This homework is due on Thursday, September 24, 2020, at 10:59PM. Self-grades are due on Thursday, October 1, 2020, at 10:59PM.

1 Eigenvalues, Eigenvectors and Diagonalization

a) Find the eigenvalues and eigenvectors for the following matrices. Are these matrices diagonalizable?

i.) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

ii.) $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

iii.) $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define P to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $P = \begin{bmatrix} \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \end{bmatrix}$.

i. Show that $AP = P\Lambda$, where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

ii. Argue that P is invertible, and therefore, $A = P\Lambda P^{-1}$.

c) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A * A * \cdots * A * A}_{k \text{ times}} \tag{1}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

i. Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$.

ii. Show that A^k has eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and conclude that A^k is diagonalizable.

2 Circuits puzzle

Prof. Sanders enjoys designing circuits. Having finished his circuits, he decided to take a break and tasked you with analyzing what these circuits were doing. Analyze the following circuits and answer the questions attached to them.

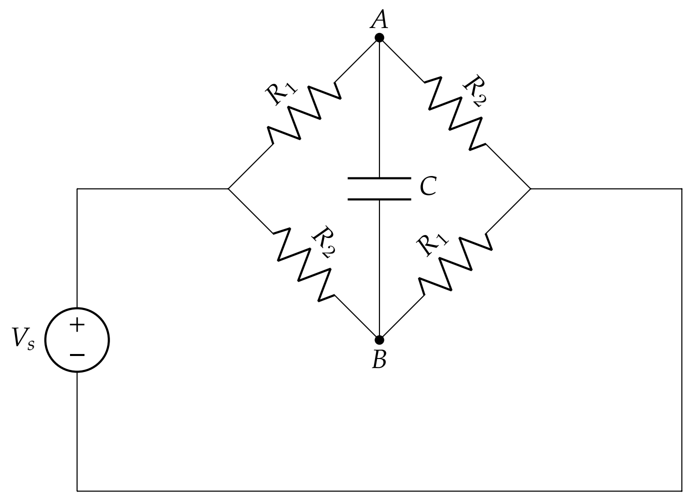

Figure 1: Modified Wheatstone bridge circuit.

a) In Figure 1, we have a modified Wheatstone bridge circuit with a capacitor in the bridge. Find the steady state voltage across the capacitor (V_{AB}). If the capacitor starts with no charge, how long will it take for it to reach 95% of its steady state value with $C = 100\text{nF}$, $R_1 = 10\Omega$, and $R_2 = 100\Omega$?

Hint: Use nodal analysis at nodes A and B.

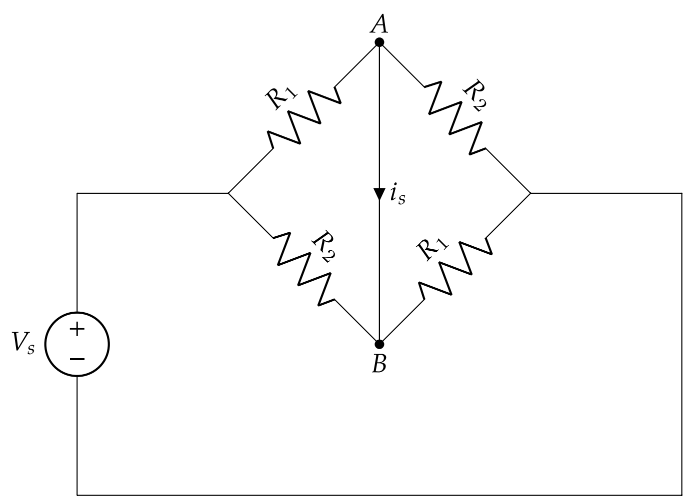

Figure 2: Modified Wheatstone bridge circuit.

b) Someone was tinkering with the Wheatstone bridge circuit and accidentally shorted the nodes A and B. In Figure 2, we have the short-circuited Wheatstone bridge. Find the steady state current through the short (i_s).

Hint: Use the current i_s in nodal analysis at nodes A and B.

c) In Figure 3, we have an operational amplifier in negative feedback. However, it seems that one of the resistors has been replaced with a capacitor. Find the relationship between the output voltage V_{out} and V_{in}. What mathematical operation does this circuit perform?

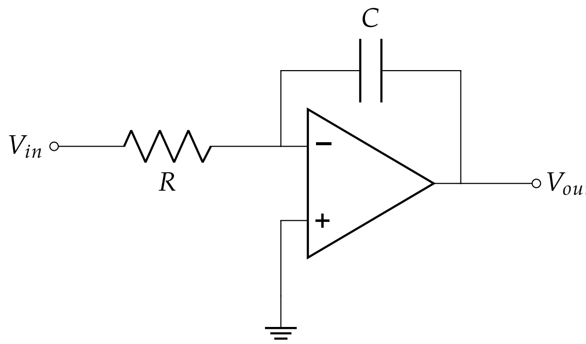

Figure 3: Operational Amplifier with RC elements.

d) In Figure 4, the positions of the resistor and the capacitor seem to have been swapped. Find the relationship between the output voltage V_{out} and V_{in}. What mathematical operation does this circuit perform?

