This homework is due on Thursday, October 22, 2020, at 10:59PM.

Self-grades are due on Thursday, October 29, 2020, at 10:59PM.

1 LED Strip

I have an LED strip with 5 red LEDs whose brightnesses I want to set. These LEDs are addressed as a queue: at each time step, I can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

a) What should we use for our state vector? What does it mean that this is a state vector? What is our input?

b) Assume that our system is linear, and write out the state equations in matrix form. Please choose a reasonable order for the state variables in the state vector.

c) Is this system controllable? Explain intuitively what this system's controllability means in terms of LED brightnesses.

d) Starting from the pattern of brightnesses (from left to right) [0, 127, 0, 255, 0], can we maintain this pattern for all future time steps? Can we display any fixed pattern of brightnesses for all time?

2 Controllability in circuits

Consider the circuit in Figure 1, where V_s is an input we can control:

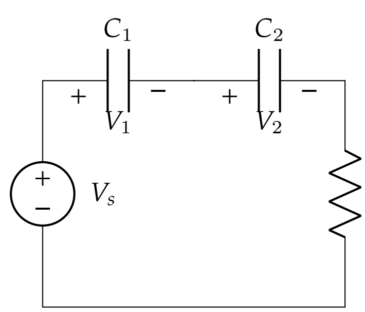


Figure 1: Controllability in circuits

a) Write the state space model for this circuit.

b) Show that this system is not controllable.

c) Explain, in terms of circuit currents and voltages, why this system isn't controllable. (Hint: think about what currents/voltages of the circuit we are controlling with V_s)

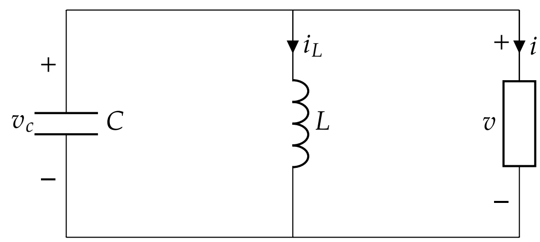
d) Draw an equivalent circuit of this system that is controllable. What quantity can you control in this system?

3 Nonlinear circuit component

This is a problem adapted from a past midterm problem (Spring 2017 midterm 2).

Consider the circuit below that consists of a capacitor, inductor, and a third element with a nonlinear voltage-current characteristic:

$$i = 2v - v^2 + 4v^3$$



a) Write a state space model of the form

$$\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t))$$

$$\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))$$

Where $x_1(t) = v_c(t)$ and $x_2(t) = i_L(t)$.

b) Linearize the state model at the equilibrium point $x_1 = x_2 = 0$ and specify the resulting A matrix.

c) Is the linearized system stable?

4 An Extension of Predator-Prey Models

On last week's homework, we looked at a model of the predator-prey dynamic in biological systems, shown below:

$$\frac{d}{dt}x(t) = (a - by)x \tag{1}$$

$$\frac{d}{dt}y(t) = (cx - d)y \tag{2}$$

This question will seek to expand on that model to create a more generalized version for systems of more than just two species. Specifically, we will look at the *Generalized Lotka-Volterra Equations*:

$$\frac{d}{dt}x_i(t) = x_i\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right), i = 1, \ldots, n \tag{3}$$

Where each x_i is the density of species i in the population, b_i is the species' growth rate, and a_{ij} are interaction parameters between species i and species j.

a) Show that, if $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ is invertible, there are at most 2^n different equilibrium points for the n-species system described by these a_{ij}.

b) What does it mean for an a_{ij} to be positive? Negative? Zero?

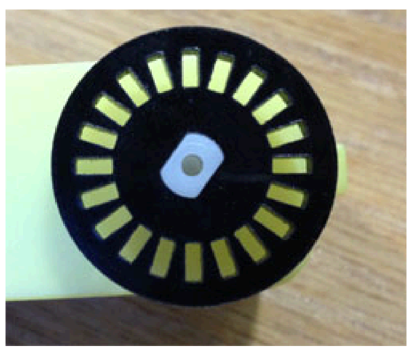
c) Let $n = 2$. Assume that a_{11} and a_{22} are both nonzero, b_1 and b_2 are both greater than zero, and that the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible. Under these assumptions, there are up to 4 equilibrium points in this system; what are they?

d) One of these points is of the form $\vec{x}^* = \begin{bmatrix} 0 \\ x_2^* \end{bmatrix}$, with nonzero x_2. Linearize the system about this point.

e) Under what conditions on A and/or $\vec{b}$ is this point stable? Does this make sense in the context of population dynamics?

