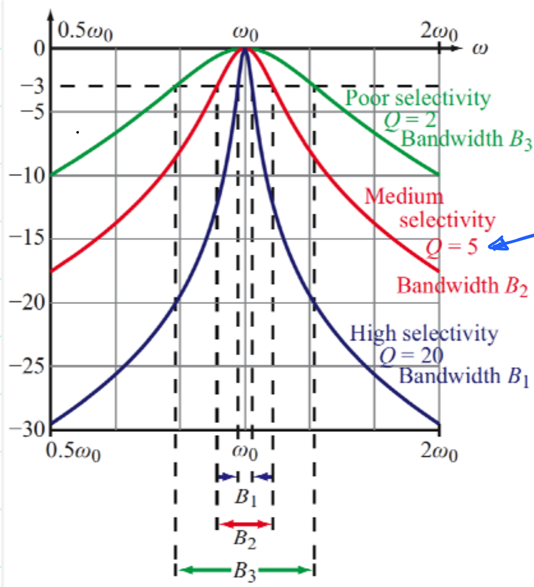
$$\omega_{c_1,c_2} = \pm \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$ ← two ω's where $|H| = \frac{1}{\sqrt{2}}$

This makes sense.



The bandwidth, B, of the bandpass filter is $B = \omega_{c2} - \omega_{c1}$

From above, $\boxed{B = \frac{R}{L}}$



what is this Q?

Note that how sharp a filter is depends on both ω_0 and B. For example, if $f_0 = 1$ GHz and B = 1 GHz, that is not a very sharp filter. In contrast, if $f_0 = 1$ GHz and B = 1 Hz (!) that is a very sharp peak.

Quality Factor (Q)

Is there a way to "score" the sharpness of a filter. Let's take a step back and look at the resonant filter above.

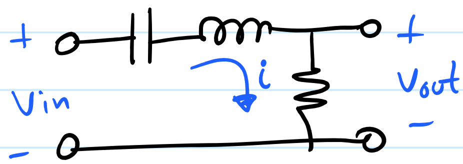
- If it is very underdamped, it rings for a long time, is close to a pure sinusoid at ω_0 and has a sharp peak in the frequency domain (i.e. on a Bode plot).
- As I increase damping, the peak widens in the frequency response.
- The key to how long a resonant circuit oscillates for in time domain is related

to how much energy is lost to heat each cycle vs. how much energy is stored.

- Let's call this ratio Q.

$$Q = 2\pi \left(\frac{U_{stored}}{U_{dissipated}} \right) \Bigg|_{\omega = \omega_0}$$

Note:



$$\tilde{I} = \frac{\tilde{V}_{in}}{Z_{total}} = \frac{\tilde{V}_{in}}{R + j\omega L + \frac{1}{j\omega C}}$$

$$\tilde{I}(\omega = \omega_0) = \frac{\tilde{V}_{in}}{R + \underbrace{j\left(\frac{1}{\sqrt{LC}}\right)L - \frac{j\sqrt{LC}}{C}}_{\text{at } \omega_0 = \frac{1}{\sqrt{LC}}, \text{ this expression} \to 0}} = \frac{\tilde{V}_{in}}{R}$$

why? Note that $L = \frac{(\sqrt{LC})^2}{C}$ and $C = \frac{(\sqrt{LC})^2}{L}$

Because $Z_{total}(\omega_0) = R$, $i(t) = \frac{V_0}{R} \cos(\omega_0 t)$

$$U_{diss} = \int_0^T i(t)^2 R = \int_0^{2\pi/\omega_0} \frac{V_0^2}{R} \cos^2(\omega_0 t) = \boxed{\frac{\pi V_0^2}{\omega_0 R}}$$

What about U_{stored}?

$$U_{stored} = U_L + U_C$$

$$U_L = \frac{1}{2} L i(t)^2 = \frac{1}{2} L \left(\frac{V_0}{R} \cos \omega_0 t \right)^2 = \frac{V_0^2 L}{2R^2} \cos^2 \omega_0 t$$

$$U_L = \frac{1}{2} L\, i(t)^2 = \frac{1}{2} L \left(\frac{V_0}{R} \cos \omega_0 t\right)^2 = \boxed{\frac{V_0^2 L}{2R^2} \cos^2 \omega_0 t}$$

$$U_C = \frac{1}{2} C v(t)^2 = \frac{1}{2} C \left(\frac{1}{C}\int i\, dt\right)^2 = \frac{1}{2} C \left(\frac{1}{C}\int \frac{V_0}{2} \cos \omega_0 t\, dt\right)^2$$

$$= \frac{1}{2} C \left(\frac{V_0}{\omega_0 R C} \sin \omega_0 t\right)^2 = \boxed{\frac{V_0^2 L}{2R^2} \sin^2 \omega_0 t}$$

$$U_{stored} = \frac{V_0^2 L}{2R^2}\left(\sin^2 \omega_0 t + \cos^2 \omega_0 t\right) = \boxed{\frac{V_0^2 L}{2R^2}}$$

So,

$$Q = 2\pi \left[\frac{\frac{V_0^2 L}{2R^2}}{\frac{\pi V_0^2}{\omega_0 R}}\right] = 2\pi \left[\frac{L\omega_0}{2\pi R}\right] = \boxed{\frac{L}{R}\omega_0}$$

$$\boxed{Q = \frac{\omega_0}{B}}$$

This is exactly what we intuited before the derivation: a comparison of ω_0 and B.

So, large Q means: very underdamped in time
very narrow, sharp peak in freq.

Returning to the numbers above, if

$$\left.\begin{array}{l} \omega_0 = 10^9 \text{ rad/s} \\ B = 10^9 \text{ rad/s} \end{array}\right\} Q = 1 \quad \text{(poor filter)}$$

$\omega_0 = 10^9$ rad/s $\}$ $Q = 10^9$ (excellent filter)

$$\left.\begin{array}{l} \omega_0 = 10^9 \text{ rad/s} \\ B = 1 \text{ rad/s} \end{array}\right\} Q = 10^9 \text{ (excellent filter)}$$