1. Op-Amp Review

Consider the circuit below:

(a) Write the KCL equations at each node (you can skip the nodes $V_c$ and $V_d$). Use Ohm’s law and the golden rules of op-amps to express $I_1$ through $I_7$ in terms of voltages and resistances.

**Solution:**

For an op-amp in negative feedback, the Golden Rules are (1) the voltage difference between the two inputs is zero ($V^+ = V^-$), and (2) no current goes into the inputs of an op-amp.

According to the first Golden Rule, we can write down:

\[
V_a = V_p \\
V_b = V_n \\
V_e = V_f 
\]
Then we write down the node current equations based on Kirchhoff’s Current Law (KCL) (the sum of total currents flowing into one node is the same as the sum of currents flowing out of that same node):

$$I_1 = I_2 \Rightarrow \frac{V_c - V_p}{R_2} = \frac{V_a - V_b}{R_1} = \frac{V_p - V_n}{R_1}$$
$$I_2 = I_3 \Rightarrow \frac{V_a - V_b}{R_1} = \frac{V_p - V_n}{R_1} = \frac{V_n - V_d}{R_2}$$
$$I_4 = I_5 \Rightarrow \frac{V_d - V_f}{R_3} = \frac{V_f}{R_4}$$
$$I_6 = I_7 \Rightarrow \frac{V_e - V_c}{R_3} = \frac{V_{out} - V_e}{R_4}$$

(b) In part (a), we used a general circuit analysis procedure to develop a full set of equations that we could then solve (as we could for any circuit). However, with this specific circuit, we can make some observations to reduce the amount of necessary calculations. Notice that there exists a symmetry between the two op-amps at the first stage of this circuit. What is the relationship between $I_1$ and $I_3$? How do $I_1$ and $I_3$ influence $I_2$?

**Solution:**
According to the branch current equations: $I_1 = I_2$ and $I_2 = I_3$, so the currents going through the two $R_2$’s are the same but with opposite directions: one is from the output of the op-amp to the inverting input, while the other is from the inverting input to the output. The current through $R_1$ is the same as the current through the $R_2$’s.

(c) Compute $I_2$.

**Solution:**
The current through $R_1$ is $I_2 = \frac{V_p - V_n}{R_1}$. If $V_p - V_n$ is negative, the current will flow in the opposite direction of what is drawn in the diagram.

(d) Compute $V_c$ and $V_d$.

**Solution:**
For the upper op-amp, $V_c = V_a + I_1 R_2$, where $I_1 = \frac{V_p - V_n}{R_1}$, and $V_a = V_p$. Therefore, the output voltage $V_c$ of the upper op-amp is $V_p + \frac{(V_p - V_n)}{R_1} R_2$.

For the lower op-amp, $V_d = V_b - I_3 R_2$, where $I_3 = \frac{V_p - V_n}{R_1}$, and $V_b = V_n$. Therefore, the output voltage $V_d$ of the lower op-amp is $V_n - \frac{(V_p - V_n)}{R_1} R_2$.

(e) Compute $V_f$.

**Solution:** From part (a), we know that $\frac{(V_d - V_f)}{R_3} = \frac{V_f}{R_4}$. Hence, we could express $V_f$ with $V_d$ as follows:

$$V_f = \frac{R_4}{R_3 + R_4} V_d$$

and plug in the value of $V_d$ we computed in (d), $V_d = V_n - \frac{(V_p - V_n)}{R_1} R_2$:

$$V_f = \frac{R_4}{R_3 + R_4} \left( V_n - \frac{(V_p - V_n)}{R_1} R_2 \right).$$
(f) Compute $V_{\text{out}}$.

**Solution:**

There are two ways to compute $V_{\text{out}}$: (1) use all known values to derive the answer, or (2) start with $V_c$ and $V_d$ as inputs (free variables) first, and then plug in the values of $V_c$ and $V_d$ in the end of computation. Here we will show you (1), and in part (h), you will see (2).

From part (a) (Golden Rules), we know that $V_e = V_f$, and from part (e), we derived $V_f = \frac{R_4}{R_3 + R_4} \left( V_n - \frac{(V_n - V_p)}{R_1} R_2 \right)$.

Also, from part (d), we showed that $V_c = V_p + \frac{(V_n - V_p)}{R_1} R_2$.

From part (a), we know that $\frac{V_e - V_c}{R_3} = V_{\text{out}} - \frac{V_e}{R_4}$.

We can express $V_{\text{out}}$ in terms of $V_e$ and $V_c$:

$$V_{\text{out}} = \left( 1 + \frac{R_4}{R_3} \right) V_e - \frac{R_4}{R_3} V_c$$

After plugging in the values for $V_c$ and $V_e$, we get $V_{\text{out}}$:

$$V_{\text{out}} = \left( 1 + \frac{R_4}{R_3} \right) \frac{R_4}{R_3 + R_4} \left( V_n - \frac{(V_p - V_n)}{R_1} R_2 \right) - \frac{R_4}{R_3} \left( V_p + \frac{(V_p - V_n)}{R_1} R_2 \right)$$

$$= R_4 \left( V_n - V_p - \frac{2 R_2 (V_p - V_n)}{R_1} \right)$$

$$= (V_n - V_p) \frac{R_4}{R_3} \left( 1 + \frac{2 R_2}{R_1} \right)$$

2. **More practice**

Consider the circuit shown below:

![Circuit Diagram](image)

Compute $V_1$ and $V_2$.

**Solution:**

Using KCL on $V_1, V_2$ yields the equations

$$0 = \frac{V_1 - 6}{4} + \frac{V_1}{4} + \frac{V_1 - V_2}{8}$$

$$0 = \frac{V_2 - V_1}{8} + \frac{V_2 - 5}{8} - 2$$

Solving for the two variables in the two equations yields $V_1 = 5 \text{ V}, V_2 = 13 \text{ V}$
3. **OPTIONAL: Even more practice**

Consider the circuit shown below:

Determine the amount of power supplied by the voltage source. Do not use superposition.

**Solution:**

We will label the currents $I_1$, $I_2$, and $I_3$ as shown in the following diagram.

\[
\begin{align*}
\text{Loop 1:} & \quad I_1 = 12 \text{ A} \\
\text{Loop 2:} & \quad (I_2 - I_1) + 2I_2 + 3(I_2 - I_3) = 0 \\
\text{Loop 3:} & \quad 3(I_3 - I_2) + 4I_3 + 6 = 0
\end{align*}
\]

Simplification leads to:

\[
\begin{align*}
6I_2 - 3I_3 &= 12 \\
-3I_2 + 7I_3 &= -6
\end{align*}
\]

Solving this system of equations gives:

\[
I_2 = 2 \text{ A} \quad I_3 = 0 \text{ A}
\]

The power supplied by the voltage source is:

\[
P = VI = 6 \cdot (0) = 0 \text{ W}
\]

4. **Write Your Own Question And Provide a Thorough Solution.**

Writing your own problems is a very important way to really learn material. The famous “Bloom’s Taxonomy” that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But
we don’t want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don’t have to achieve this every week. But unless you try every week, it probably won’t ever happen.