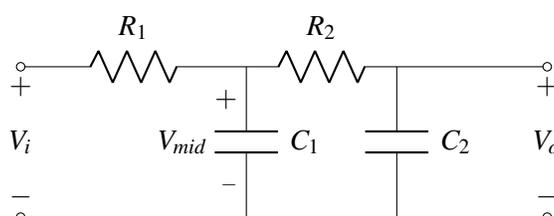


This homework is due on Monday, October 7, 2019, at 11:59PM.

Self-grades and HW Resubmission are due on Wednesday, October 9, 2019, at 11:59PM.

1. Transfer functions and why loading is annoying

Consider the circuit below.



The circuit has an input phasor voltage \tilde{V}_i at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage \tilde{V}_o at output terminals.

- (a) We are going to construct the transfer function $H(\omega) = \frac{\tilde{V}_o}{\tilde{V}_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{\tilde{V}_{mid}}{\tilde{V}_i}$ and $H_2(\omega) = \frac{\tilde{V}_o}{\tilde{V}_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{\tilde{V}_o}{\tilde{V}_i} = H_1(\omega)H_2(\omega)$.

This approach is valid since the \tilde{V}_{mid} cancel.

For the first step, **find the intermediate transfer function** $H_2(\omega) = \frac{\tilde{V}_o}{\tilde{V}_{mid}}$. Have your expression be in terms of Z_{R_2} and Z_{C_2} , that is the impedances of R_2 and C_2 .

- (b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{\tilde{V}_{mid}}{\tilde{V}_i}$. Have your expression be in terms of Z_{R_1} , Z_{R_2} , Z_{C_1} , and Z_{C_2} . (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.) *hint: Applying KCL at the V_{mid} node would be a good place to start. You should try to find an expression for H_1 that has factors that H_2 can cancel out.*

Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega) = \frac{\tilde{V}_o}{\tilde{V}_i} = H_1(\omega)H_2(\omega)$.

- (c) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for** $H(\omega) = \tilde{V}_o/\tilde{V}_i$ **in the form**

$$H(\omega) = \frac{V_o}{V_i} = \frac{1}{1 + 2\xi \frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\Omega$, $R_2 = 4\Omega$, $C_1 = \frac{9}{2}$ F, and $C_2 = 1$ F. What are the values of ξ and ω_c ?

- (d) **For the previous case, what is the magnitude of the transfer function at the $\omega = \omega_c$ you calculated?**

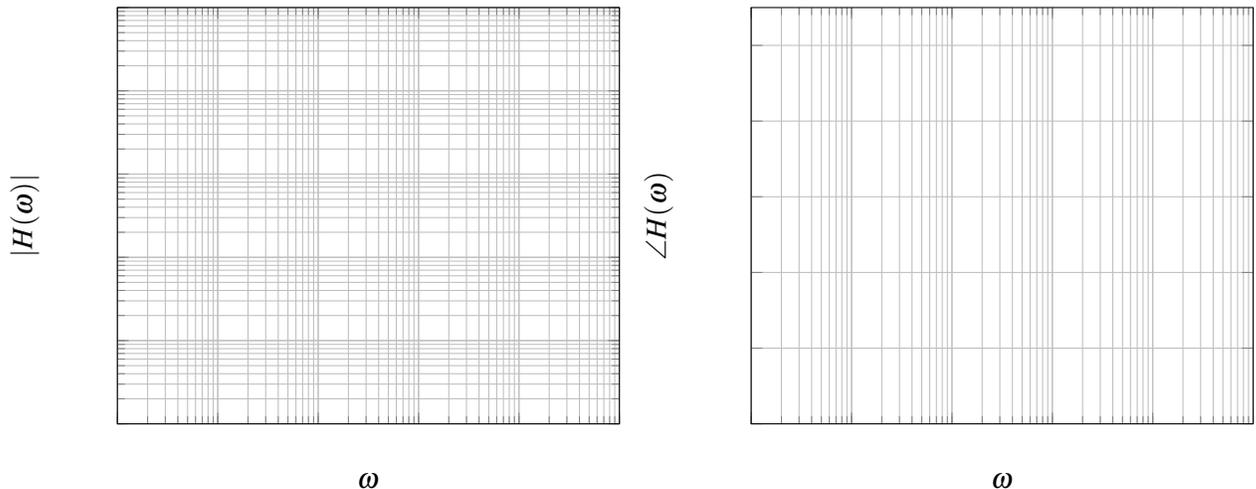
This is here so that you can see that just because we called it ω_c doesn't mean that the amplitude here is $\frac{1}{\sqrt{2}}$.

