EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet **Discussion 3B**

1 Complex Numbers

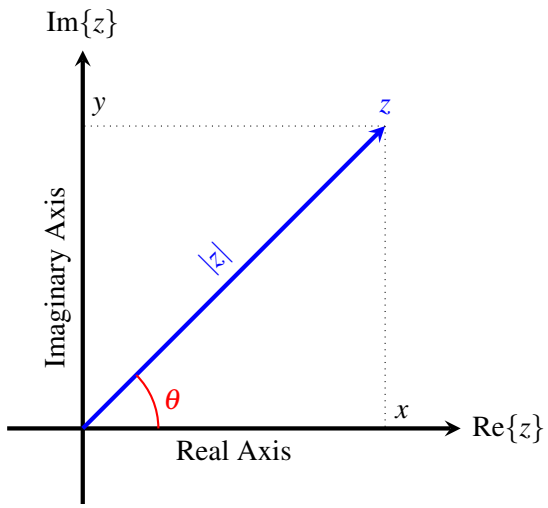


Figure 1: Complex number z represented as a vector in the complex plane.

A complex number z is an ordered pair (x,y), where x and y are real numbers, written as $z = x + jy$ where $j = \sqrt{-1}$. A complex number can also be written in polar form as follows:

$$z = |z|e^{j\theta}$$

The magnitude z is denoted as $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}.$$

The phase or argument of a complex number is denoted as θ and is given by

$$\theta = \text{atan2}(y,x).$$

Here, $\text{atan2}(y,x)$ is a function that returns the angle from the positive x-axis to the vector from the origin to the point (x,y) [1].

The complex conjugate of a complex number z is denoted by $\bar{z}$ (or might also be written z^*) and is given by

$$\bar{z} = x - jy.$$

Euler's Identity is

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

[1] See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at `https://en.wikipedia.org/wiki/Atan2`.

With this definition, the polar representation of a complex number will make more sense. Note that

$$|z|e^{j\theta} = |z|\cos(\theta) + j\,|z|\sin(\theta).$$

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case $|z|$ is the magnitude and $e^{j\theta}$ is the unit vector that defines the direction.

2 Useful Identities

Complex Number Properties

Rectangular vs. polar forms: $z = x + jy = |z|e^{j\theta}$

where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2+y^2}$, $\theta = \text{atan2}(y,x)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j},\ \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Complex conjugate: $\overline{z} = x - jy = |z|e^{-j\theta}$

$$\overline{(z+w)} = \overline{z} + \overline{w}, \overline{(z-w)} = \overline{z} - \overline{w}$$

$$\overline{(zw)} = \overline{z}\,\overline{w},\ \overline{(z/w)} = \overline{z}/\overline{w}$$

$$\overline{z} = z \Leftrightarrow z \text{ is real}$$

$$\overline{(z^n)} = (\overline{z})^n$$

Complex Algebra

Let $z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}$, $z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}$.

(Note that we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $z_1 z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)}$

Division: $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$

Power: $z_1^n = |z_1|^n e^{jn\theta_1}$

$$z_1^{\frac{1}{2}} = \pm|z_1|^{\frac{1}{2}}e^{j\frac{\theta_1}{2}}$$

Useful Relations

$$-1 = j^2 = e^{j\pi} = e^{-j\pi}$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1}$$

$$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1+j)}{\sqrt{2}}$$

$$\sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \frac{\pm(1-j)}{\sqrt{2}}$$

3 Phasors

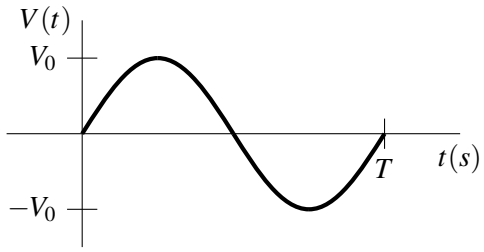
We consider sinusoidal voltages and currents of a specific form:

Voltage	$v(t) = V_0\cos(\omega t + \phi_v)$
Current	$i(t) = I_0\cos(\omega t + \phi_i)$

where,

(a) V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.

(b) ω is the angular frequency of oscillation. ω is related to frequency by $\omega = 2\pi f$. Frequency f is the number of oscillation cycles that occur per second. If T is the *period* of the sinusoid (that is, the amount of time it takes for one complete cycle to occur) the frequency is $f = \frac{1}{T}$.



(c) ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time, of the sinusoid.

