1 Circulant Matrices

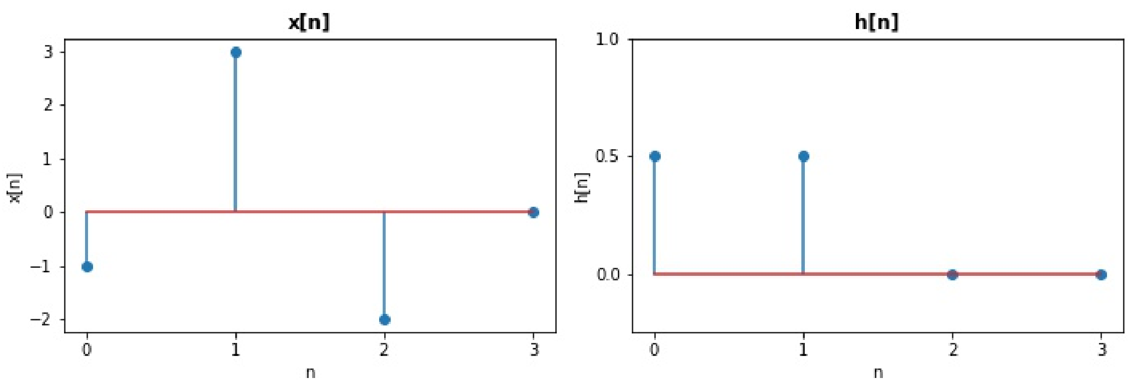
A square matrix C_h is circulant if each row vector is rotated one element to the right relative to the preceding row vector.

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & \vdots & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \tag{1}$$

2 Circulant Matrices & Convolution

Consider the signal $x[n]$ of length 3 and an impulse response $h[n]$ of length 2. You may assume that they are zero everywhere else.

$$\vec{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T \qquad \vec{h} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \tag{2}$$



a) What is the convolution $y[n] = x[n] * h[n]$? Also what is the length of this output signal?

b) Now write each term of the output signal $y[n]$ as a sum using the convolution formula and set up a matrix equation $\vec{y} = A\vec{x}$. What is the size of this matrix?

c) Add elements to the matrix A and zeros to the vector $\vec{x}$ to create a square matrix C_h that is circulant.

d) What is the importance behind this result? Compare the runtimes between convolution and the Fast Fourier Transform (FFT) which takes $O(N \log N)$ operations.

3 Complex Inner Product

For the complex vector space $\mathbb{C}^n$, we can no longer use our conventional real dot product as a valid inner product for $\mathbb{C}^n$. This is because the real dot product is no longer positive-definite for complex vectors. For example, let $\vec{v} = \begin{bmatrix} j \\ j \end{bmatrix}$. Then, $\vec{v} \cdot \vec{v} = j^2 + j^2 = -2 < 0$.

Therefore, for two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, we define the complex inner product to be:

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} \overline{u}_i v_i = \overline{\vec{u}^T}\vec{v} = \vec{u}^*\vec{v}$$

where $\overline{\cdot}$ denotes the complex conjugate and $\vec{\cdot}^*$ denotes the complex conjugate transpose.

The complex inner product satisfies the following properties:

- Conjugate Symmetry: $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$
- Scaling: $\langle \vec{u}, c\vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$ and $\langle c\vec{u}, \vec{v} \rangle = \overline{c}\langle \vec{u}, \vec{v} \rangle$
- Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

Recall that the complex conjugate of a complex number $z = a + jb = re^{j\theta}$ is equal to $\overline{z} = a - jb = re^{-j\theta}$. The conjugate transpose of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is $\vec{v}^* = \begin{bmatrix} \overline{v}_1 & \cdots & \overline{v}_n \end{bmatrix}$.

Adjoint of a Matrix

The **adjoint** or **conjugate-transpose** of a matrix A is the matrix A^* such that $A^*_{ij} = \overline{A_{ji}}$. From the complex inner product, one can show that

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle \tag{3}$$

A matrix is **self-adjoint** or **Hermitian** if $A = A^*$. Self-adjoint matrices are the complex extension of real-symmetric matrices. There is an equivalent version of the **Spectral Theorem** for such self-adjoint matrices.

Unitary Matrices

A **unitary** matrix is a square matrix whose columns are orthonormal with respect to the complex inner product.

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}, \qquad \vec{u}_i^*\vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that $U^*U = UU^* = I$, so the inverse of a unitary matrix is its conjugate transpose $U^{-1} = U^*$.

Since the columns of a unitary matrix are orthonormal vectors, we can interpret these matrices as "rotation" and "reflection" matrices of the standard axes. This also implies that $\|U\vec{v}\| = \|\vec{v}\|$ for any vector $\vec{v}$.

4 Everything is Complex

In this question, we explore the similarities and differences between real and complex vector spaces.

a) Suppose that we are given a matrix U that has orthornomal columns.

$$U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} \tag{4}$$

We would like to change coordinates to express $\vec{x}$ as a linear combination of these orthonormal basis vectors. Find scalars α_i in terms of $\vec{x}$ and $\vec{u}_i$ so that

$$\vec{x} = \alpha_1 \vec{u}_1 + \ldots + \alpha_n \vec{u}_n \tag{5}$$

b) Show that the eigenvalues λ of a unitary matrix U must always have magnitude 1.

c) Let A be an $m \times n$ matrix with complex entries. Show that A^*A must have non-negative eigenvalues. *Hint: Think back to what we did for the real case. Consider the quantity* $\|A\vec{v}\|^2$ *where* $\vec{v}$ *is an eigenvector of* A^*A.