1   Transfer Functions

When analyzing circuits in the phasor domain, we have always told you what the input voltage to the circuit was. However, sometimes we have many input sinusoids and we would like to see how the circuit generically responds to a sinusoid input of frequency ω. We want to see how an input sinusoid "transfers" into an output sinusoid. How do we do this?

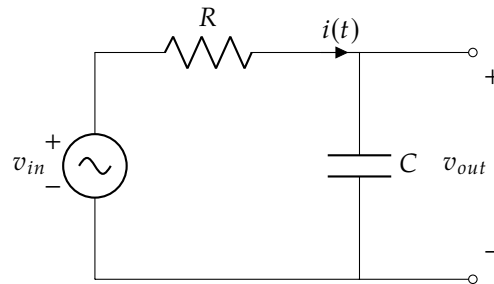Let's start with a simple RC circuit.

Figure 1: First Order RC Low Pass Filter

In the phasor domain, the impedance of the capacitor is $Z_C = \frac{1}{j\omega C}$ and the impedance of the resistor is $Z_R = R$. Because we treat impedances the same as resistances, this circuit looks like a voltage divider in the phasor domain. Remember we must also represent v_{in} as a phasor V_{in}; transfer functions are in the phasor domain only, not the time domain.

$$V_{out} = \frac{Z_C}{Z_R + Z_C} V_{in} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} V_{in} = \frac{1}{j\omega RC + 1} V_{in}$$

We define the frequency response as

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

Now, given an arbitrary input sinusoid, if we multiply it by the frequency response, we can get the output sinusoid. What this allows us to do is to model any arbitrary circuit as a single-input-single-output black box. The transfer function completely defines how our circuit works. We'll take a look at a couple of examples in the discussion to understand what this means.

2 Circuit Filters

Let's start with a simple RC circuit with a sinusoidal input $v_{in}(t)$.

For this question, we will define the transfer function $H(\omega)$ as the ratio of the input and output phasors

$$H(\omega) = \frac{V_{out}}{V_{in}} \tag{1}$$

The resistor $R = 1\,\text{k}\Omega$ and the capacitor $C = 1\,\mu\text{F}$.

a) Compute the transfer function $H(\omega)$.

b) What are the magnitude and phase of the transfer function $H(\omega)$?

c) What happens to the magnitude and phase of $\omega \to 0$? How about when $\omega \to \infty$? Lastly, explain why this circuit is often called a *high-pass filter*.

d) Plot the magnitude and phase of $H(\omega)$ using a computer.

e) Now let's take a look at a couple of outputs given some input sinusoids. Let $R = 1\,\mathrm{k\Omega}$ and $C = 1\,\mathrm{\mu F}$. Compute the steady state output voltage for the following inputs

   i) $v_{in}(t) = 4\cos\left(1000t + \frac{\pi}{3}\right)$ [V]

   ii) $v_{in}(t) = 2\sin\left(10^6 t - \frac{\pi}{4}\right)$ [V]

   iii) $v_{in}(t) = 5\,\mathrm{V}$

3   Rational Transfer Function Form

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = \frac{z(\omega)}{p(\omega)} = \frac{(j\omega)^{N_{z0}}}{(j\omega)^{N_{p0}}} \left(\frac{(j\omega)^n \alpha_n + (j\omega)^{n-1} \alpha_{n-1} + \cdots + j\omega\alpha_1 + \alpha_0}{(j\omega)^m \beta_m + (j\omega)^{m-1}\beta_{m-1} + \cdots + j\omega\beta_1 + \beta_0} \right)$$

$$= K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles", and N_{z0}, N_{p0} are the number of zeros and poles at $\omega = 0$

4   Poles and Zeros

Identify all of the zeros and poles for the following transfer functions

a) $H_1(\omega) = \dfrac{1}{1 + j\omega/10}$

b) $H_2(\omega) = \dfrac{1 + j\omega/50}{(1 + j\omega/100)(1 + j\omega/5)}$

c) $H_3(\omega) = \dfrac{j\omega \cdot 100}{1 + j\omega \cdot 1000}$

d) $H_4(\omega) = \dfrac{20 + j\omega \cdot 40}{(1 + j\omega \cdot 10)^2}$

e) $H_5(\omega) = \dfrac{10 + j\omega}{1000 + 110j\omega + (j\omega)^2}$