This homework is due on Thursday, October 29, 2020, at 10:59PM. Self-grades are due on Thursday, November 5, 2020, at 10:59PM.

Clarification: You will have to submit code for Q2 and Q3. You can either attach a screenshot to your written submission or you can submit the .ipynb to the HW 8 Code Submission assignment on Gradescope.

1 Eigenvalue Placement

Consider the following linear discrete time system

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & -6 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{1}$$

a) What are the eigenvalues of this system? Is this system stable?

b) Show that this system is controllable.

c) Using state feedback $u(t) = K\vec{x}(t) = \begin{bmatrix} k_0 & k_1 & k_2 \end{bmatrix} \vec{x}(t)$ design a controller to place the closed-loop system's eigenvalues at $0, 1/2, -1/2$.

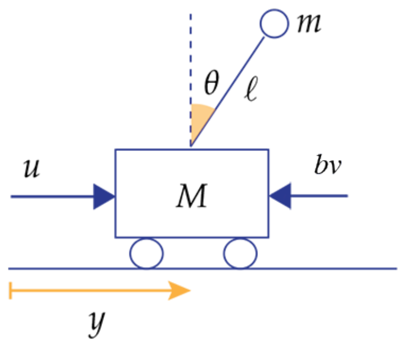
d) Upon implementing this controller in the lab, we run into issues since our hardware can only produce K values in between -5 and 5. **Is it still possible to design a controller that can stabilize the original system given in (1)?**

2 Inverted Pendulum on a Rolling Cart

Note: In this problem we use the state-feedback policy $\vec{u} = -K\vec{x}$ to stabilize the eigenvalues of $A - BK$ instead of $A + BK$ in order to use external libraries to solve for K. This should not change any of the empirical results, but the values of K will be negated if you use $\vec{u} = K\vec{x}$ to pick the eigenvalues.

Consider the inverted pendulum depicted below, which is placed on a rolling cart and whose equations of motion are given by:

$$\ddot{y} = \frac{1}{M + m\sin^2\theta}\left(u - m\dot{\theta}^2\ell\sin\theta + mg\sin\theta\cos\theta - b\dot{y}\right)$$
$$\ddot{\theta} = \frac{1}{\ell(M + m\sin^2\theta)}\left(u\cos\theta + m\dot{\theta}^2\ell\sin\theta\cos\theta + (M+m)g\sin\theta - b\dot{y}\cos\theta\right).$$



The dots above variables are shorthand for derivatives – that is ($\dot{x} = \frac{d}{dt}x$ and $\ddot{x} = \frac{d^2}{dt^2}x$).

a) Write out the state model using the variables $x_1 = y, x_2 = \dot{y}, x_3 = \theta, x_4 = \dot{\theta}$.

b) Linearize this model at the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and $u = 0$, and indicate the resulting A and B matrices.

   Hint: When taking partial derivatives, all other variables will be constant. For example, if we need to evaluate $\frac{\partial f}{\partial x_1}$, we can plug in $x_2 = x_3 = x_4 = u = 0$ before taking any derivatives.

c) Let $m = 1\,\text{kg}, M = 5\,\text{kg}, L = 2\,\text{m}, g = 10\,\frac{\text{m}}{\text{s}^2}, b = 1\,\frac{\text{kg}}{\text{s}}$. Show that the linearized model is controllable but unstable. Then plot the response of the system in the Jupyter notebook and explain how the cart-pendulum system behaves if we push the cart with a small input force.

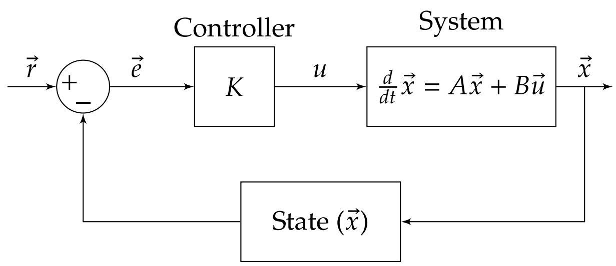
d) Since the system is controllable, we can design a state-feedback controller $u = -K\vec{x}$ to assign the eigenvalues of $A_{cl} = A - BK$. Find the values of K using the Jupyter Notebook or by hand to place the closed-loop eigenvalues at $\begin{bmatrix} -2 & -1.1 & -1.2 & -1.3 \end{bmatrix}$.

e) Now that we have stabilized our system, let's analyze how we can move our system to a target state $\vec{r}$. To do this, we define $\vec{e} = \vec{r} - \vec{x}$ as the error between the state $\vec{x}$ and the target $\vec{r}$.

