EE16B
Designing Information Devices and Systems II

Lecture 10A
Geometry of SVD
Election-Day Special

Intro

- Last time:
  - Described the SVD in
    - Compact matrix form: $U_1 S V_1^T$
    - Full form: $U \Sigma V^T$
  - Showed a procedure to SVD via $A^T A$
  - Show procedure via AA^T
- Today:
  - Uniquness of SVD
  - Geometry of SVD
  - Continue proofs (symmetric matrices)
  - PCA (maybe)

Uniqueness of the SVD

Find SVD of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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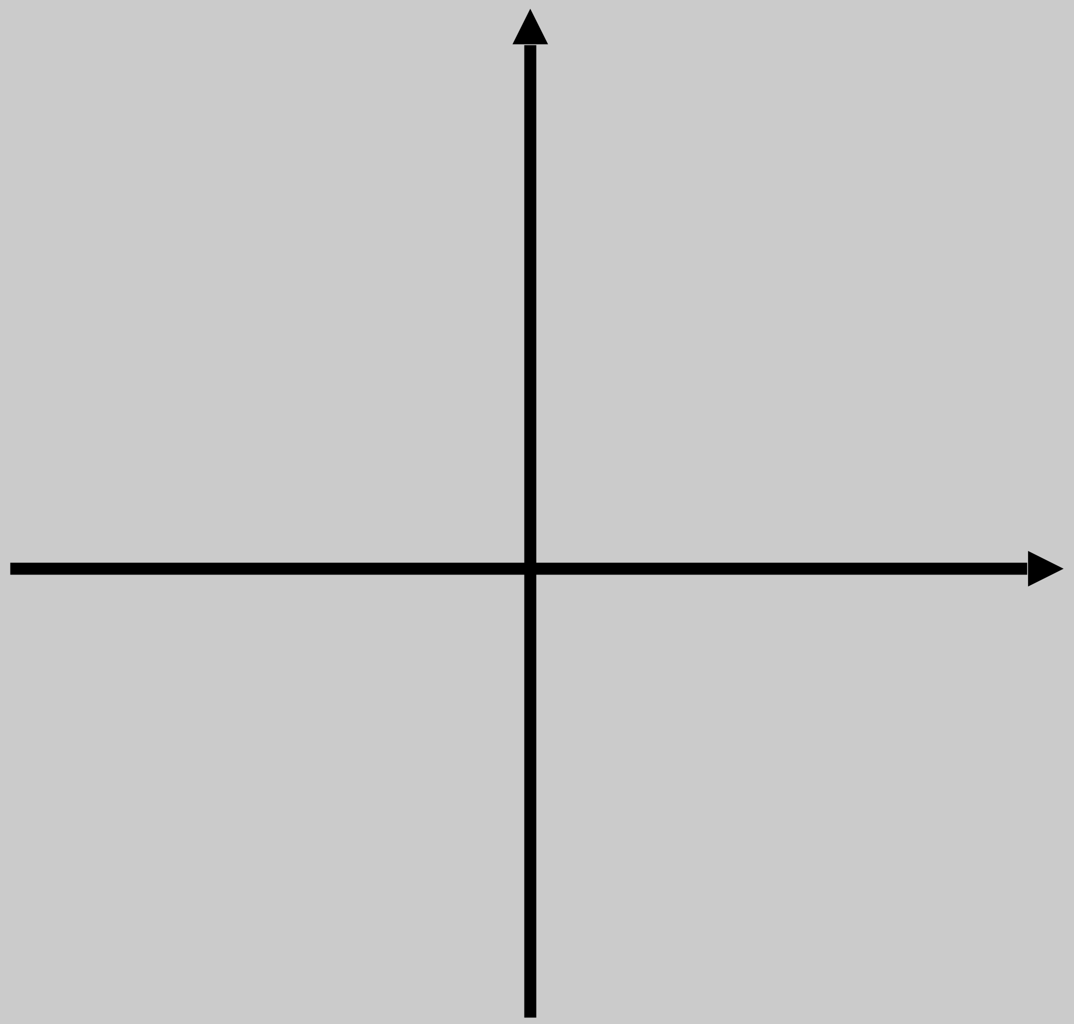
$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



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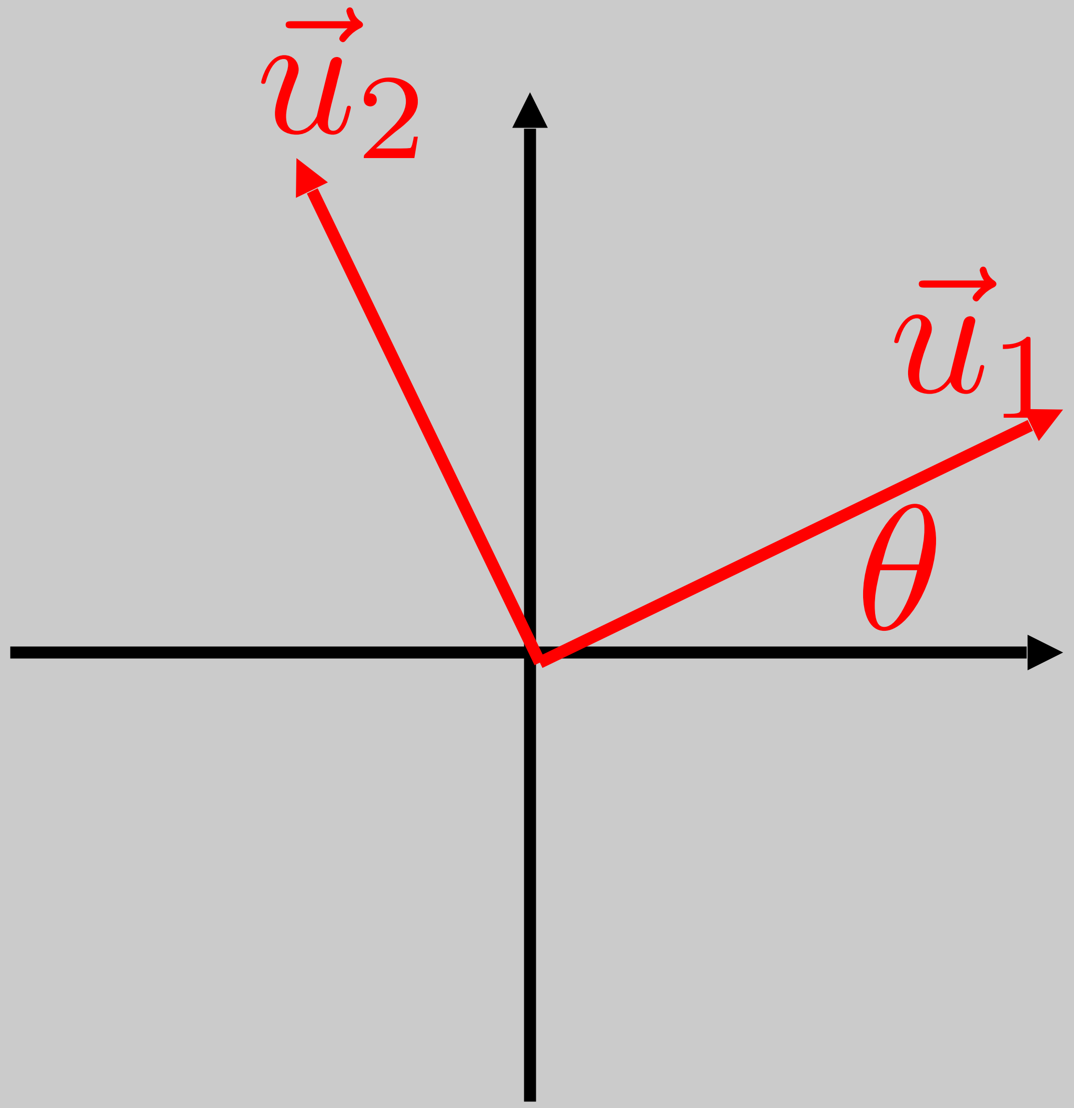

