EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet Discussion 13B

Questions

1. Lagrange interpolation and polynomial basis

In practice, to approximate some unknown or complex function $f(x)$, we take n evaluations/samples of the function, denoted by $\{(x_i, y_i \triangleq f(x_i));\ 0 \leq i \leq n-1\}$. With the Occam's razor principle in mind, we try to fit a polynomial function of least degree (which is $n-1$) that passes through all the given points.

(a) Using the polynomial basis $\{1, x, x^2, \cdots, x^{n-1}\}$, the fitting problem can be cast into finding the coefficients $a_0, a_1, \ldots, a_{n-1}$ of the function

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

such that $g(x_i) = y_i,\ \forall i = 0, 1, \ldots, n-1$. **Find out the set of equations that need to be satisfied, and write them in a matrix form $A\vec{a} = \vec{y}$, with $\vec{a} = [a_0, a_1, \ldots, a_{n-1}]^T$ and $\vec{y} = [y_0, y_1, \ldots, y_{n-1}]^T$**

(b) Observe that in order to find those coefficients, we need to calculate $\vec{a} = A^{-1}\vec{y}$. Solving that system of equations might be tricky if n is large.

The idea of Lagrange interpolation is to use a different basis $\{L_0(x), L_1(x), \cdots, L_{n-1}(x)\}$ for the subspace of polynomials of the desired degree, which has the property that

$$L_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

With that the fitting problem becomes finding the coefficients $b_0, b_1, \ldots, b_{n-1}$ of the function

$$h(x) = b_0 L_0(x) + b_1 L_1(x) + b_2 L_2(x) + \cdots + b_{n-1} L_{n-1}(x)$$

such that $h(x_i) = y_i,\ \forall i = 0, 1, \ldots, n-1$. **Find the set of equations that need to be satisfied, and write them in matrix/vector form. What do you observe?**

(c) **Show that if we define**

$$L_i(x) = \prod_{j=0;j\neq i}^{n-1} \frac{(x-x_j)}{(x_i-x_j)}$$

then the property required in part (b) is satisfied. What is the intuition behind this construction?

(d) Based on the previous two parts, write down the explicit form of $h(x)$ that passes through the samples $\{(x_i, y_i);\ 0 \leq i \leq n-1\}$. The resulting formula is the so called Lagrange polynomial which passes through the n sampled points.

(e) Explicitly find the Lagrange polynomial given evaluated samples $f(-1) = 3, f(0) = -4, f(1) = 5, f(2) = -6$.

(f) Suppose that we want to model a function using a degree-at-most $n-1$ polynomial. Our data set for the behavior of the system includes m sample/data points, where $m > n$. **Write out the set of equations that satisfy this system. How can we 'solve' the over-constrained set of equations?**

(g) By hand plot what $1, x, x^2, x^3, \ldots$ look like on the interval from -1 to $+1$ and then from -2 to $+2$. Look at the endpoints, the values at ± 1, and other points. What do you notice? What might go wrong when we try to learn high-degree polynomials over such intervals from data?

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