EE16B
Designing Information Devices and Systems II

Lecture 11B
Discrete Signals and Systems
Intro

• Mitterm II yesterday…

• Last time:
  – Sampling Theorem
  – Aliasing
  – Discrete Signals

• Today
  – Discrete systems
Complex Frequencies

- Sinusoids are sums of left and right rotating complex exponentials

\[ 2 \cos(\omega t) = e^{j\omega t} + e^{-j\omega t} \]

“Positive” and “Negative” frequencies

Discrete frequencies with period N:

\[ y[n] = e^{j2\pi n/N} \]

\[ W_N \triangleq e^{j2\pi/N} \implies y[n] = W_N^n \]
Complex Frequencies

\[ W_N \triangleq e^{j2\pi/N} \Rightarrow y[n] = W_N^n \]

• N = 4 \[ y[n] = W_4^n \]

• N = 6, neg. freq. \[ y[n] = W_6^{-n} \]
Discrete Time Systems

- Causality
- Linearity
- Stability
- Time/shift invariance

*WARNING: Going to interchange $x[n]$ and $u[n]$ as inputs
$x[n]$ will be a state, not input
$U[n]$ is unit step, not to be confused with $u[n]$
Properties of D.T. Systems

• Causality:
  – $y[n_0]$ depends only on $x[n]$ for $\infty \leq n \leq n_0$

Causal?

\[
\begin{align*}
\ddot{x}[n + 1] &= A\ddot{x}[n] + Bu[n] \\
y[n] &= C\ddot{x}[n] \\
\ddot{x}[n] &= A^n \ddot{x}[0] + \sum_{k=0}^{n-1} A^{n-1-k} Bu[k] \\
y[n] &= CA^n \ddot{x}[0] + \sum_{k=0}^{n-1} CA^{n-1-k} Bu[k]
\end{align*}
\]
Properties of D.T. Systems

\[ y[n] = F\{x[n]\} \]

• Linearity
  – Homogeneity: scaling the input, scales the output

\[ F\{ax[n]\} = aF\{x[n]\} = ay[n] \]
Properties of D.T. Systems

\[ y[n] = F\{x[n]\} \]

• Linearity
  – Homogeneity: scaling the input, scales the output
    \[ F\{ax[n]\} = aF\{x[n]\} = ay[n] \]
  – Superposition: sum of inputs ⇒ sum of outputs
    \[ F\{x_1[n] + x_2[n]\} = F\{x_1[n]\} + F\{x_2[n]\} = y_1[n] + y_2[n] \]
Example:

\[
\begin{align*}
\tilde{x}[n + 1] &= A\tilde{x}[n] + Bu[n] \\
y[n] &= C\tilde{x}[n]
\end{align*}
\]

Linear?

\[
y[n] = CA^n\tilde{x}[0] + \sum_{k=0}^{n-1} CA^{n-1-k}Bu[k]
\]
Properties of D.T. Systems

\[ y[n] = F \{ x[n] \} \]

• BIBO Stability
  - If \( x[n] \) is bounded, then \( y[n] \) is bounded

\[ |x[n]| < M < \infty \quad \forall n \Rightarrow \quad |y[n]| < P < \infty \quad \forall n \]

BIBO stable?

\[ y[n] = CA^n x[0] + \sum_{k=0}^{n-1} CA^{n-1-k} Bu[k] \]
Properties of D.T. Systems

\[ y[n] = F\{x[n]\} \]

- Time Invariance: Shifted input \( \Rightarrow \) shifted output

\[ y[n - n_0] = F\{x[n - n_0]\} \]

Time Invariant?

\[ \ddot{x}[n + 1] = A\ddot{x}[n] + Bu[n] \]
\[ y[n] = C\ddot{x}[n] \]

\[ y[n] = CA^n\ddot{x}[0] + CBu[n - 1] + CABu[n - 2] + \cdots + CA^{n-1}Bu[0] \]
Linear Time Invariant Systems

- Linear Time/Shift Invariant (LTI/LSI) systems are completely characterized by their impulse response $h[n]$

\[ \delta[n] \rightarrow \text{LTI} \rightarrow h[n] \]

$h[n]$ is the “DNA” of an LTI system
Knowing $h[n]$ is enough to find $y[n]$ for ANY $x[n]$!
Linear Time Invariant Systems

- Decompose $x[n]$: $x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m] = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

- Compute output: $y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] = x[n] * h[n]$  
  
  Sum of weighted, delayed impulse responses!
Example:

Infinite impulse response (IIR)

\[ y[n] = ay[n - 1] + x[n] \]
\[ h[n] = \begin{cases} 
  a^n & n \geq 0 \\
  0 & n < 0 
\end{cases} \]

finite impulse response (FIR)

\[ y[n] = x[n] - x[n - 1] \]
\[ h[n] = \delta[n] - \delta[n - 1] \]
Convolution Sum

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]

- What is \( h[n-m] \) for different \( n \)'s?

\[ h[0 - m] \]

\[ h[n] = \delta[n] - \delta[n - 1] \]
Convolution Sum

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] = x[n] * h[n] \]

• What is \( h[n-m] \) for different \( n \)'s?