$$\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

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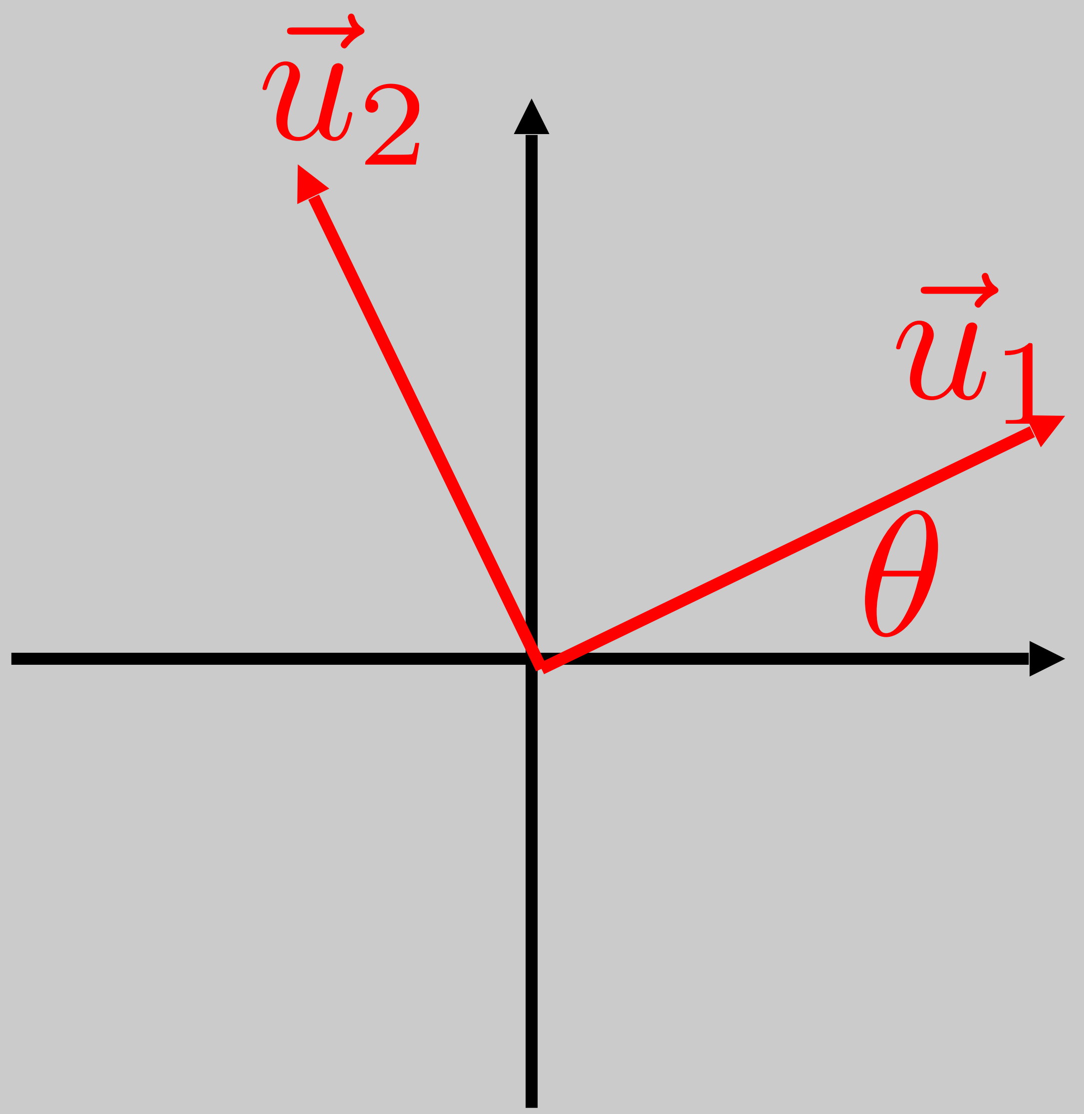

$$\vec{v}_1 = \frac{1}{\sigma_1} A^T \vec{u}_1 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -\sin\theta \\ -\cos\theta \end{bmatrix}$$

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$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

