EECS 16B Fall 2019 — Designing Information Devices and Systems II — Discussion Worksheet

5A: Inputs

1. A system governed by differential equations being controlled with piecewise constant inputs

Working through this question will help you understand better differential equations with inputs and the sampling of a continuous-time system of differential equations into a discrete-time view. This is important for control, since it is often easier to think about doing what we want in discrete-time.

(a) Consider the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t). \tag{1}$$

Suppose that our $u(t)$ of interest is *constructed* to be piecewise constant over durations of width Δ, which we assume to be 1 for this problem. In other words:

$$u(t) = u(i) \text{ if } t \in [i, i+1) \tag{2}$$

Given that we start at $x(i)$, where do we end up at $x(i+1)$?

(b) Suppose that $x(0) = x_0$. **Unroll the implicit recursion you derived in the previous part to write $x(i+1)$ as a sum that involves x_0 and the $u(j)$ for $j = 0, 1, \ldots, i$.**

For this part, feel free to just consider the discrete-time system in a simpler form

$$x(i+1) = ax(i) + bu(i) \tag{3}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.

Your derivation here is actually an example of a simple proof by induction.

(c) Suppose we have a system of differential equations with an input that we express in vector form:

$$\frac{d}{dt}\vec{x}_c(t) = A\vec{x}_c(t) + \vec{b}u(t) \tag{4}$$

where $\vec{x}_c(t)$ is n-dimensional.

Suppose further that the matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Collect the eigenvectors together into a matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$.

If we apply a piecewise constant control input $u(t)$ as in (2), and sample the system $\vec{x}(i) = \vec{x}_c(i)$, **what are the corresponding A_d and $\vec{b}_d$ in:**

$$\vec{x}(i+1) = A_d\vec{x}(i) + \vec{b}_d u(i). \tag{5}$$

(d) Suppose that $\vec{x}(0) = \vec{x}_0$. **Unroll the implicit recursion you derived in the previous part to write $\vec{x}(i+1)$ as a sum that involves $\vec{x}_0$ and the $u(j)$ for $j = 0, 1, \ldots, i$.**

For this part, feel free to just consider the discrete-time system in a simpler form

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) \tag{6}$$

and you don't need to worry about what A and $\vec{b}$ actually are in terms of the original parameters.

2. Controlling states by designing sequences of inputs

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider a the following peculiar matrices chosen for intuitive ease of understanding what is going on:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's assume we have a *discrete-time* system that follows the following "difference equation."

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{b}u(t).$$

(a) We are given the initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}(m) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $m \geq 0$. We don't need to stay there, we just want to be in this state at that time. **What is the smallest m such that this is possible? What is our choice of sequence of inputs $u(i)$?**

(b) **What if we started from $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest m and what is our choice of $u(i)$?**

(c) **What if we started from $\vec{x}(0) = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$? What is the smallest m and what is our choice of $u(i)$?**

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