This homework is due on Thursday, October 1, 2020, at 10:59PM.

Self-grades are due on Thursday, October 8, 2020, at 10:59PM.

1 Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + jy \text{ with } \mathrm{Re}\{z\} = x \text{ and } \mathrm{Im}\{z\} = y$$

See Figure 1 for a visualization of z in the complex plane.

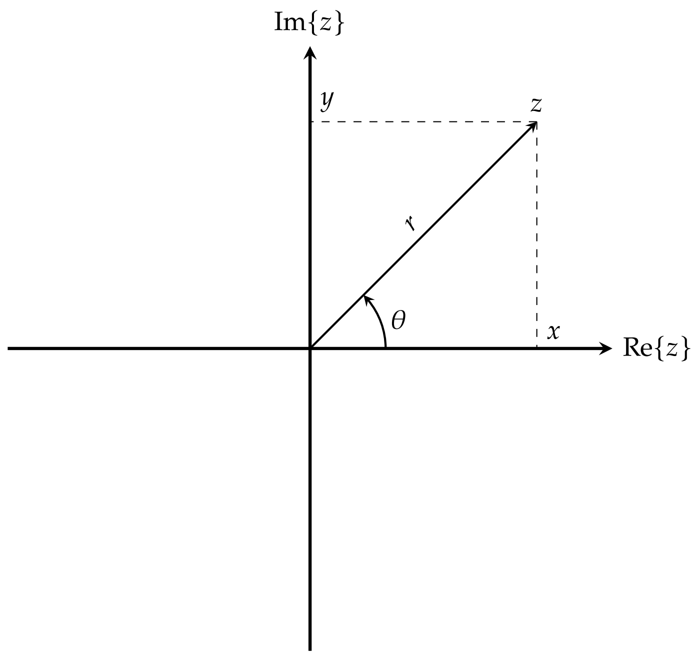


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r. *Hint.* Use the Pythagorean theorem.

b) **Represent x, the real part of z, and y, the imaginary part of z, in terms of r and θ.**

c) **Substitute for x and y in z.** Use Euler's identity[1] $e^{j\theta} = \cos\theta + j\sin\theta$ to **conclude that,**

$$z = re^{j\theta}$$

[1] also known as de Moivre's Theorem.

d) In the complex plane, **draw out all the complex numbers such that $|z| = 1$. What are the z values where the figure intersects the real axis and the imaginary axis?**

e) If $z = re^{j\theta}$, **prove that** $\bar{z} = re^{-j\theta}$. Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

f) **Show that,**

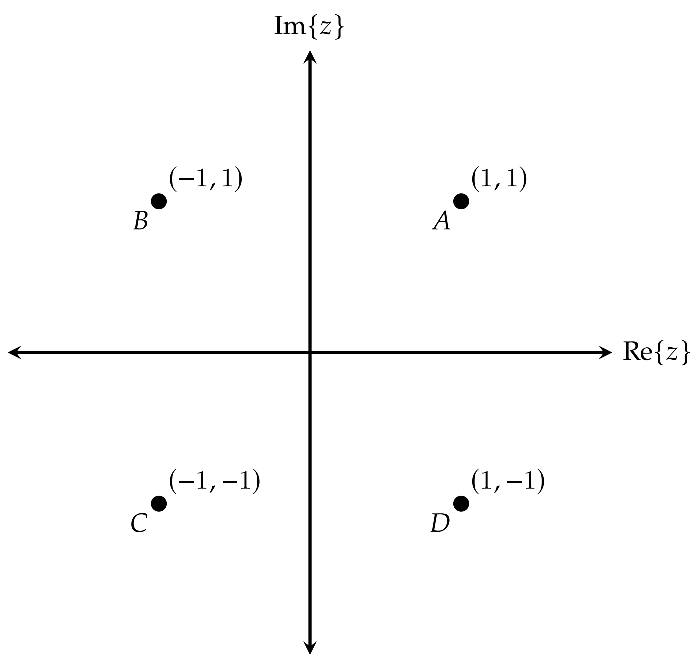
$$r^2 = z\bar{z}$$

g) Intuitively **argue that**

$$\sum_{k=0}^{2} e^{j\frac{2\pi}{3}k} = 0$$

Do so by drawing out the different values of the sum in the complex plane and making an argument based on the vector sum.