- (e) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

- (f) Use a computer and then **draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$** . A blank plot is provided here for you to use.



2. Bandpass Filter: Lowpass and Highpass Cascade

In lecture, you heard about how you can go through the design of a bandpass filter by cascading lowpass and highpass filters via buffers (Op-Amps in unity-gain negative feedback to prevent loading effects). In this problem, you will do this for yourself.

Consider an input signal that is composed of the superposition of:

- 20mV level pure tone at 60Hz corresponding to power line noise.
- 1mV level pure tone at 600Hz corresponding to a voice signal.
- 10mV level pure tone at 60kHz corresponding to fluorescent light noise.

This is the signal that you want to filter.

- (a) We would like to keep the 600Hz tone, which could correspond to a voice signal, for example. Ignoring any phase offset for each signal (i.e. set the phases to zero), **write the $V_{in}(t)$ that describes the above input in time domain.**
- (b) **What are the radian/sec frequencies ω s involved and the phasors associated with each tone?**
- (c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the knee or cutoff-frequency for the lowpass filters?**

- (d) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the knee or cutoff-frequency for the highpass filters?**
- (e) Suppose that you only had $1\mu\text{F}$ capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**
- (f) Suppose you decide to use three lowpass filters in a row followed by three highpass filters — all cascaded through unity-gain buffers (op-amps in voltage-follower negative feedback) to prevent loading effects.

What are the phasors for each of the frequency tones after all three lowpass filters?

Feel free to use a computer to help you evaluate both the magnitudes and the phases here.

- (g) **What are the phasors for each of the frequency tones after the whole filter?**

Feel free to use a computer to help you evaluate both the magnitudes and the phases here.

- (h) **Write the final time domain voltage waveform that would be present after the filter.**

- (i) The included jupyter notebook sets up the same problem described above. In the notebook, you can use the slider bars to play around with:

- highpass cutoff frequency (i.e. the knee frequency of the highpass filters)
- lowpass cutoff frequency (i.e. the knee frequency of the lowpass filters)
- Filter order N . Filter order means the number of lowpass filters and highpass filters that are used in a row. Here, N means that there are N lowpass filters and N highpass filters, so the overall order of the entire filter is actually $2N$.

The notebook will plot the Bode Plots for the magnitude and phase, the input voltage waveform, and the output waveform at the end of the filter.

Play around with the values for the highpass and lowpass cutoff frequencies, and N .

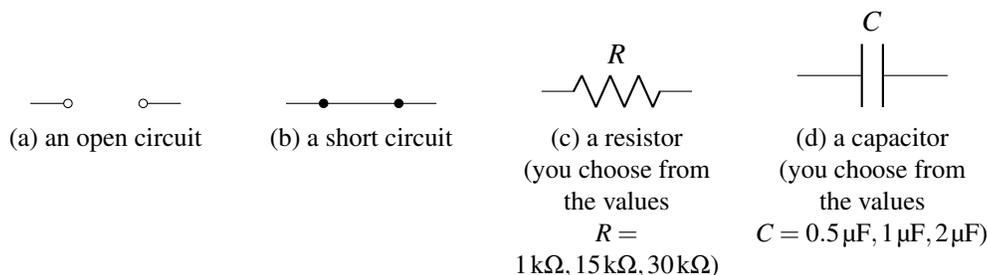
Observe the waveforms at the output of the filter. **Comment on the limits of $f_{lowpass}$, $f_{highpass}$, and N that you can use to successfully isolate the desired 600Hz tone. What happens if you keep $f_{lowpass}$ and $f_{highpass}$ constant, and just increase N ?**