$$= \begin{bmatrix} \quad \end{bmatrix} + \begin{bmatrix} \quad \end{bmatrix}$$

Uniqueness of the SVD

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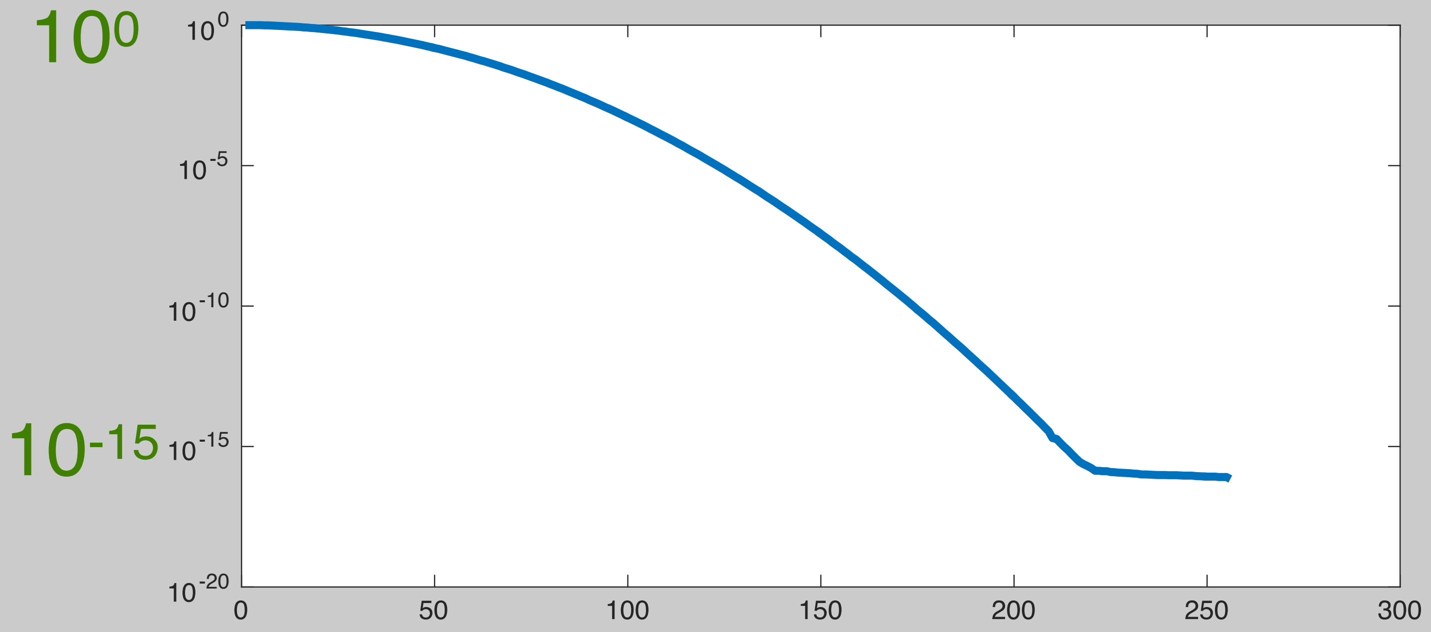
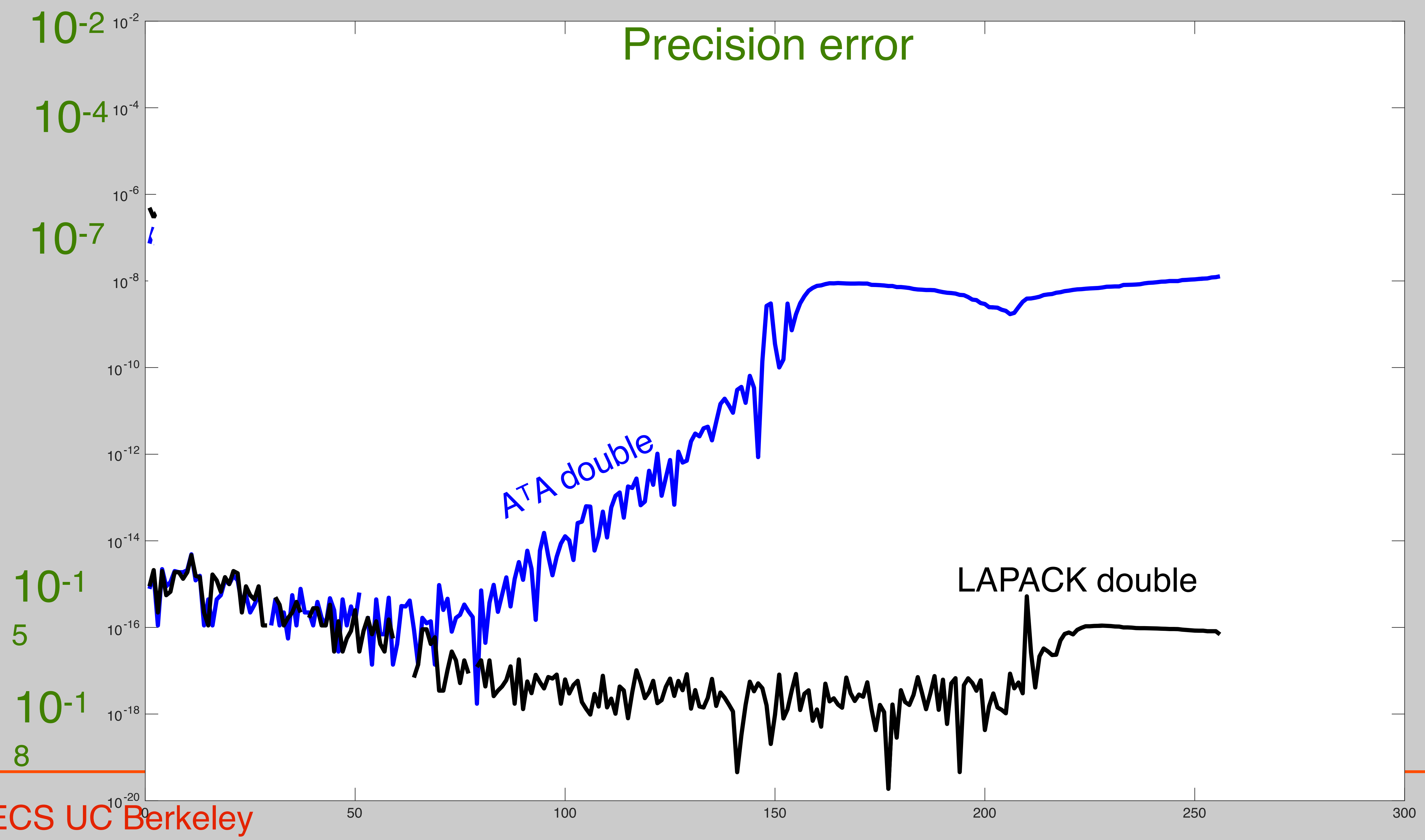
$$= \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Accuracy with Finite Precision

Consider matrix $A \in R^{512x256}$ with the following singular values:





	sign	exponent	mantissa	exponent	significant
format	bit	bits	bits	excess	digits
IEEE 32-bit	1	8	23	127	6
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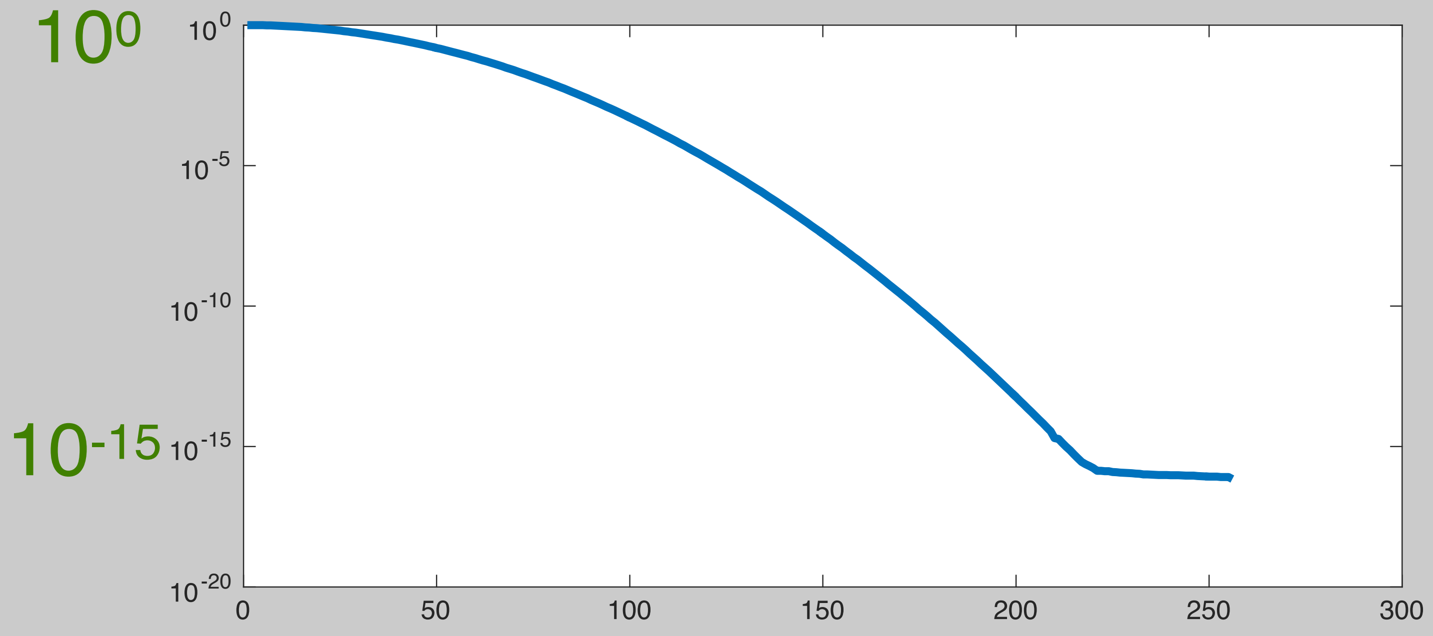