We know from Euler's identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

Extending this to our voltage signal from above:

$$v(t) = V_0\cos(\omega t + \phi_v) = \frac{V_0}{2}e^{j\omega t + j\phi_v} + \frac{V_0}{2}e^{-j\omega t - j\phi_v} = \frac{V_0}{2}e^{j\phi_v}e^{j\omega t} + \frac{V_0}{2}e^{-j\phi_v}e^{-j\omega t} \tag{1}$$

The coefficient of the $e^{j\omega t}$ term in Equation 1 is called the phasor form of this signal:

$$\widetilde{V} = \frac{1}{2}V_0 e^{j\phi_v}$$

The complex conjugate of the phasor is the coefficient of the $e^{-j\omega t}$ term in Equation 1 and is given by:

$$\overline{\widetilde{V}} = \frac{1}{2}V_0 e^{-j\phi_v}$$

The phasor representation is a constant that contains the magnitude and phase information of the signal. The time-varying part of the signal does not need to be explicitly represented, because it is given by $e^{j\omega t}$, which is always implicit when using phasors. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like (16A style) circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

Note: We can only use phasors if we know that all of our signals are sinusoids. We will see later in 16B (Module 4) why this is not too restrictive a condition.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\widetilde{V}$ is the phasor.

$$V_0\cos(\omega t + \phi_v) = \widetilde{V}e^{j\omega t} + \overline{\widetilde{V}}e^{-j\omega t}$$

The standard forms for voltage and current phasors are given below:

Voltage	$\widetilde{V} = \frac{1}{2}V_0 e^{j\phi_v}$
Current	$\widetilde{I} = \frac{1}{2}I_0 e^{j\phi_i}$

We define the *impedance* of a circuit component to be $Z = \frac{\widetilde{V}}{\widetilde{I}}$, where $\widetilde{V}$ and $\widetilde{I}$ represent the voltage across and the current through the component, respectively. For capacitors and inductors, this impedance will generally depend on $j\omega$.

3.1 Phasor Relationship for Resistors

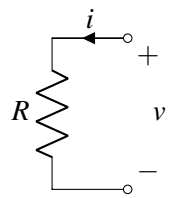


Figure 2: A simple resistor circuit

Consider a simple resistor circuit as in Figure 2, with current being

$$i(t) = I_0 \cos(\omega t + \phi) = \frac{I_0}{2} e^{j\phi} e^{j\omega t} + \frac{I_0}{2} e^{-j\phi} e^{-j\omega t}$$

By Ohm's law,

$$\begin{aligned} v(t) &= i(t)R \\ &= I_0 R \cos(\omega t + \phi) = \frac{I_0 R}{2} e^{j\phi} e^{j\omega t} + \frac{I_0 R}{2} e^{-j\phi} e^{-j\omega t} \end{aligned}$$

In the phasor domain, for both $s = +j\omega$ and $s = -j\omega$

$$\widetilde{V} = R\widetilde{I}$$

We usually refer to the impedance of the resistor, Z_R, in the phasor domain. Since $Z_R = R$, we can also write:

$$\widetilde{V} = Z_R \widetilde{I}$$

3.2 Phasor Relationship for Capacitors

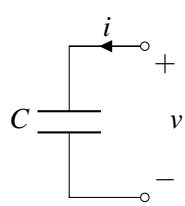


Figure 3: A simple capacitor circuit

Consider a capacitor circuit as in Figure 3, with voltage being

$$v(t) = V_0 \cos(\omega t + \phi)$$

By the capacitor equation, and expanding cos in the complex exponential domain:

$$\begin{aligned} v(t) &= V_0 \cos(\omega t + \phi) \\ &= \frac{V_0}{2} e^{j(\omega t + \phi)} + \frac{V_0}{2} e^{-j(\omega t + \phi)} \\ &= \frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \\ &= \widetilde{V} e^{j\omega t} + \overline{\widetilde{V}} e^{-j\omega t} \end{aligned}$$

$$\begin{aligned} i(t) &= C \frac{d}{dt} v(t) \\ &= C \frac{d}{dt} \left(\frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \right) \\ &= j\omega C \frac{V_0}{2} e^{j\phi} e^{j\omega t} - j\omega C \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \end{aligned}$$

Considering the $s = +j\omega$ term:

$$\widetilde{I} = j\omega C \frac{V_0}{2} e^{j\phi} = j\omega C \widetilde{V}$$

This can also be derived in the sinusoid domain:

$$\begin{aligned} i(t) &= C \tfrac{\mathrm{d}v}{\mathrm{d}t}(t) \\ &= -CV_0 \omega \sin(\omega t + \phi) \\ &= -CV_0 \omega \left(-\cos\left(\omega t + \phi + \frac{\pi}{2} \right) \right) \\ &= CV_0 \omega \cos\left(\omega t + \phi + \frac{\pi}{2} \right) \\ &= (\omega C) V_0 \cos\left(\omega t + \phi + \frac{\pi}{2} \right) \end{aligned}$$