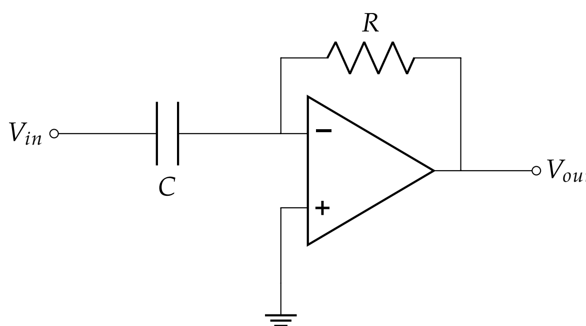

Figure 4: Operational Amplifier with RC elements.

3 Matrix Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{2}$$

where x, y are variables depending on time t, and A is a 2×2 matrix with constant coefficients. We call (2) a matrix differential equation.

a) Suppose we have a system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 7x - 8y \tag{3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 5y \tag{4}$$

Write this in the form of (2).

b) Compute the eigenvalues of the matrix A from the previous part.

c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_0 t} + c_3 e^{\lambda_1 t} \end{bmatrix},$$

where c_0, c_1, c_2, c_3 are constants, and λ_0, λ_1 are the eigenvalues of A. Suppose that the initial conditions are $x(0) = 1, y(0) = -1$. Solve for the constants c_0, c_1, c_2, c_3.

Hint: What are $\frac{dx}{dt}(0)$ *and* $\frac{dy}{dt}(0)$?

d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (3), (4).

e) We now apply the method above to solve another second-order ordinary differential equation. Suppose we have the system

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} - 5\frac{\mathrm{d}z(t)}{\mathrm{d}t} + 6z(t) = 0, \tag{5}$$

Write this in the form of (2), by choosing appropriate variables $x(t)$ and $y(t)$.

f) Solve the system in (5) with the initial conditions $z(0) = 1$, $\frac{\mathrm{d}z}{\mathrm{d}t}(0) = 1$, using the method developed in parts (b) and (c).

4 Multi-Capacitor Circuit

Consider the circuit below

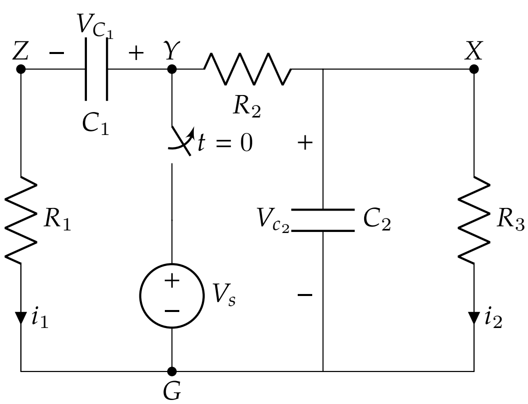

Figure 5: Circuit with multiple capacitors.

The resistors shown in the circuit have the same value $R_1 = R_2 = R_3 = R$. Capacitors C_1 and C_2 have the same capacitance $C_1 = C_2 = C$. Further, $RC = 1s$.

a) Assume that the switch shown in Figure 5 was held in the closed position for a long time before $t = 0$. At $t = 0$, immediately after the switch is opened, what are the capacitor voltages $V_{C_1}(0)$ and $V_{C_2}(0)$?

b) How are the current i_2 and the capacitor voltage V_{C_2} related?

c) Using KCL on node X and the relationship above, write an equation relating V_{C_2} and i_1.

d) Using KVL on the loop comprising of both capacitors C_1 and C_2, find a relationship between V_{C_1}, V_{C_2} and i_1.

e) Rewrite the equations derived above, eliminating the current i_1 to obtain a system of differential equations involving V_{C_1} and V_{C_2}. Write this system of equations in a matrix form

$$\frac{d}{dt}\begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = A \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix}$$

What is the matrix A and what are its eigenvalues?

Hint: You can use the relation $i_1 = C_1 \frac{d}{dt} V_{C_1}$ in addition to the relations we have derived so far.

f) In order to solve for the capacitor voltages, we finally need the initial values of the voltage derivatives. Immediately after the switch is opened, what are the voltage derivatives for the two capacitors, $\frac{\mathrm{d}V_{C_1}}{\mathrm{d}t}(0)$ and $\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t}(0)$?

Hint: Calculate the currents i_1 and i_2 immediately after the switch if opened at $t = 0$. While the capacitor voltages do not change immediately, the current through them will change.

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**