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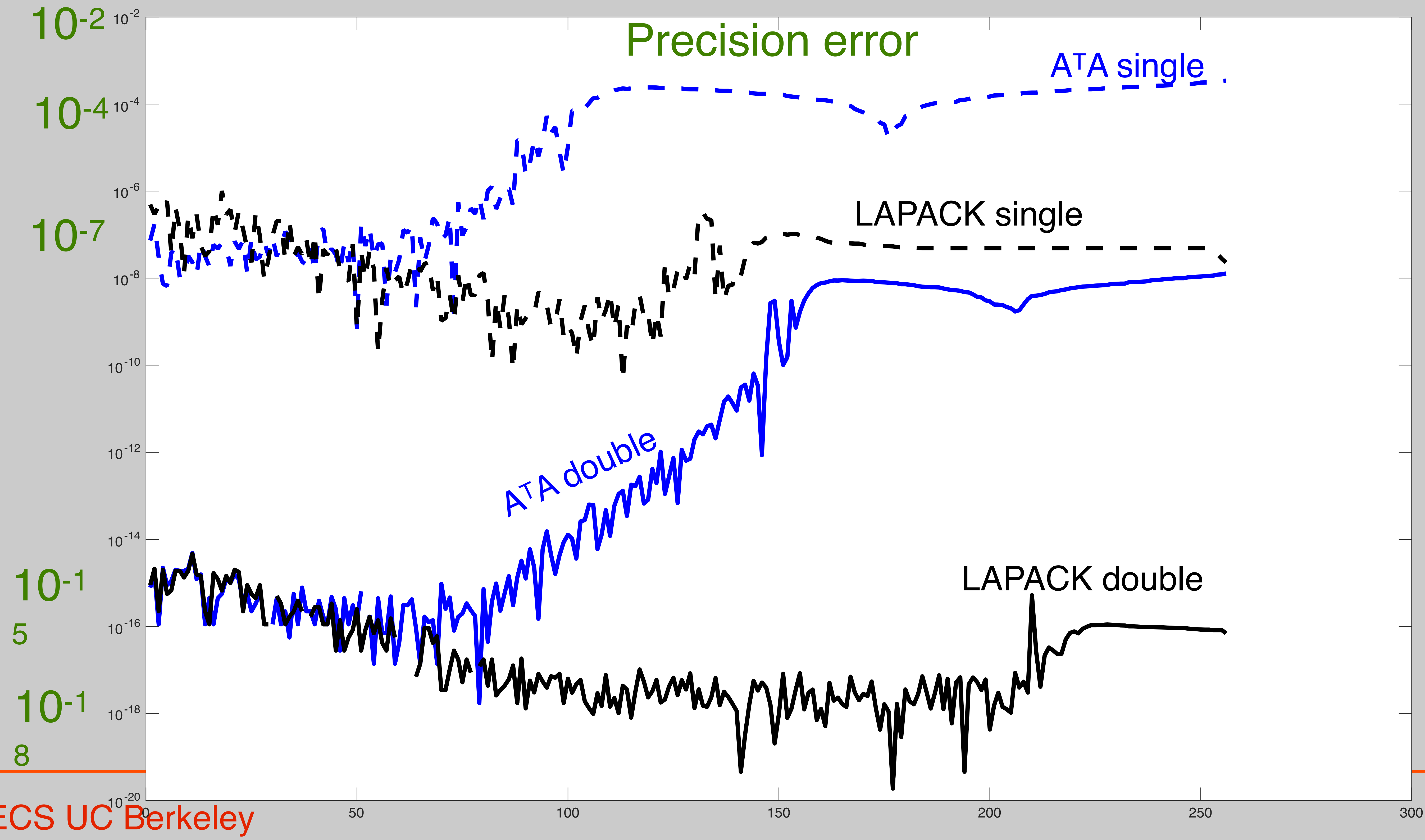


Full Matrix Form of SVD

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix}$$

$m \times r$

$$S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

$r \times r$

$$V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$

$n \times r$

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

$m \times m$

$$\Sigma = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & 0 \end{array}\right]$$

$m \times n$

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$n \times n$

$$A = U \Sigma V^T$$

$$U^T U = I_{m \times m}$$
$$V^T V = I_{n \times n}$$

Unitary Matrices

Multiplying with unitary matrices does not change the length

$$||U\vec{x}|| = \sqrt{(U\vec{x})^T(U\vec{x})} = \sqrt{\vec{x}^T U^T U \vec{x}} = \sqrt{\vec{x}^T \vec{x}} = ||\vec{x}||$$

Example: Rotation, or reflection matrices

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Geometric Interpretation

$$A = U\Sigma V^T$$

$$A\vec{x} = U\Sigma V^T\vec{x}$$

1) $V^T\vec{x}$ re-orients $\vec{x}$ without changing length.

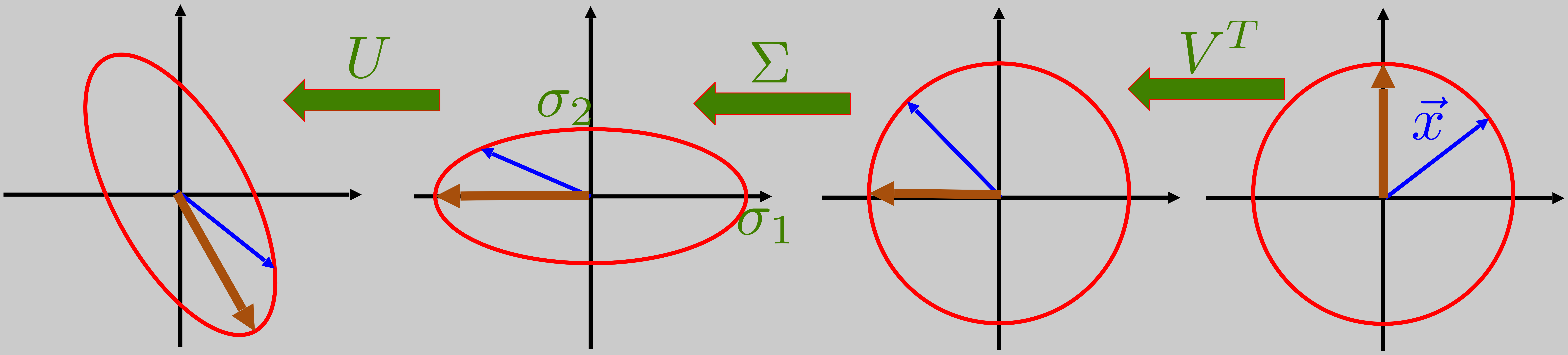
2) $\Sigma(V^T\vec{x})$ Stretches along the axis with singlular values

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 x_1 \\ \sigma_2 x_2 \end{bmatrix}$$

3) $U(\Sigma V^T\vec{x})$ re-orients again without changing length

Geometric Interpretation

$$A = U\Sigma V^T \qquad A\vec{x}$$



$$||A\vec{x}|| \le \sigma_1 ||\vec{x}||$$

Q: What vector would amplify the most?

