1. Differential equations with piecewise constant inputs

Working through this question will help you understand better differential equations with inputs. Along the way, we will also touch a bit on going from continuous-time into a discrete-time view. This problem also provides a vehicle to review relevant concepts from calculus.

(a) Consider the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t). \tag{1}$$

Suppose that our input $u(t)$ of interest is *constructed* to be piecewise constant over durations of width Δ. In other words:

$$u(t) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta) \tag{2}$$

Given that we start at $x(i\Delta) = x_d[i]$, where do we end up at $x_d[i+1] = x((i+1)\Delta)$?

(b) Suppose that $x_d[0] = x_0$. **Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves x_0 and the $u[j]$ for $j = 0, 1, \ldots, i$.**

For this part, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu[i] \tag{3}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.

(c) **For a given time t in continuous real time, what is the discrete i interval that corresponds to it?**

(d) Now, we are going to turn this around. Suppose that the $u[i]$ is actually a sample of a desired input $u_c(t)$ in continuous time. Namely that $u[i] = u_c(i\Delta)$. Recall what the a and b mean in (3) and **approximate $x(t)$ if we apply this piecewise constant input $u(t)$ to the system** (1)**.** You can assume that Δ is small enough that $x(t)$ and $u_c(t)$ do not change too much over an interval of length Δ.

(e) **Draw a picture of what is going on and then further approximate the previous expression by considering $n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ where needed and treating $\Delta \approx \frac{t}{n}$.** This is a meaningful approximation when we think about n large enough.

(f) Consider the term in from of the sum in the above expression for $x(t)$:

$$\frac{1 - e^{-\lambda\Delta}}{\lambda}$$

How can we approximate this term as $\Delta \to 0$?

(g) **Now take the limit of $x(t)$ as $\Delta \to 0$ by taking the limit $n \to \infty$. What is the expression you get for $x(t)$?**

(HINT: Remember your definition of definite integrals as limits of Riemann sums in calculus.)

(h) **Verify the analytic solution for $x(t)$ found in (g) for $u(t) = 0$ and for $u(t) = u_0$.**

(i) **Verify the analytic solution found in (g) by plugging it back into the differential equation.**

(j) **If input $u(t)$ is the linear sum of two other inputs $u(t) = c_1 u_1(t) + c_2 u_2(t)$, what does the solution look like? What does this mean?**

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