h) In modern wireless communication, signals are sent as complex exponentials $e^{j\omega t}$, with receivers detecting both the cosine and sine components of the signal. One common scheme for encoding digital data, such as what is used in your phone and in WiFi, is known as quadrature phase-shift keying (QPSK). In this technique, there are four points of interest on the complex plane (see figure below):

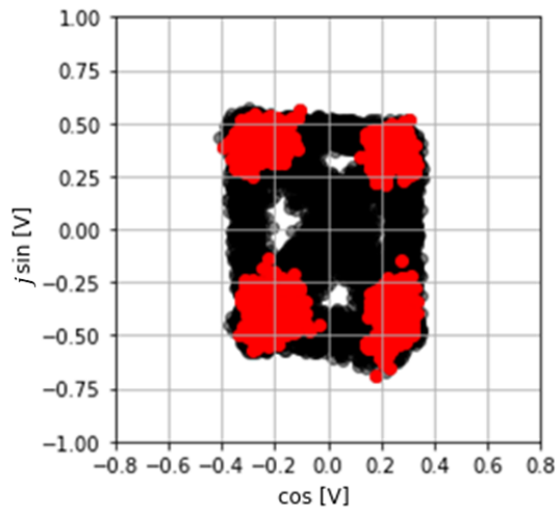


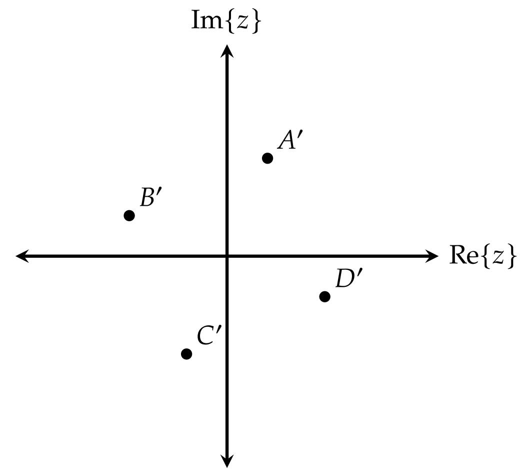
Write each of the points A, B, C, and D in polar form.

i) Due to amplitude error and phase noise, in practice these points often arrive at the receiver scaled and rotated (see the figure on the left). Suppose the received points are as follows:

$$A' = 1e^{j\frac{3\pi}{8}} \qquad C' = 1e^{j\frac{-5\pi}{8}}$$

$$B' = 1e^{j\frac{7\pi}{8}} \qquad D' = 1e^{j\frac{-\pi}{8}}$$

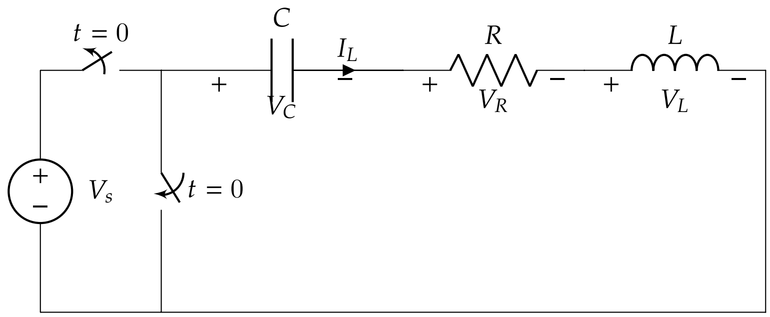




Find a corrective value $r_x e^{j\theta_x}$ that when multiplied with A', B', C', D' recovers the original points A, B, C, D. What is the original noise $r_n e^{j\theta_n}$ that created the shift?

