Questions

1. Lagrange interpolation and polynomial basis

In practice, to approximate some unknown or complex function \( f(x) \), we take \( n \) evaluations/samples of the function, denoted by \( \{(x_i, y_i) \overset{\Delta}{=} f(x_i) \}; \ 0 \leq i \leq n - 1 \}. \) With the Occam’s razor principle in mind, we try to fit a polynomial function of least degree (which is \( n - 1 \)) that passes through all the given points.

(a) Using the polynomial basis \( \{1, x, x^2, \ldots, x^{n-1}\} \), the fitting problem can be cast into finding the coefficients \( a_0, a_1, \ldots, a_{n-1} \) of the function

\[
g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}
\]

such that \( g(x_i) = y_i, \ \forall i = 0, 1, \ldots, n - 1. \) Find out the set of equations that need to be satisfied, and write them in a matrix form \( A\vec{a} = \vec{y} \), with \( \vec{a} = [a_0, a_1, \ldots, a_{n-1}]^T \) and \( \vec{y} = [y_0, y_1, \ldots, y_{n-1}]^T \).

(b) Observe that in order to find those coefficients, we need to calculate \( \vec{a} = A^{-1}\vec{y} \). Solving that system of equations might be tricky if \( n \) is large.

The idea of Lagrange interpolation is to use a different basis \( \{L_0(x), L_1(x), \ldots, L_{n-1}(x)\} \) for the subspace of polynomials of the desired degree, which has the property that

\[
L_i(x_j) = \begin{cases} 
1 & \text{if } j = i \\
0 & \text{if } j \neq i 
\end{cases}
\]

With that the fitting problem becomes finding the coefficients \( b_0, b_1, \ldots, b_{n-1} \) of the function

\[
h(x) = b_0 L_0(x) + b_1 L_1(x) + b_2 L_2(x) + \cdots + b_{n-1} L_{n-1}(x)
\]

such that \( h(x_i) = y_i, \ \forall i = 0, 1, \ldots, n - 1. \) Find the set of equations that need to be satisfied, and write them in matrix/vector form. What do you observe?
(c) **Show that if we define**

\[ L_i(x) = \prod_{j=0, j \neq i}^{n-1} \frac{(x-x_j)}{(x_i-x_j)} \]

**then the property required in part (b) is satisfied. What is the intuition behind this construction?**

(d) Based on the previous two parts, write down the explicit form of \( h(x) \) that passes through the samples \( \{(x_i, y_i); 0 \leq i \leq n - 1\} \). The resulting formula is the so called Lagrange polynomial which passes through the \( n \) sampled points.

(e) Explicitly find the Lagrange polynomial given evaluated samples \( f(-1) = 3, f(0) = -4, f(1) = 5, f(2) = -6 \).

(f) Suppose that we want to model a function using a degree-at-most \( n - 1 \) polynomial. Our data set for the behavior of the system includes \( m \) sample/data points, where \( m > n \). **Write out the set of equations that satisfy this system. How can we ‘solve’ the over-constrained set of equations?**

(g) By hand plot what \( 1, x, x^2, x^3, \ldots \) look like on the interval from \(-1\) to \(+1\) and then from \(-2\) to \(+2\). Look at the endpoints, the values at \( \pm 1 \), and other points. What do you notice? What might go wrong when we try to learn high-degree polynomials over such intervals from data?

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