In the phasor domain,

$$\widetilde{I} = \omega C e^{j\frac{\pi}{2}} \widetilde{V} = j\omega C \widetilde{V}$$

The impedance of a capacitor is an abstraction to model the capacitor in a manner similar to a resistor in the phasor domain. This is denoted Z_C, which is given by:

$$Z_C = \frac{\widetilde{V}}{\widetilde{I}} = \frac{1}{j\omega C} = \frac{1}{sC}$$

for $s = j\omega$.

1. **Cosine vs Sine Relation**

   (a) **What value of ϕ makes the following relation true?**

$$\sin(\omega t) = \cos(\omega t + \phi)$$

2. Inductor Impedance

Given the voltage-current relationship of an inductor $V = L\frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

3. Complex Matrix Inverse

Consider a complex matrix

$$M = M_r + jM_i$$

and its inverse

$$N = N_r + jN_i$$

(a) **Show that the inverse of $\overline{M} = M_r - jM_i$ (the complex conjugate of M) is equal to $\overline{N} = N_r - jN_i$ (the complex conjugate of N).**

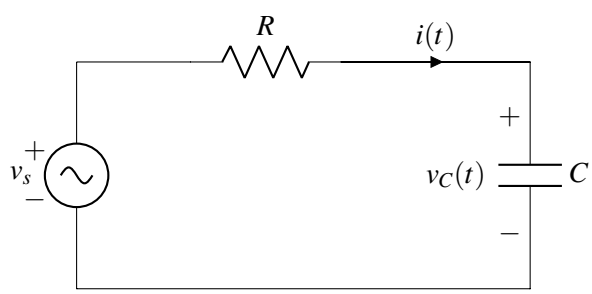
4. Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}, \tag{2}$$

where $\widetilde{X}$ is a time-independent function called the **phasor** representation of $x(t)$ (recall that $\overline{a}$ denotes the complex conjugate of a). Note that 1) $\widetilde{X}$ and $\overline{\widetilde{X}}$ are complex conjugates of each other, 2) $e^{j\omega t}$ and $e^{-j\omega t}$ are complex conjugates of each other, and 3) that $\widetilde{X}\,e^{j\omega t}$ and $\overline{\widetilde{X}}\,e^{-j\omega t}$ are also complex conjugates of each other.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{3}$$

with $\omega = 1 \times 10^3\ \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

(a) **Step 1: Write sources as exponentials: $\widetilde{A}e^{j\omega t} + \overline{\widetilde{A}}e^{-j\omega t}$**

All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. Convert $v_s(t)$ into a exponential and write down its phasor representation $\widetilde{V}_s$.

(b) **Step 2: Transform circuits to phasor domain for $s = +j\omega$**
The voltage source is represented by its phasor $\widetilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\widetilde{I}$ by

$$i(t) = \widetilde{I}e^{j\omega t} + \overline{\widetilde{I}}e^{-j\omega t}.$$

What are the impedances of the resistor, Z_R, and capacitor, Z_C? We sometimes also refer to this as the "phasor representation" of R and C.

(c) **Step 3: Cast the branch and element equations in phasor domain**
Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

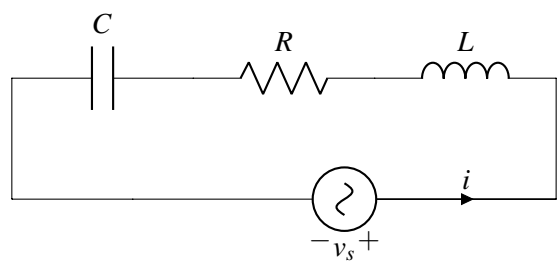
(d) **Step 4: Solve for unknown variables**
Solve the equation you derived in Step 3 for $\widetilde{I}$ and $\widetilde{V}_C$. What is the polar form of $\widetilde{I}$ and $\widetilde{V}_C$? Polar form is given by $Ae^{j\theta}$, where A is a positive real number.

(e) **Step 5: Transform solutions back to time domain**
To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?

5. RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

$$v_s(t) = B\cos(\omega t - \phi)$$



(a) Write out the phasor representations of $v_s(t)$, R, C, and L.

(b) Use Kirchhoff's laws to write down a loop equation relating the phasors in the previous part.

(c) Solve the equation in the previous step for the current $\widetilde{I}$. What is the magnitude and phase of the polar form of $\widetilde{I}$?

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