3. Circuit Design

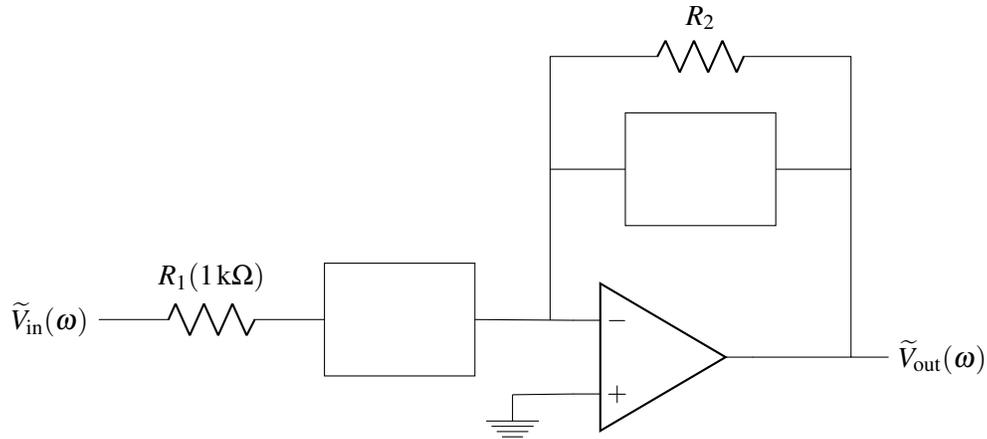
In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. *A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.*

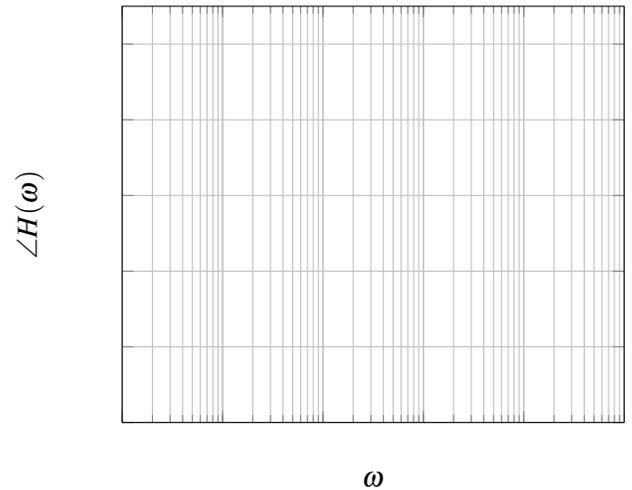
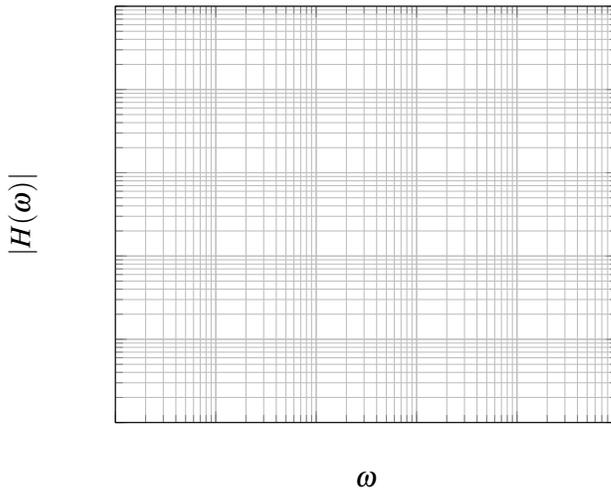
You have at your disposal *only one of each* of the following components (not including R_1 and R_2):



Consider the circuit below. The voltage source $V_{in}(t)$ has the form $V_{in}(t) = v_0 \cos(\omega t + \phi)$. The labeled voltages $\tilde{V}_{in}(\omega)$ and $\tilde{V}_{out}(\omega)$ are the phasor representations of $v_{in}(t)$ and $v_{out}(t)$. The transfer function $H(\omega)$ is defined as $H(\omega) = \frac{\tilde{V}_{out}(\omega)}{\tilde{V}_{in}(\omega)}$.