Symmetric Matrices

We assumed before that,

A^TA has only real eigenvalues, r of them are positive and the rest are zero
A^TA has orthonormal eigenvectors (to be proven next time)

For symmetric matrices: $Q^T = Q$

$$(AB)^T = B^T A^T$$

$$(A^T A)^T = A^T A$$

$$(AA^T)^T = AA^T$$

Properties of Symmetric Matrices

1) A real-valued symmetric matrix has real eigenvalues and eigenvectors

$$Qx = \lambda x \qquad \lambda = a + ib \qquad \overline{\lambda} = a - ib$$

Somehow we need to use the symmetric and real-ness property of Q to show that b==0

$$Q\overline{x} = \overline{\lambda}\overline{x}$$

$$\overline{x}^T Q = \overline{\lambda}\overline{x}^T$$

$$\overline{x}^T Q x = \overline{\lambda}\overline{x}^T x \qquad \overline{x}^T Q x = \lambda \overline{x}^T x$$

$$\overline{\lambda}\overline{x}^T x = \lambda \overline{x}^T x \quad \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathrm{R}$$

Properties of Symmetric Matrices

$$Qx = \lambda x$$

$$\underbrace{(Q - \lambda I)}_{\text{real}} x = 0$$

So x is real as well

✅A real-valued symmetric matrix has real eigenvalues and eigenvectors

Properties of Symmetric Matrices

2) Eigenvectors of a symmetrix matrix can be chosen to be orthonormal

Choose two distinct eigenvalues and vectors $\lambda_1 \neq \lambda_2$

$$Qx_1 = \lambda_1 x_1 \qquad Qx_2 = \lambda_2 x_2$$

$$x_2^T Q x_1 = \lambda_1 x_2^T x_1 \qquad x_1^T Q x_2 = \lambda_2 x_1^T x_2$$

$$(\lambda_1 - \lambda_2) x_2^T x_1 = 0$$

$$\lambda_1 \neq \lambda 2 \Rightarrow x_2^T x_1 = 0$$

✅ Eigenvectors of a symmetrix matrix can be chosen to be orthonormal

Positiveness of Eigenvalues

3) If Q can be written as Q = $R^T R$ for real R, then Q is positive semidefinite – eigenvalues greater of equal to zero

$$Qx = \lambda x$$

$$R^T R x = \lambda x$$

$$x^T R^T R x = \lambda x^T x$$

$$(Rx)^T (Rx) = \lambda x^T x$$

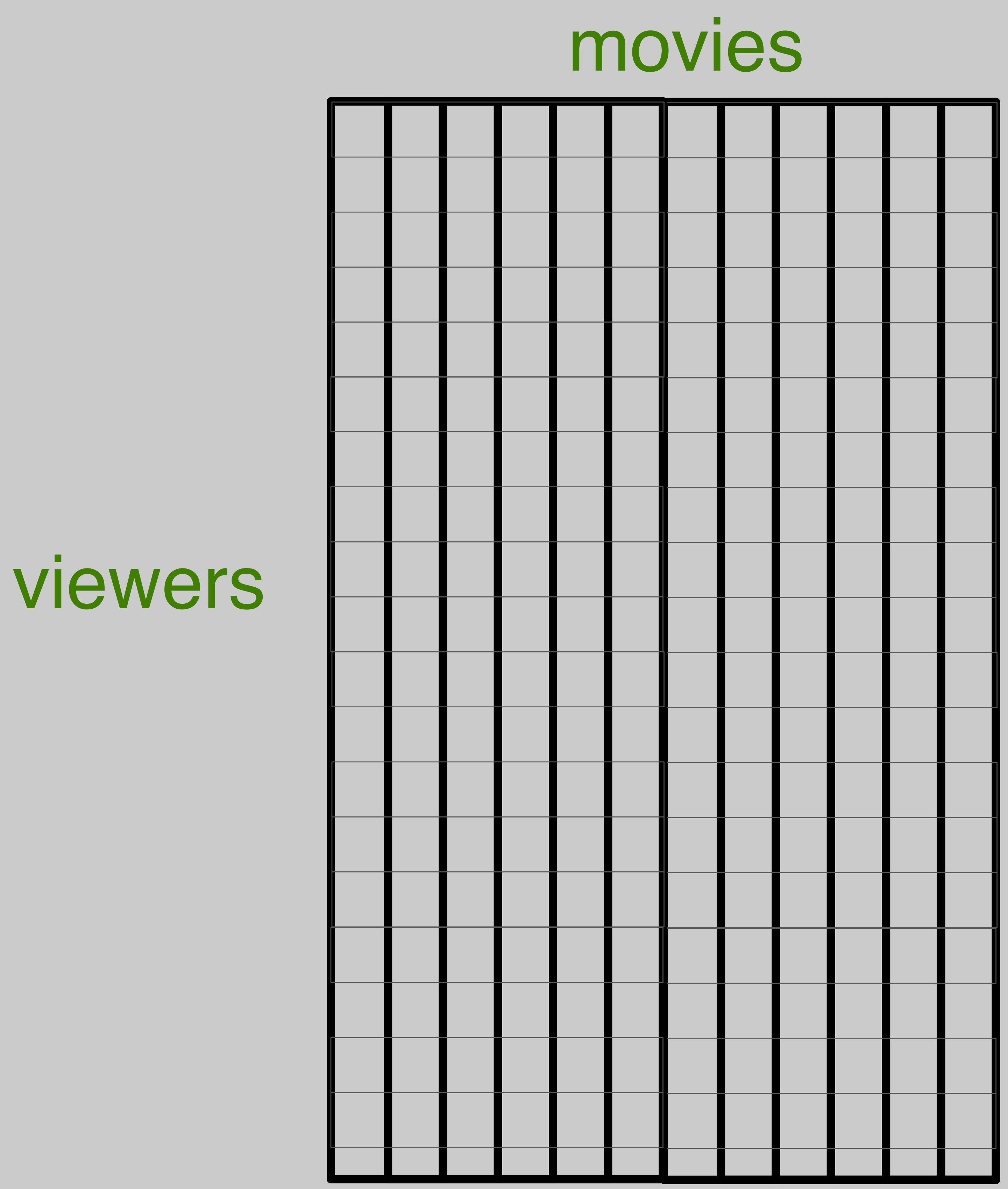
$$||Rx||^2 = \lambda ||x||^2 \quad \Rightarrow \lambda \geq 0$$

✅ If Q can be written as Q = $R^T R$ for real R, then Q is positive semidefinite – eigenvalues greater of equal to zero

