EECS 16B Fall 2019 — Designing Information Devices and Systems II, UC Berkeley — HW 14

1. Basic Orthonormality Proofs

In this problem, we ask you to establish several important properties of orthonormal bases in the complex case. This is designed to both sharpen your understanding of these properties as well as practice doing proofs/derivations.

Let $U = \begin{bmatrix} \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{n-1} \end{bmatrix}$ be an n by n matrix, where its columns $\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{n-1}$ form an orthonormal basis. One example of this is the orthonormal DFT basis.

(a) **Show that $U^{-1} = U^*$, where U^* is the conjugate transpose of U.**

(b) **Show that U preserves inner products, i.e. if $\vec{v}, \vec{w}$ are vectors of length n, then**

$$\langle \vec{v}, \vec{w} \rangle = \langle U\vec{v}, U\vec{w} \rangle .$$

Recall that the inner-product is defined to be $\langle \vec{v}, \vec{w} \rangle = \vec{w}^* \vec{v}$.

Also remember that for any matrices A, B or appropriate size so that their multiplication makes sense, that $(AB)^* = B^* A^*$. (This latter fact can be seen by looking at the entry in the ith row and jth column of $(AB)^*$. This is complex conjugate of the entry in the jth row and ith column of AB. This entry is just $A_j \vec{b}_i$ where A_j is the jth row of A and $\vec{b}_i$ is the ith column of B. Notice that the complex conjugate of this is just $\vec{b}_i^* A_j^*$ since the complex conjugate of a product is just the product of complex conjugates (most easily seen in polar form) and the complex conjugate of a sum is the sum of complex conjugates. This is thus the ith row and jth column of the product $B^* A^*$.)

This fact is called Parseval's relation when applied in signal processing contexts and it helps us see orthogonality as well as think about energy in different bases.

(c) **Show that $\vec{u}_0, \ldots, \vec{u}_{n-1}$ must be linearly independent.**

(Hint: Suppose $\vec{w} = \sum_{i=0}^{n-1} \alpha_i \vec{u}_i$, then first show that $\alpha_i = \langle \vec{w}, \vec{u}_i \rangle$. From here ask yourself whether a nonzero linear combination of the $\{\vec{u}_i\}$ could ever be identically zero.)

This basic fact shows how orthogonality is a very nice special case of linear independence.

(d) Let M be a matrix which can be diagonalized by U, i.e. $M = U \Lambda U^*$, where Λ is a diagonal matrix with the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ along the diagonal. **Show that M^* has the same set of eigenvectors U, while the eigenvalues of M^* are $\overline{\lambda_0}, \ldots, \overline{\lambda_{n-1}}$.**

(e) Let V be another n by n matrix, where the columns also form an orthonormal basis. **Show that the columns of the product, UV, also form an orthonormal basis.** (This turns out to be very helpful when we are defining a two-dimensional DFT or thinking about image or video processing generally.)

2. Adapting Proofs to the Complex Case

At many points in the course, we have made assumptions that various matrices or eigenvalues are real while discussing various theorems. If you have noticed, this has always happened in contexts where we have invoked orthonormality during the proof or statement of the result. Now that you understand the idea of orthonormality for complex vectors, and how to adapt Gram-Schmitt to complex vectors, you can go back and remove those restrictions. This problem asks you to do exactly that.

Unlike many of the problems that we have given you in 16A and 16B, this problem has far less hand-holding — there aren't multiple parts breaking down each step for you. Fortunately, you have the existing proofs in your notes to work based on. So this problem has a secondary function to help you solidify your understanding of these earlier concepts ahead of the final exam.

There is one concept that you will need beyond the idea of what orthogonality means for complex vectors as well as the idea of conjugate-transposes of vectors and matrices. The analogy of a real symmetric matrix S that satisfies $S = S^\top$ is what is called a Hermitian matrix H that satisfies $H = H^*$ where $H^* = \overline{H^\top}$ is the conjugate-transpose of H.

(a) The upper-triangularization theorem for all (potentially complex) square matrices A says that there exists an orthonormal (possibly complex) basis V so that V^*AV is upper-triangular.

Adapt the proof from the real case with assumed real eigenvalues to prove this theorem.

Feel free to assume that any square matrix has an (potentially complex) eigenvalue/eigenvector pair. You don't need to prove this. But you can make no other assumptions.