2 RLC Responses

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

In this problem, the current through the inductor and the voltage across the capacitor are the natural physical state variables since these are what correlate to how energy is actually stored in the system. (A magnetic field through the inductor and an electric field within the capacitor.)

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s, L, R, and C.**

b) Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

c) Find the eigenvalues of the A matrix symbolically.

(Hint: the quadratic formula will be involved.)

d) Under what condition on the circuit parameters R, L, C are the eigenvalue(s) of A

i) distinct and real eigenvalues.

ii) distinct purely imaginary eigenvalues.

iii) distinct complex eigenvalues with nonzero real and imaginary parts.

iv) a single real eigenvalue.

e) **Overdamped Case.** In the provided Jupyter notebook, move the sliders to approximately $R = 1k\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane.

f) **Undamped Case.** In the provided Jupyter notebook, move the sliders to approximately $R = 0\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?

Note: Because there is no resistance, this is called the "undamped" case.

g) **Underdamped Case.** In the provided Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

h) **Critically Damped Case**. In the provided Jupyter notebook, move the sliders to the resistance value of $R = 100\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R?

3 Phasors for circuit

In the lectures, we introduced phasors. That is, for a given sinusoidal $x(t) = A\cos(\omega t + \phi)$ with $\omega \neq 0$ and A being the magnitude, we can represent it as a phasor $X = Ae^{j\phi}$ such that $x(t) = \mathrm{Re}(Ae^{j\phi}e^{j\omega t}) = \mathrm{Re}(Xe^{j\omega t})$.

a) Show that $x(t)$ can be written as $x(t) = \frac{1}{2}(Xe^{j\omega t} + \bar{X}e^{-j\omega t})$, where $\bar{X}$ is the complex conjugate of X.

b) It is straight forward that if the phasor $X = 0$ then $x(t) = \mathrm{Re}(Xe^{j\omega t}) = 0$ for every t. The converse is also true. Show if $x(t) = 0$ for all t, then $X = 0$.

   Hint: Pick appropriate t to show that the real part and imaginary part of the phasor is 0. That is, what t makes $e^{j\omega t} = 1$ and what t makes $e^{j\omega t} = j$?

c) With the results from the previous part, show that KCL on phasors are equivalent to KCL on the time waveform. That is, for current $i_1(t), i_2(t), \ldots, i_n(t)$ and their phasors $I_1, I_2, \ldots, I_n$, $I_1 + I_2 + \cdots + I_n = 0$ if and only if $i_1(t) + i_2(t) + \cdots + i_n(t) = 0$ for any t.

   Hint: The linearity holds that the time waveform $x_1(t) + x_2(t)$ has corresponding phasor $X_1 + X_2$.

d) Show the same result holds for KVL. That is, for voltages $v_1(t), v_2(t), \ldots, v_n(t)$ and their phasors $V_1, V_2, \ldots, V_n$, $V_1 + V_2 + \cdots + V_n = 0$ if and only if $v_1(t) + v_2(t) + \cdots + v_n(t) = 0$ for any t.

   Since both KCL and KVL hold on phasor form, we may employ the same circuit analysis tools on phasors.

e) Consider the following circuit where the power supply outputs $u(t) = \mathrm{Re}\left\{V_0 e^{j\omega t}\right\}$ for some $\omega \neq 0$. Employ nodal analysis using phasors and only the voltage-current relation, $V = IZ$. In the circuit, $Z_1 = \frac{1}{j\omega C}$ is the impedance of the capacitor and $Z_2 = R$ is the impedance of the resistor. Then express V_1 in terms of V_0, Z_1, and Z_2. Note that V_0, V_1 are both phasors.