Principal Component Analysis

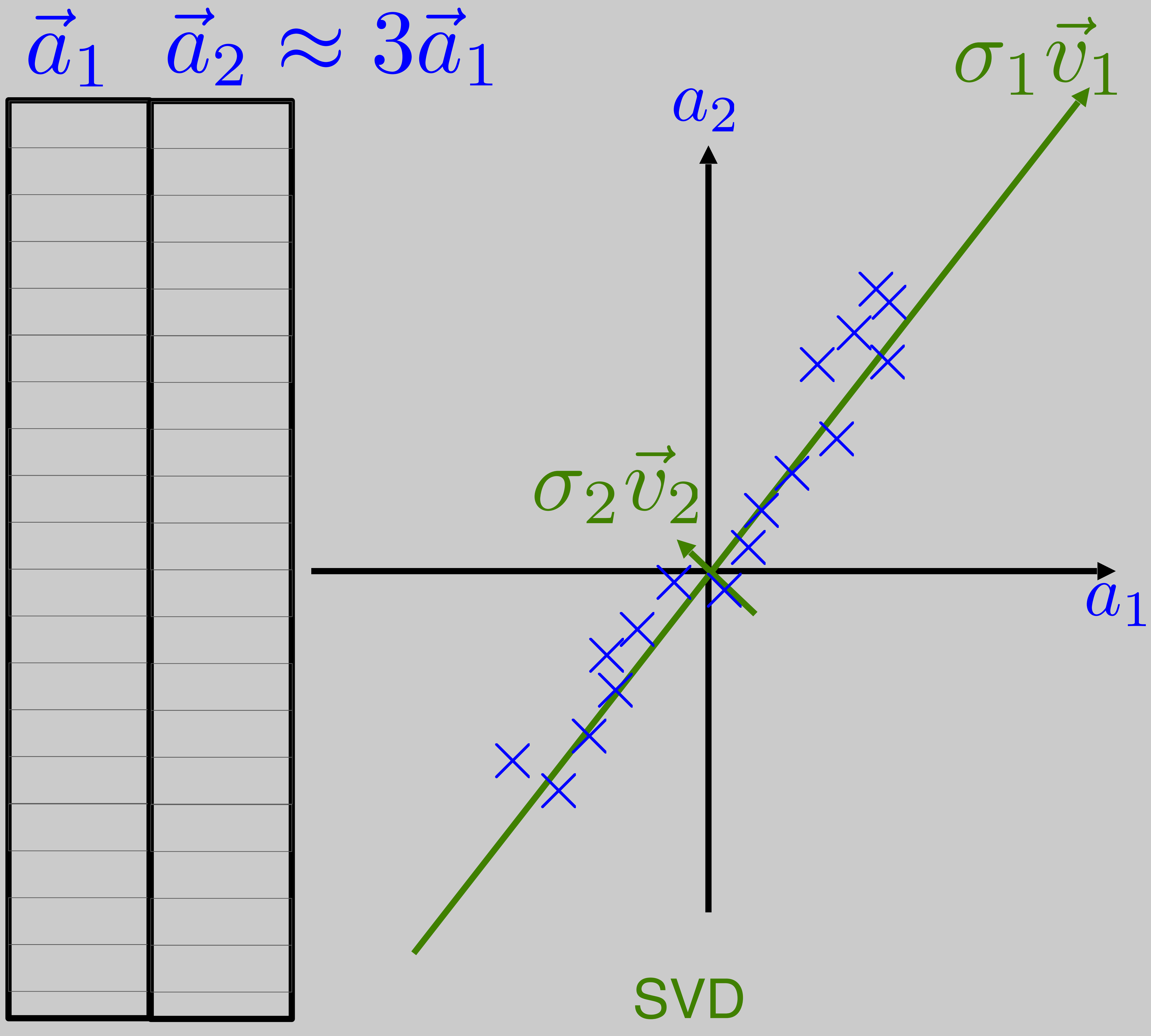
Application of the SVD to datasets to learn features

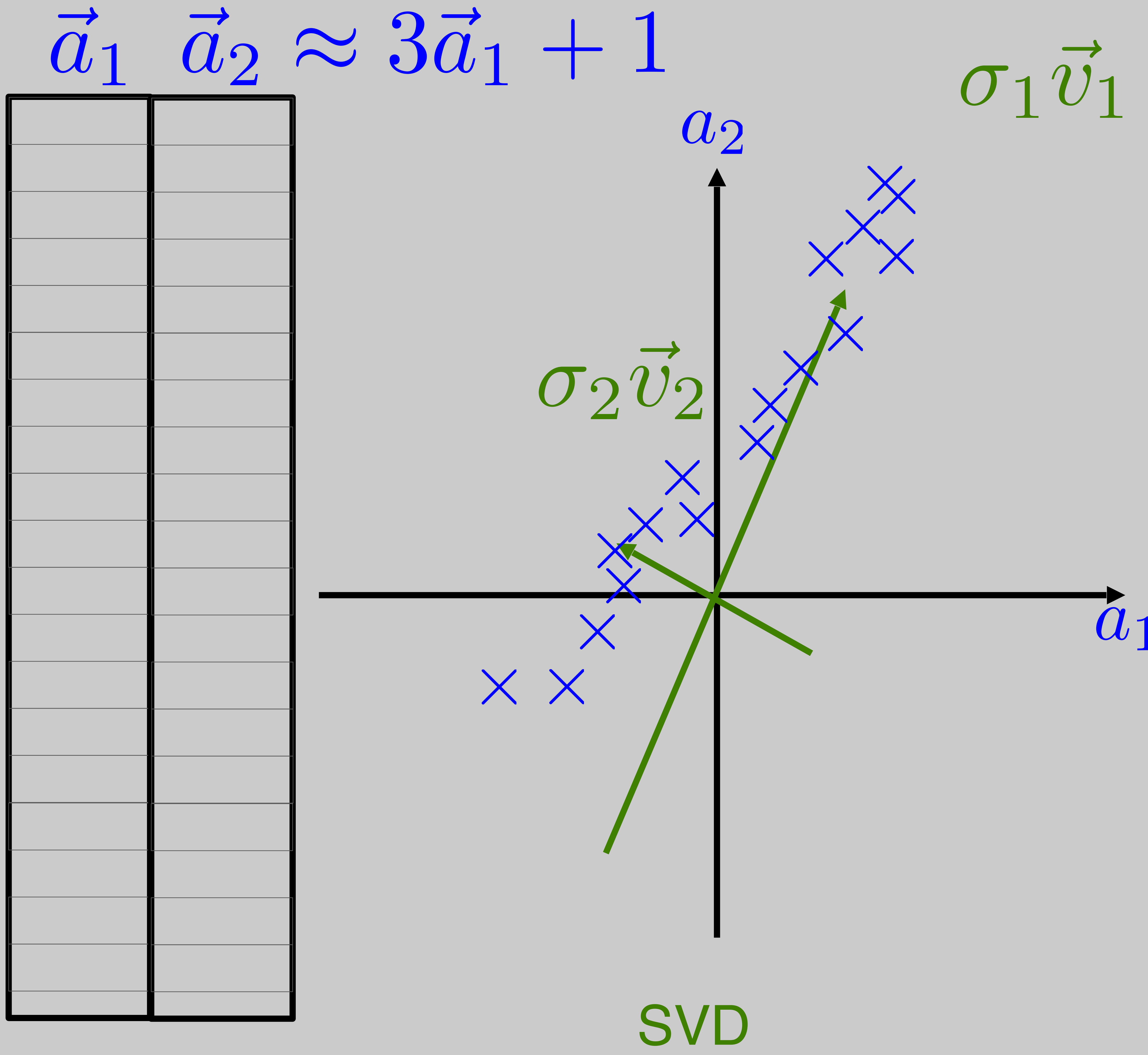
PCA is a tool in statistics and machine learning, which can be computed using SVD