(HINT: Use the exact same argument as before, just use conjugate-transposes instead of transposes.)

Congratulations, once you have completed this part you essentially can solve all systems of linear differential equations based on what you know, and you can also complete the proof that having all the eigenvalues being stable implies BIBO stability.

(b) The spectral theorem for Hermitian matrices says that a Hermitian matrix has all real eigenvalues and an orthonormal set of eigenvectors.

Adapt the proof from the real symmetric case to prove this theorem.

(HINT: As before, you should just leverage upper-triangularization and use the fact that $(ABC)^* = C^*B^*A^*$. There is a reason that this part comes after the first part.)

(c) The SVD for complex matrices says that any rectangular (potentially complex) matrix A can be written as $A = \sum_i \vec{u}_i \sigma_i \vec{w}^*$ where σ_i are real positive numbers and the collection $\{\vec{u}_i\}$ are orthonormal (but potentially complex) as well as $\{\vec{w}_i\}$.

Adapt the derivation of the SVD from the real case to prove this theorem.

Feel free to assume that A is wide. (Since you can just conjugate-transpose everything to get a tall matrix to become wide.)

(HINT: Analogously to before, you're going to have to show that the matrix A^*A is Hermitian and that it has non-negative eigenvalues. Use the previous part. There is a reason that this part comes after the previous parts.)

3. Lagrange Interpolation by Polynomials

Given n distinct points and the corresponding sampling of a function $f(x)$, $(x_i, f(x_i))$ for $0 \le i \le n-1$, the Lagrange polynomial interpolation is the polynomial of the least degree that passes through all of the given points.

Given n distinct points and the corresponding evaluations, $(x_i, f(x_i))$ for $0 \le i \le n-1$, the Lagrange polynomial interpolation is the $n-1^{th}$ degree polynomial

$$P(x) = \sum_{i=0}^{i=n-1} f(x_i) L_i(x),$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{j=n-1} \frac{(x-x_j)}{(x_i-x_j)} = \frac{(x-x_0)}{(x_i-x_0)} \cdots \frac{(x-x_{i-1})}{(x_i-x_{i-1})} \frac{(x-x_{i+1})}{(x_i-x_{i+1})} \cdots \frac{(x-x_{n-1})}{(x_i-x_{n-1})}. \tag{1}$$

Here is an example: for two data points, $(x_0, f(x_0))$ = (0, 4), $(x_1, f(x_1))$ = (-1, -3), we have

$$L_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-(-1)}{0-(-1)} = x+1$$

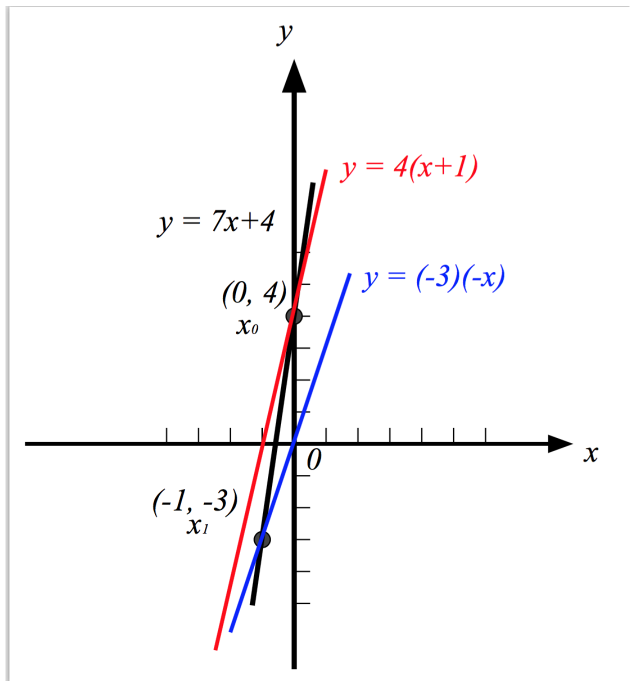
and

$$L_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-(0)}{(-1)-(0)} = -x.$$

Then

$$P(x) = f(x_0)L_0(x) + f(x_1)L_1(x) = 4(x+1) + (-3)(-x) = 7x+4.$$

We can sketch those equations on the 2D plane as follows:



In the figure above, the red line is the 0^{th} interpolating polynomial L_0 weighted by the 0^{th} function values $f(x_0)$, $y = f(x_0)L_0 = 4(x+1)$. The blue line is the 1^{st} interpolating polynomial L_1 weighted by the 1^{st} function values $f(x_1)$, $y = f(x_1)L_1 = (-3)(-x) = 3x$. The black line is the interpolated signal, $P(x) = 7x+4$.