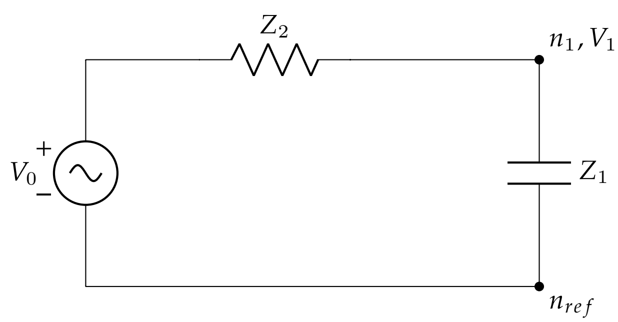


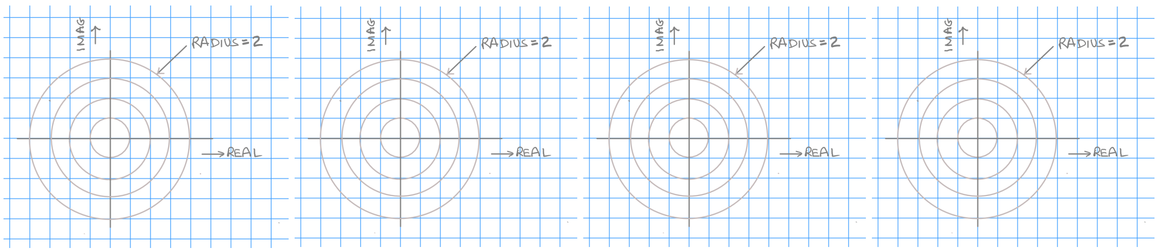
Figure 2: Equivalent Circuit in the Phasor Domain

f) For $V_0 = A$ where A is a real number, the power supply outputs time waveform $u(t) = A\cos(\omega t)$. Write the time waveform $v_1(t)$ for V_1 based on the phasor you obtained for the circuit in the previous part.

4 Phasors

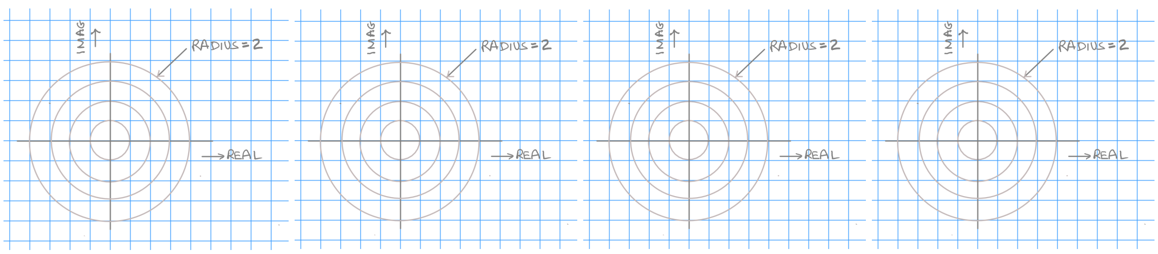
a) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1\text{F}$), and an inductor ($L = 1\text{H}$) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

b) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**



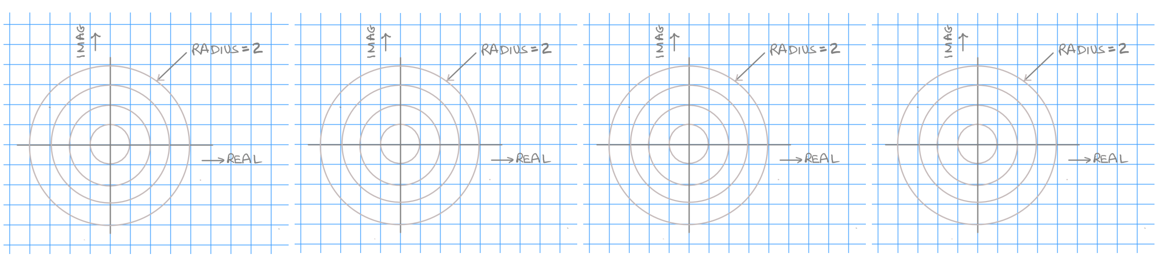
(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

c) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**



(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

d) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**



(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

e) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**