Then we will use the control policy $u = K\vec{e} = K(\vec{r} - \vec{x})$. **Show that if $\vec{r} \in$ Nul(A), we can write out the following closed-loop state-space model for $\vec{e}$.**

$$\frac{d}{dt}\vec{e} = (A - BK)\vec{e} \tag{2}$$

Explain why the state $\vec{x}$ converges to the reference state $\vec{r}$ as $t \to \infty$. The block-diagram drawn below illustrates the behavior of our new system outlined in (2).

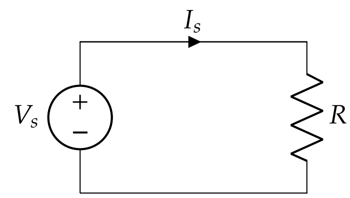


f) Pick different eigenvalues for the closed-loop matrix and run the simulation of the cart-pendulum system. Then explain the effect that the eigenvalues have on the system.

3 System Identification

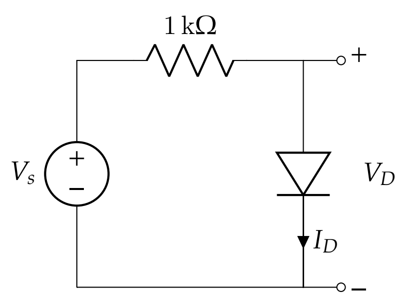
Taejin is building his SIXT33N car in the lab and needs your help on identifying some unknown parameters. Each part will require you to set up and solve a Least-Squares problem in the provided Jupyter Notebook.

a) While building the front-end circuit, he notices that one of his resistors has no color bands. Therefore, he decides to apply voltages V_s and measure the current I_s across the resistor.



Using the measurement data provided in the Jupyter Notebook, **set up and solve a least-squares problem to estimate the resistance** R.

b) Taejin continues to build the front-end circuit, but he comes across a circuit device he's never seen before. After consulting Ramsey, the circuits expert, he learns that $I_D = I_s e^{V_D/V_{TH}}$.



Here I_s is the saturation current and V_{TH} is the thermal voltage both of which depend on the device physics. To obtain experimental data, Taejin applies voltages V_s and measures voltages and currents V_D and I_D across the mystery device. **Set up and solve a least-squares problem in the Jupyter Notebook to estimate** I_s **and** V_{TH}.

c) Now Taejin moves onto building the car. His lab partner, Nick, tells him that the position of the car can be formulated as a discrete-time system of the form

$$\begin{bmatrix} d_L(t+1) \\ d_R(t+1) \end{bmatrix} = A \begin{bmatrix} d_L(t) \\ d_R(t) \end{bmatrix} + B \begin{bmatrix} u_L(t) \\ u_R(t) \end{bmatrix} \tag{3}$$

The car's wheels start at $d_L(0) = d_R(0) = 0$. They apply the sequence of inputs

$$u_L(0), u_R(0), u_L(1), u_R(1), \ldots, u_L(98), u_R(98), u_L(99), u_R(99),$$

and measure the car's position

$$d_L(1), d_R(1), d_L(2), d_R(2), \ldots, d_L(99), d_R(99), d_L(100), d_R(100),$$

at each time step. Using the measurement data provided in the notebook, **set up and solve a least-squares problem to find what the A and B matrices are.**

d) Upon solving for A and B and providing new inputs to the system, they notice that the car isn't responding to the inputs properly. Take a look at the data-points and explain why this might be the case. **How can Taejin and Nick fix this issue to make their car run smoothly?**

4 Open-Loop Control of SIXT33N

Last time, we learned that the ideal input PWM for running a motor at a target velocity v^* is:

$$u(t) = \frac{v^* + \beta}{\theta}$$

In this problem, we will extend our analysis from one motor to a two-motor car system and evaluate how well our open-loop control scheme does.

$$v_L(t) = d_L(t+1) - d_L(t) = \theta_L u_L(t) - \beta_L$$
$$v_R(t) = d_R(t+1) - d_R(t) = \theta_R u_R(t) - \beta_R$$

a) In reality, we need to "kickstart" electric motors with a pulse in order for them to work. That is, we can't go straight from 0 to our desired input signal for $u(t)$, since the motor needs to overcome its initial inertia in order to operate in accordance with our model.

