1 System Identification and Linear Control

A scalar discrete-time system has the following dynamics:

$$x(t+1) = \lambda x(t) + g(u(t)),$$

where $g : \mathbb{R} \to \mathbb{R}$ not necessarily linear.

a) If g is approximated to order 2 around the operating point $u^* = 0$, so that

$$x(t+1) \approx \lambda x(t) + \beta_0 + \beta_1 u(t) + \beta_2 u^2(t),$$

what should β_0, β_1, and β_2 be, when $u(t)$ is small?

b) Suppose that $x(0) = 0$. We apply a sequence of inputs

$$\big(u(0), u(1), \ldots, u(N-1)\big) \tag{1}$$

and observe states $x(1), x(2), \ldots, x(N)$. Derive the least-squares estimates of λ, β_0, β_1, and β_2.

2 System Identification

Let's now look at how System Identification works in the vector case. Again you are given an unknown discrete-time system. We don't know its specifics but we know that it takes one scalar input and as two observable states.
We would like to find a linear model of the form

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) + \vec{w}(t),$$

where $\vec{w}(t)$ is an error term due to unseen distrubances and noise, $u(t)$ is a scalar input, and

$$A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \quad \vec{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}.$$

To identify the system parameters from measured data, we need to find the unknowns: a_0, a_1, a_2, a_3, b_0 and b_1, however, you can only interact with the system via a blackbox model. The model allows you to view the states $\vec{x}(t) = \begin{bmatrix} x_0(t) & x_1(t) \end{bmatrix}^\top$ and it takes a scalar input $u(t)$ that drives the system to the next state $\vec{x}(t+1) = \begin{bmatrix} x_0(t+1) & x_1(t+1) \end{bmatrix}^\top$.

a) **Write scalar equations for the new states,** $x_0(t+1)$ **and** $x_1(t+1)$ **in terms of** a_i, b_i**, the states** $x_0(t)$**,** $x_1(t)$**, and the input** $u(t)$. Here, assume that $\vec{w}(t) = \vec{0}$ (i.e. the model is perfect).

b) Now we want to identify the system parameters. We observe the system at the initial state $\vec{x}(0) = \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix}$, input $u(0)$ and observe the next state $\vec{x}(1) = \begin{bmatrix} x_0(1) \\ x_1(1) \end{bmatrix}$. We can continue this for a sequence of inputs. Say we feed in a total of 4 inputs $u(0), u(1), u(2), u(3)$ into our blackbox. This allows us to observe $x_0(0), x_0(1), x_0(2), x_0(3), x_0(4)$ and $x_1(0), x_1(1), x_1(2), x_1(3), x_1(4)$, which we can use to identify the system.

To identify the system we need to set up an approximate (because of potential disturbances) matrix equation

$$D\vec{p} \approx \vec{y}$$

using the observed values above and the unknown parameters we want to find. Suppose you are given the form of $\vec{p}$ and $\vec{y}$ as follows:

$$\vec{y} = \begin{bmatrix} x_0(1) \\ x_0(2) \\ x_0(3) \\ x_0(4) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} a_0 \\ a_1 \\ b_0 \\ a_2 \\ a_3 \\ b_1 \end{bmatrix} \tag{2}$$

For the $\vec{p}$ and $\vec{y}$, **what is the D so that $D\vec{p} \approx \vec{y}$ makes sense?**

c) Now that we have set up $D\vec{p} \approx \vec{y}$, **explain how you would use this approximate equation to estimate the unknown values a_0, a_1, a_2, a_3, b_0 and b_1 assuming the columns of D are linearly independent.**

d) Suppose instead of 4 inputs, we have m inputs: $\vec{x}(0), \ldots, \vec{x}(m-1)$ and $u(0), \ldots, u(m-1)$. And observe $\vec{x}(m)$.

What is the minimum value of m you need to identify the system parameters?

e) **What could go wrong in the previous cases with 4 inputs? What kind of inputs would make least-squares fail to give you the parameters you want?**