Example -- PCA

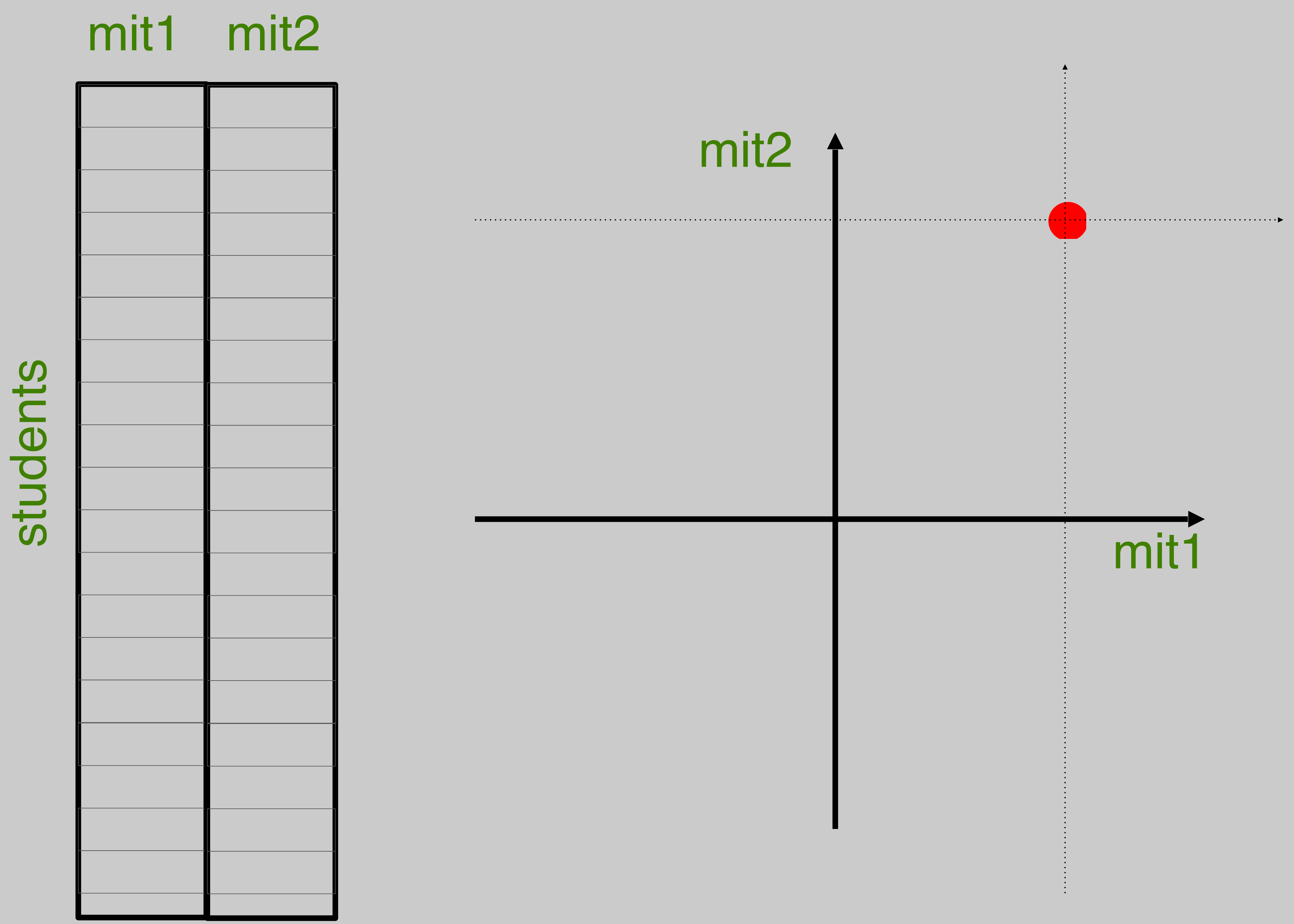
Consider data s.t.





Example -- PCA

Consider miterm data



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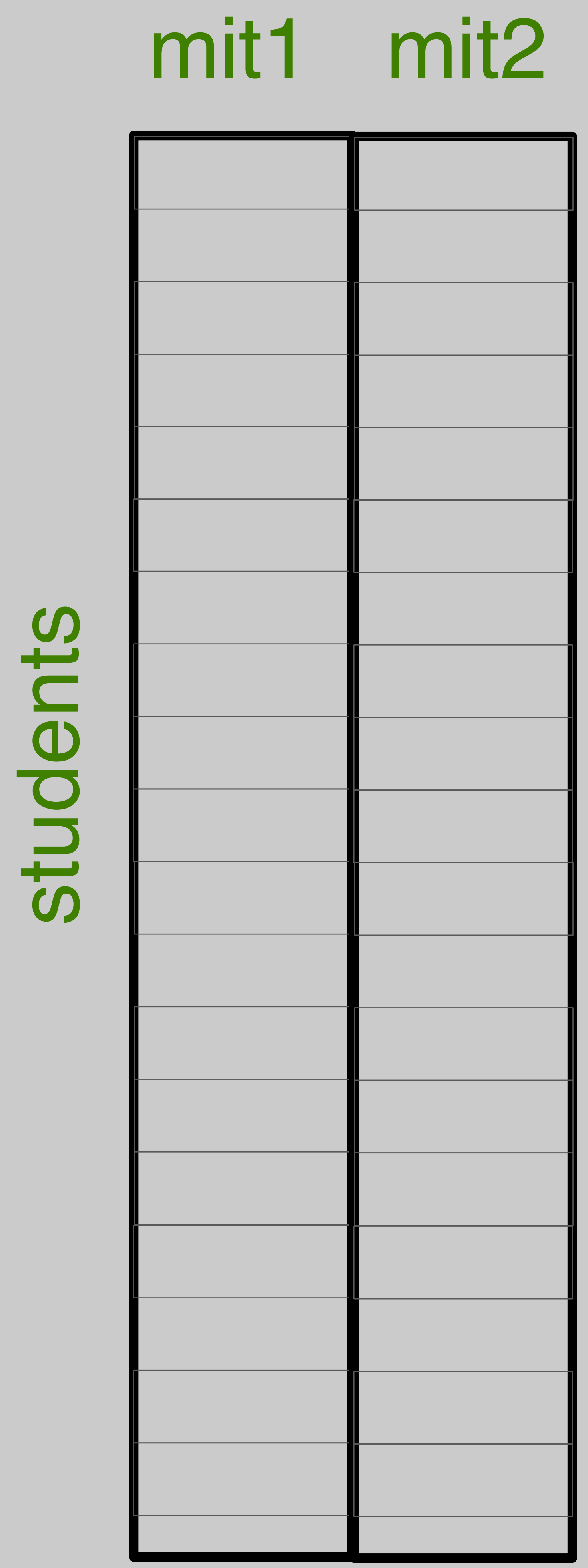