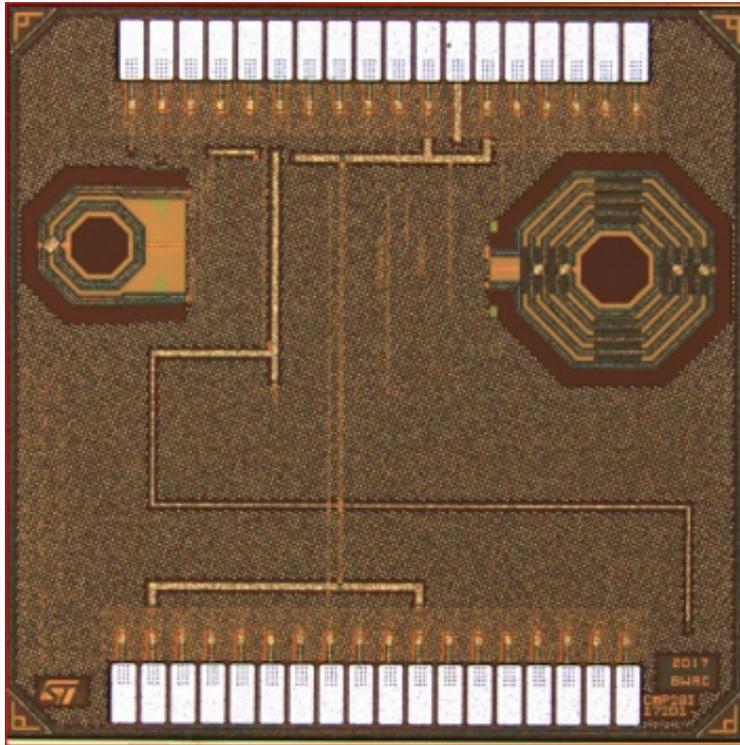
- (a) Let R_1 be $1\text{ k}\Omega$. **Fill in the boxes and determine the value of R_2 , such that the circuit's transfer function $H(j\omega)$ has the following properties:**
- It is a high-pass filter.
 - $|H(\infty)| = 10$.
 - $|H(j10^3)| = \sqrt{50}$.
 - R_2 must be one of the three values listed above.
- (b) Use a computer to help you and **draw the Bode plot of this transfer function.** A blank plot is provided here for you to use.



4. LC Tank

The chip below was designed by students in EE194 in Spring 2017. It is 1.1 mm on a side, and contains a 32 bit RISC-V microprocessor, 64kB RAM, and a 2.4 GHz Bluetooth Low Energy radio for communicating with cell phones.

In this problem, we will use a model for on-chip inductors to help design the oscillator for the chip.



An ideal inductor follows the relationship

$$V = L \frac{d}{dt} I$$

from which we obtain the phasor impedance

$$Z_L = j\omega L$$

Real-world inductors are made by forming loops out of metal wires. In the chip above, there are two inductors that look like octagonal loops of metal. The inductor on the left has a single loop, while the inductor on the right has multiple loops of wire.

In real world inductors, metal that makes up the inductor loops has some resistance, which appears in series with the inductance. Because of this series wire resistance, the phasor impedance of a real world inductor can be modeled by

$$Z = R + j\omega L$$

In other words, there is always a resistance R that is present.

There are two inductors on the chip above.

The inductor on the right is part of an impedance-matching circuit for the antenna interface. We will refer to it as the *antenna interface inductor*. Its inductance is 9nH, and its parasitic resistance is $R_{L_2} = 17.0\Omega$.

The inductor on the left is part of the “local oscillator”, a circuit at the core of the radio’s transmit- and receive-circuits. We will refer to it as the *local oscillator inductor*. It has an inductance of 1nH, and its parasitic resistance is $R_{L_1} = 2.01\Omega$.

There is a capacitor in parallel with each of the two inductors, which is what we need to make things resonate.

These circuit diagrams are meant to be representative of the two resonators described, which are a small part of the overall circuit seen in the chip above.

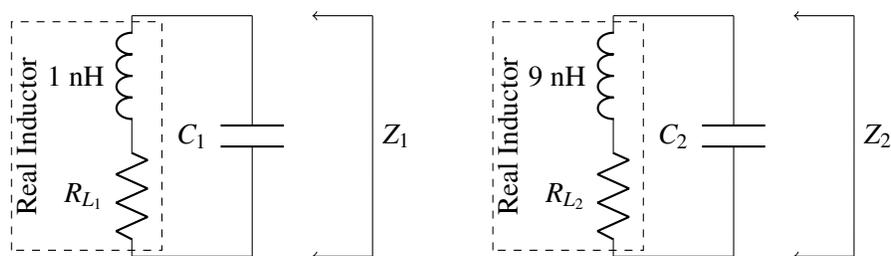
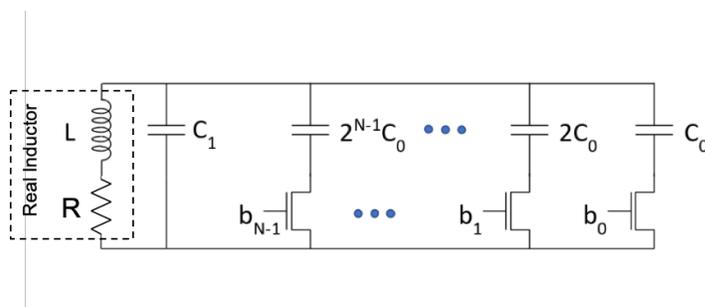


Figure 2: Circuit models of the two inductors on chip. Left: local-oscillator inductor. Right: antenna interface inductor.