(a) Before we find the Lagrange interpolation, let us first use interpolation by global polynomials so we can verify our Lagrange interpolation results. Using the polynomial function basis $\{1, x, x^2, \cdots x^{n-1}\}$, the interpolation problem can be cast into finding the coefficients $a_0, a_1, a_2, \cdots, a_{n-1}$ of the function

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

such that $g(x_i) = f(x_i)$ for n samples of a function $(x_i, f(x_i))$ with $i \in \{0, 1, 2, ..., n-1\}$.

Given three data points, $(2,3)$, $(0,-1)$, **and** $(-1,-6)$, **find a polynomial** $g(x) = a_2x^2 + a_1x + a_0$ **fitting the three points using global polynomial interpolation. Is this polynomial unique?** That is, is it the only second degree polynomial that fits this data?

It is computationally expensive to do this process for large numbers of points, which is why we use the Lagrange interpolation method.

(b) The set of Lagrange polynomials $\{L_i(x)\}$, $i \in \{0,1,2,...,n-1\}$ is a new function basis for the subspace of degree $n-1$ or lower polynomials. **Find the** $L_i(x)$ **given by Eq. 1 corresponding to the three sample points in (a).** Show your work.

(c) $P(x)$ is the sum of the Lagrange polynomials weighted by the function value at the corresponding points, giving the Lagrange interpolation of the given points. **Find the Lagrange polynomial interpolation** $P(x)$ **that goes through the three points in (a). Compare the result to the global polynomial interpolation of the same points, which you calculated in (a). Are they different from each other? Why or why not?**

(d) **Plot** $P(x)$ **and each** $f(x_i)L_i(x)$**.** You can use a plotting utility (e.g. matplotlib) and or plot by hand.

(e) **Show that** $P(x_i) = f(x_i)$ **for all** x_i**.** That is, show that the Lagrange interpolation passes through all given data points. Show this symbolically in the general case, not just for the example above.

4. Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In particular, this basis can be viewed as being the standard monomials x^j for $j = 0, \ldots N-1$ evaluated at the N-th roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number ω satisfying the equation $\omega^N = 1$ (or equivalently $\omega^N - 1 = 0$).

(a) **Show that the polynomial** $z^N - 1$ **factors as**

$$z^N - 1 = (z-1)\left(\sum_{k=0}^{N-1} z^k\right).$$

(b) **Show that any complex number of the form** $\omega_N^k = e^{j\frac{2\pi}{N}k}$ **for** $k \in \mathbb{Z}$ **is an** N**-th root of unity.** From here on, let $\omega_N = e^{j\frac{2\pi}{N}}$.

(c) For a given integer $N \geq 2$, using the previous questions, **give the complex roots of the polynomial** $1 + z + z^2 + ... + z^{N-1}$**.**

(d) **Draw the fifth roots of unity in the complex plane. How many unique fifth roots of unity are there?**

(e) **For** $N = 5$, $\omega_5 = e^{j\frac{2\pi}{5}}$**. What is another expression for** ω_5^{42}**?**

(f) **What is the complex conjugate of** ω_5**? What is the complex conjugate of** ω_5^{42}**? What is the complex conjugate of** ω_5^4**?**

(g) **Compute** $\sum_{m=0}^{N-1} W_N^{km}$ **where** W_N **is an** N**th root of unity.** Does the answer make sense in terms of the plot you drew?

(h) **Write the expression for the N basis vectors** $\vec{u}_0, \ldots, \vec{u}_{N-1} \in \mathbb{R}^N$ **for the DFT of a signal of length N in terms of** ω_N^k**.**

(i) **Show that the DFT basis vectors** $\vec{u}_i, \vec{u}_j$ **are orthogonal if** $i \neq j$**.**

5. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

6. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **Who did you work on this homework with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

(d) **Roughly how many total hours did you work on this homework?**

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