5 Understanding the SIXT33N Car Control Model

As the students in the lab continue along the process of making the awesome SIXT33N cars, we'd like to better understand the car model that they will be using to develop a control scheme. As a wheel on the car turns, there is an encoder disc (see below) that also turns as the wheel turns. The encoder shines a light though the encoder disc, and as the wheel turns, the light is continually blocked and unblocked, allowing the encoder to detect how fast the wheel is turning by looking at the number of times that the light "ticks" between being blocked and unblocked over a specific time interval.



The following model applies separately to each wheel (and associated motor) of the car:

$$v[k] = d[k+1] - d[k] = \theta u[k] - \beta$$

Meet the variables at play in this model:

- k - The current timestep of the model. Since we model the car as a discrete system, this will advance by 1 on every new sample in the system.
- $d[k]$ - The total number of ticks advanced by a given encoder (the values may differ for the left and right motors–think about when this would be the case).

- $v[k]$ - The discrete-time velocity (in units of ticks/timestep) of the wheel, measured by finding the difference between two subsequent tick counts ($d[k+1] - d[k]$).
- $u[k]$ - The input to the system. The motors that apply force to the wheels are driven by an input voltage signal. This voltage is delivered via a technique known as pulse width modulation (PWM), where the average value of the voltage (which is what the motor is responsive to) is controlled by changing the duty cycle of the voltage waveform. The duty cycle, or percentage of the square wave's period for which the square wave is HIGH, is mapped to the range $[0, 255]$. Thus, $u[k]$ takes a value in $[0, 255]$ representing the duty cycle. For example, when $u[k] = 255$, the duty cycle is 100 %, and the motor controller just delivers a constant signal at the system's HIGH voltage, delivering the maximum possible power to the motor. When $u[k] = 0$, the duty cycle is 0 %, and the motor controller delivers 0 V.
- θ - Relates change in input to change in velocity: if the wheel rotates through n ticks in one timestep for a given $u[k]$ and m ticks in one timestep for an input of $u[k] + 1$, then $\theta = m - n = \frac{\Delta v[k]}{\Delta u[k]} = \frac{v_{u_1[k]}[k] - v_{u_0[k]}[k]}{u_1[k] - u_0[k]}$. **Its units are ticks/(timestep · duty cycle).** Since our model is linear, we assume that θ is the same for every unit increase in $u[k]$. This is empirically measured using the car: θ depends on many physical phenomena, so for the purpose of this class, we will not attempt to create a mathematical model based on the actual physics. However, you can conceptualize θ as a "sensitivity factor", representing the idiosyncratic response of your wheel and motor to a change in power (you will have a separate θ for your left and your right wheel).
- β - Similarly to θ, β is dependent upon many physical phenomena, so we will empirically determine it using the car. β represents a constant offset in velocity, and hence **its units are ticks/timestep**. Note that you will also have a different β for your left and your right wheel.

In this problem (except parts (c) and (e)) we will assume that the wheel conforms perfectly to this model to get an intuition of how the model works.

a) If we wanted to make the wheel move at a certain target velocity v^*, what input $u[k]$ should we provide to the motor that drives it? Your answer should be symbolic, and in terms of v^*, $u[k]$, θ, and β.

b) Even if the wheel and the motor driving it conform perfectly to the model, our inputs still limit the range of velocities. Given that $0 \leq u[k] \leq 255$, determine the maximum and minimum velocities possible for the wheel. How can you slow the car down?

c) Our intuition tells us that a wheel on a car should eventually stop turning if we stop applying any power to it. Find $v[k]$ assuming that $u[k] = 0$. Does the model obey your intutition? What does that tell us about our model?

d) In order to characterize the car, we need to find the θ and β values that model your left and right wheels and their motors: θ_l, β_l, θ_r, and β_r. How would you determine θ and β empirically? What data would you need to collect? *Hint:* keep in mind we also know the input $u[k]$ for all k.

e) How can you use the data you collected in part (d) to mitigate the effect of the system's nonlinearity and/or minimize model mismatch? *Hint:* you will only wind up using a small range of the possible input values in practice. There are several reasons this is true, but one is that each motor has a characteristic attainable velocity range, and for your car to drive straight, we need the wheels to rotate with the same velocity.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**