- (a) Recall that resonance occurs at the frequency where the imaginary part of the impedance is 0. **For the antenna interface inductor on the right, what is the capacitance (to one significant figure) needed to make it resonant at 2.4 GHz?** In this case, the impedance is measured by a circuit across the capacitor, indicated by Z_2 .
- (b) **For the local oscillator inductor on the left, what capacitance (to four significant figures) is needed to make it resonant at 4.8 GHz? How about at 5 GHz?** Recall that resonance occurs at the frequency where the imaginary part of the impedance (Z_1) is 0.
- (c) Suppose that we chose the capacitance in the previous part so that the circuit resonates at 5GHz. Recall that at resonance, there is no imaginary part to the impedance. **How much can you change the frequency by (in units of MHz) until the imaginary part of the impedance is of the same magnitude as the real part?**
- (d) In the previous parts, you should have seen that changing the capacitance by a little bit allows you to change the frequency of the resonance.

Therefore, to be able to tune the resonance, we need to have a tunable capacitance. While physically, we saw in 16A that we can adjust capacitance by changing the area of charged plates or by adjusting the separation between them, this sort of mechanical reconfiguration is not usually practical on a small chip. So how can we adjust the capacitance?

This is implemented using an array of capacitors that can be digitally connected or disconnected to tune the overall capacitance. As shown below, the array consists of capacitors $C_0, 2C_0, 4C_0, \dots, (2^{N-1})C_0$, where N is the number of bits that we dedicate to the variable capacitor. The microprocessor on the chip will control each bit: a digital binary signal will turn the transistor “on” or “off”. When the transistor is off, the capacitor contributes 0 capacitance to the overall capacitance of the array. When the transistor is on, the capacitor’s capacitance contributes to the overall capacitance of the array. Assume that the transistors operate as ideal switches, with no parasitic resistance or capacitance of their own. The capacitor array is placed in parallel with a fixed capacitor C_1 .



Our goal is to have the microprocessor set the frequency to any value between 4.8 GHz and 5 GHz with frequency error of no more than 100 kHz. **What value of C_1 should be used?**

(HINT: What is the smallest value of capacitance that you will need?)

- (e) **What is the minimum number of bits N that is necessary to achieve the tunability goal?**

(HINT: How many bins do you need between your two limiting frequencies so that you are always within 100 kHz of your target?)

- (f) **What is the best value for C_0 ?**

Don't worry if this seems too small.

(HINT: Remember that once you pick C_0 , powers of two will set the capacitance for the other capacitors in the array.)

5. Uniqueness justification for phasor-style solutions

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof that tells us why we can trust the solution that we have found as being the one true solution to the problem at hand.

In the previous homeworks, you saw that phasor-style techniques give rise to convenient solutions to the matrix-vector differential equations of the form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \tilde{U}e^{st} \quad (1)$$

with some initial condition $\vec{x}(0) = \vec{x}_0$.

However, all the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

- (a) Suppose that the $n \times n$ matrix A has n distinct eigenvalues and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, so that the matrix $V = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$ has linearly independent columns.

Show that if you are given any valid solution for the original differential equation, you can change coordinates and get a valid solution for a collection of scalar differential equations with specified initial conditions.

(Hint: First mathematize the statement you are asked to show. Give names and symbols to things. It will be easier to proceed once you have done so.)

- (b) You have already proved the uniqueness of solutions for any scalar differential equation of the form $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$. How can you use this fact to now argue that the solution must be unique for the matrix/vector differential equation making the same assumptions as the previous part?

Give a proof by contradiction style argument.

(Hint: start by assuming that you have two distinct solutions. Use the linear independence of V and the fact that if two vectors are different, they must be different in some coordinate.)

We will see later in the course how the assumption we made on the eigenstructure of A is not actually needed for this proof to hold. But for now, it is important to understand this special case first.

6. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous ‘‘Bloom’s Taxonomy’’ that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create.

Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

7. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- (a) **What sources (if any) did you use as you worked through the homework?**
- (b) **Who did you work on this homework with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
- (c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
- (d) **Roughly how many total hours did you work on this homework?**

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