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EECS 16B      Designing Information Devices and Systems II  
 Fall 2019      Discussion Worksheet      Discussion 11B

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## Controller Canonical Form

When working with systems using a state-space representation, you may have noticed that a single system can be represented in many different forms, depending on lots of choices that we can make — such as how you ordered your state vector. Writing out systems in certain **canonical forms** often allows engineers to quickly determine system behavior.

The **controllable canonical form** for a system with a scalar input, which guarantees controllability and simplifies eigenvalue placement, takes the following form:

$$A_z = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}, \quad \vec{b}_z = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (1)$$

You will verify later that this essentially means that every single system that is controllable can be viewed as having a higher-order recurrence-relation inside (for the discrete-time case) or a higher-order differential equation inside (for the continuous-time case).

The notation here is slightly different from lecture, but better matches the reader.

## Change of Basis to Controller Canonical Form

Given a **controllable** system of the form  $\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t)$ , we can transform it into controllable canonical form by choosing some basis  $T$ , such that:

$$\vec{z} = T^{-1}\vec{x}, \quad A_z = T^{-1}AT, \quad \text{and} \quad \vec{b}_z = T^{-1}\vec{b}$$

for matrices  $A_z$  and  $\vec{b}_z$  of the form shown above.

Here  $T$  is the new basis that we are using. In that basis, the system looks like it is in controllable canonical form.

In lecture, you saw a way to compute this transformation as the composition of two different transformations. It is also possible to understand it more directly. Of course, the original controllability matrix still plays an essential role.

We can calculate this  $T$  using  $R_n = [\vec{b} \quad A\vec{b} \quad \cdots \quad A^{n-1}\vec{b}]$ , the controllability matrix of the original form using  $A$  and  $\vec{b}$ . Note that  $R_n$  is invertible, because the original system is controllable. Calculate  $R_n^{-1}$  — this maps vectors into the basis given by the columns of  $R_n$ . Therefore  $\vec{a} = R_n^{-1}A^n\vec{b}$  gives the coefficients of the characteristic polynomial of  $A$ .

With those coefficients in hand, we can compute a new matrix  $\tilde{R}_n = \begin{bmatrix} \vec{b}_z & A_z \vec{b}_z & \cdots & A_z^{n-1} \vec{b}_z \end{bmatrix}$ , the controllability matrix of the CCF form using  $A_z$  and  $\vec{b}_z$ . It turns out that once again  $\vec{a} = \tilde{R}_n^{-1} A_z^n \vec{b}_z$ .

Then, we can cascade them together to get the transformation  $T^{-1} = \tilde{R}_n R_n^{-1}$ . This maps from the original coordinates to coordinates for where the CCF system representation lives.

Also notice that when we place our system in feedback using  $u(t) = \tilde{K}\vec{z} = K\vec{x}$  with  $\tilde{K} = KT$ , we get the closed loop matrix

$$T^{-1}(A + BK)T = A_z + \vec{b}_z \tilde{K}$$

The eigenvalues of both systems are the same, and we can arbitrarily assign the eigenvalues of  $\tilde{A} + \tilde{B}\tilde{K}$  with the choice of  $\tilde{K}$ . Therefore, we just proved that *controllability enables arbitrary eigenvalue assignment* in any state space system. Note that it is *not* necessary to bring the system to the controller canonical form to assign its eigenvalues.

## Questions

### 1. Control Canonical Form - Introduction

- (a) Show that for any square matrix  $A$  and any invertible matrix  $T$ , the matrix  $T^{-1}AT$  has the same characteristic polynomial as  $A$ .

- (b) Consider the matrix

$$A_z = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

Verify that

$$\vec{v}_\lambda = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$$

is an eigenvector of  $A$ . What does this imply about  $\lambda^n$ ?

(c) Compute the characteristic polynomial of

$$A_z = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \\ a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \end{bmatrix}$$

(d) Show that a discrete-time system in controllable canonical form is essentially a higher order scalar recurrence relation with scalar input.

(e) Show that a continuous-time system in controllable canonical form is essentially a higher order scalar differential equation with scalar input.

## 2. Controllable Canonical Form - Practice and Eigenvalues Placement

Consider the following linear discrete time system

$$\vec{x}(t+1) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

(a) Is this system controllable?

(b) Is the linear discrete time system stable?

(c) Bring the system to the controllable canonical form

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

using transformation  $\vec{z}(t) = T^{-1}\vec{x}(t)$ . Compute the basis  $T$ .

(d) Using state feedback  $u(t) = [\tilde{k}_1 \quad \tilde{k}_2 \quad \tilde{k}_3] \vec{z}(t)$  place the eigenvalues at  $0, 1/2, -1/2$ .

(e) What is  $u(t) = [k_1 \quad k_2 \quad k_3] \vec{x}(t)$  in the original basis?

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