\( h[1 - m] \)

\[ h[n] = \delta[n] - \delta[n - 1] \]
Convolution Sum

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]

• What is \( h[n-m] \) for different \( n \)’s?

\[ h[2 - m] \]

\[ h[n] = \delta[n] - \delta[n - 1] \]
Convolution Sum

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]

• What is \( h[n-m] \) for different \( n \)'s?

\[ h[3 - m] \]

\[ h[n] = \delta[n] - \delta[n - 1] \]
Graphical Example of Convolution

\[ x[n] \]

\[ h[n] \]

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]
Graphical Example of Convolution

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Graphical Example of Convolution

\[ x[m] \]

\[ h[n - m] \]

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]
Graphical Example of Convolution

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]
Graphical Example of Convolution

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] \]
Example

Awekward

Friendly

Mean
Finite Sequences

- Consider a finite sequence of length $N$

$x[n] = \begin{cases} 
\text{something} & 0 \leq n < N \\
0 & \text{otherwise}
\end{cases}$

- Can also be written as a vector

\[
\tilde{x} = \begin{bmatrix}
x(0) \\
x(1) \\
\vdots \\
x(N-1)
\end{bmatrix}
\]
Why?

• To compute this:
Finite Sequences as Vectors

- Define an inner-product (for $\mathbb{R}^N$):

$$< \vec{x}, \vec{y} > = \vec{x} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]y[n] =$$

$$= \vec{x}^T \vec{y}$$

So,

$$< \vec{x}, \vec{x} > = \sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} x^2[n] = ||\vec{x}||^2$$

$$\Rightarrow \vec{x}^T \vec{x} = ||\vec{x}||^2$$
Finite Sequences as Vectors

• What about complex?

\[ x \cdot x = x^2 = (x_r + jx_i)(x_r + jx_i) = x_r^2 - x_i^2 + 2jx_rx_i \neq ||x||^2 \]

but,

\[ x^* \cdot x = (x_r - jx_i)(x_r + jx_i) = x_r^2 + x_i^2 = ||x||^2 \]

• Transpose vs Transpost conjugate

\[ \vec{x} = \begin{bmatrix} 1 \\ j \\ 1 + j \end{bmatrix} \quad \vec{x}^T = \begin{bmatrix} 1 & j & 1 + j \end{bmatrix} \]

\[ \vec{x}^* = \begin{bmatrix} 1 & -j & 1 - j \end{bmatrix} \]
Finite Sequences as Vectors

- Define Complex inner product

\[ \langle \vec{x}, \vec{y} \rangle = \overline{\vec{x}} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = \vec{x}^* \vec{y} = \vec{x}^H \vec{y} \]

\[ \vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \quad \Rightarrow \quad \vec{x}^* \vec{x} = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = 2 \]
Projections

• Orthogonality:
  \[ \langle \vec{x}, \vec{y} \rangle = \sum_{n=0}^{N-1} x[n] * y[n] = 0 \]

• Unit vector:
  \[ ||\hat{x}|| = 1 \]
  \[ \hat{x} = \frac{\vec{x}}{||\vec{x}||} \]

• Define projection as:
  \[ \vec{y}^* \vec{x} \]
  \[ \frac{||\vec{y}||}{||\vec{y}||} \]
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

\[
\hat{e}_1^* \vec{x} = [1 \ 0] \vec{x} = x_1 \\
\hat{e}_2^* \vec{x} = [0 \ 1] \vec{x} = x_2
\]
Change of Coordinates (Basis)

• We can compute new coordinates by projections onto orthonormal basis vectors

New coordinates:

\[
\begin{bmatrix}
\hat{b}_1^* \vec{x} \\
\hat{b}_2^* \vec{x}
\end{bmatrix} = \begin{bmatrix}
\hat{b}_1 & \hat{b}_2
\end{bmatrix} \vec{x}
\]

\[
\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x}) \hat{b}_1 + (\hat{b}_2^* \vec{x}) \hat{b}_2
\]
Change of basis

\[ \hat{b}_1 \quad \frac{1}{\sqrt{8}} \quad = \quad \hat{b}_1 + \hat{b}_2 + \hat{b}_3 + \hat{b}_4 + \hat{b}_5 + \hat{b}_6 + \hat{b}_7 + \hat{b}_8 \]