Let us model the pulse as having a width (in timesteps) of t_p. In order to model this phenomenon, we can say that $u(t) = 255$ for $t \in [0, t_p - 1]$[1]. In addition, the car initially (at $t = 0$) hasn't moved, so we can also say $d(0) = 0$.

Firstly, let us examine what happens to d_L and d_R at $t = t_p$, that is, right after the kickstart pulse has passed. **Find $d_L(t_p)$ and $d_R(t_p)$.** (*Hint:* If it helps, try finding $d_L(1)$ and $d_R(1)$ first and then generalizing your result to the t_p case.)

Note: It is very important that you distinguish θ_L and θ_R as the motors we have are liable to vary in their parameters, just as how real resistors vary from their ideal resistance.

b) Let us define $\delta(t) = d_L(t) - d_R(t)$ as the difference in positions between the two wheels. If both wheels of the car are going at the same velocity, then this difference δ should remain constant since no wheel will advance by more ticks than the other. As a result, this will be useful in our analysis and in designing our control schemes.

Find $\delta(t_p)$. For both an ideal car ($\theta_L = \theta_R$ and $\beta_L = \beta_R$) where both motors are perfectly ideal and a non-ideal car ($\theta_L \neq \theta_R$ and $\beta_L \neq \beta_R$), did the car turn compared to before the pulse?

Note: Since $d(0) = d_L(0) = d_R(0) = 0$, $\delta(0) = 0$.

c) We can still declare victory though, even if the car turns a little bit during the initial pulse (t_p will be very short in lab), so long as the car continues to go straight afterwards when we apply our control scheme; that is, as long as $\delta(t \to \infty)$ converges to a constant value (as opposed to going to $\pm\infty$ or oscillating).

Let's try applying the open-loop control scheme we learned last week to each of the motors independently, and see if our car still goes straight.

$$u_L(t) = \frac{v^* + \beta_L}{\theta_L}$$
$$u_R(t) = \frac{v^* + \beta_R}{\theta_R}$$

Let $\delta(t_p) = \delta_0$. Find $\delta(t)$ for $t \geq t_p$ in terms of δ_0. (*Hint:* As in part (a), if it helps you, try finding $\delta(t_p + 1)$, $\delta(t_p + 2)$, etc., and generalizing your result to the $\delta(t)$ case.)

Does $\delta(t \to \infty)$ deviate from δ_0? Why or why not?

[1] $x \in [a, b]$ means that x goes from a to b inclusive.

d) Unfortunately, in real life, it is hard to capture the precise parameters of the car motors like θ and β, and even if we did manage to capture them, they could vary as a function of temperature, time, wheel conditions, battery voltage, etc. In order to model this effect of **model mismatch**, we consider model mismatch terms (such as $\Delta\theta_L$), which reflects the discrepancy between the model parameters and actual parameters.

$$v_L(t) = d_L(t+1) - d_L(t) = (\theta_L + \Delta\theta_L)u_L(t) - (\beta_L + \Delta\beta_L)$$
$$v_R(t) = d_R(t+1) - d_R(t) = (\theta_R + \Delta\theta_R)u_R(t) - (\beta_R + \Delta\beta_R)$$

Let us try applying the open-loop control scheme again to this new system. Note that **no model mismatch terms appear below** – this is intentional since our control scheme is derived from the model parameters for θ and β, not from the actual $\theta + \Delta\theta$, etc.[2]

$$u_L(t) = \frac{v^* + \beta_L}{\theta_L}$$
$$u_R(t) = \frac{v^* + \beta_R}{\theta_R}$$

As before, let $\delta(t_p) = \delta_0$. Find $\delta(t)$ for $t \geq t_p$ in terms of δ_0.

Does $\delta(t \to \infty)$ change from δ_0? Why or why not, and how is it different from the previous case of no model mismatch?

You may have noticed that open-loop control is insufficient in light of non-idealities and mismatches. Next time, we will analyze a more powerful form of control (closed-loop control) which should be more robust against these kinds of problems.

[2] Why not just do a better job of capturing the parameters, one may ask? Well, as noted above, the mismatch can vary as a function of an assortment of factors including temperature, time, wheel conditions, battery voltage, and it is not realistic to try to capture the parameters under every possible environment, so it is up to the control designer to ensure that the system can tolerate a reasonable amount of mismatch.

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**