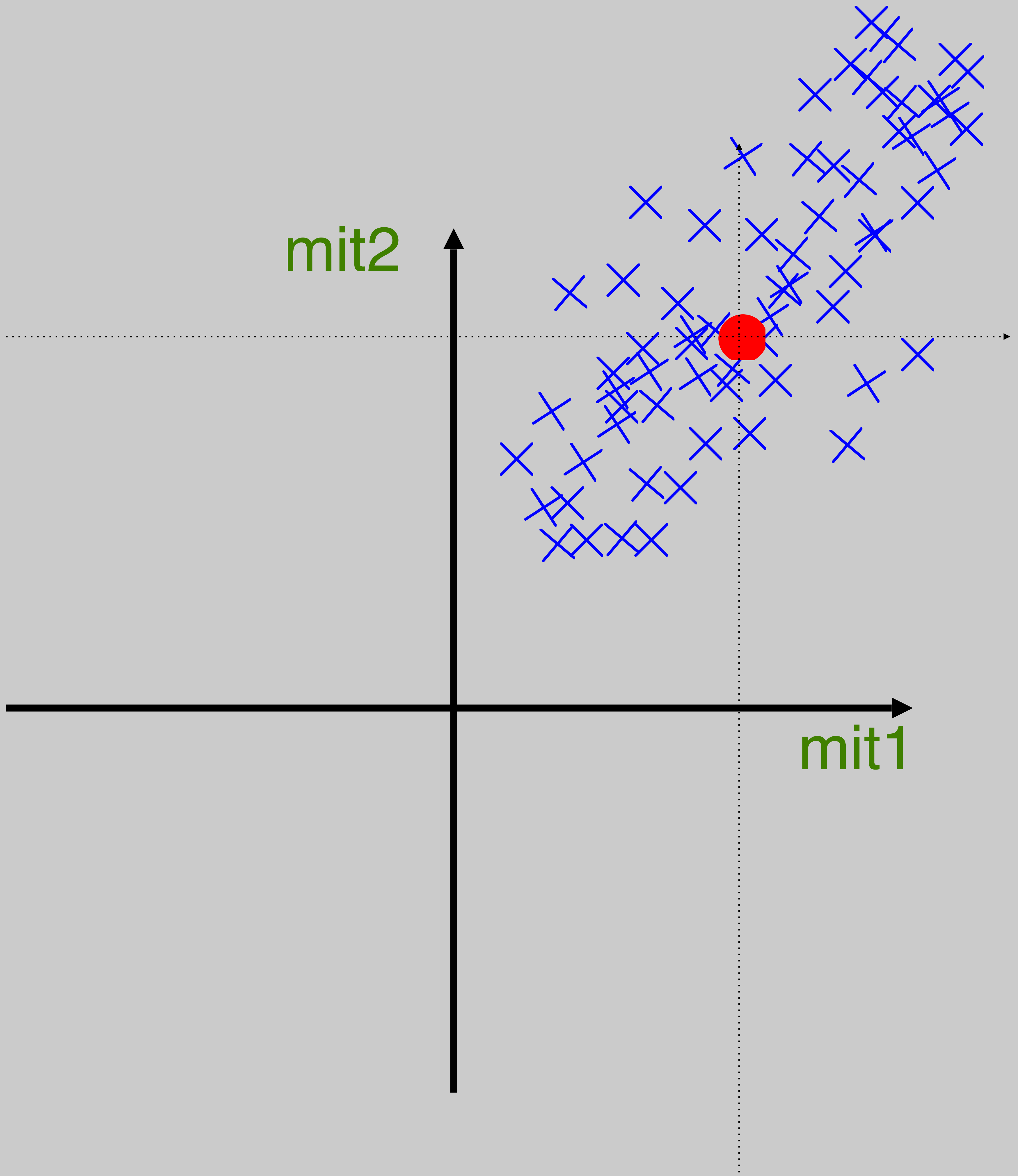

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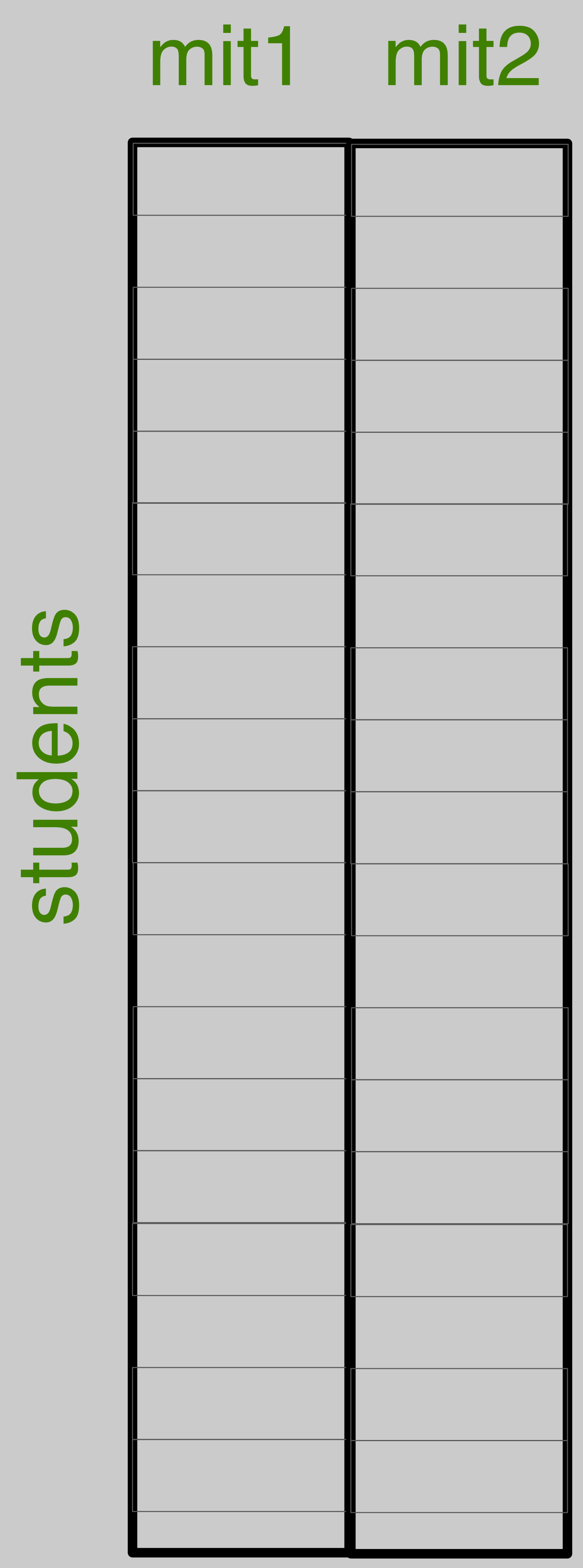

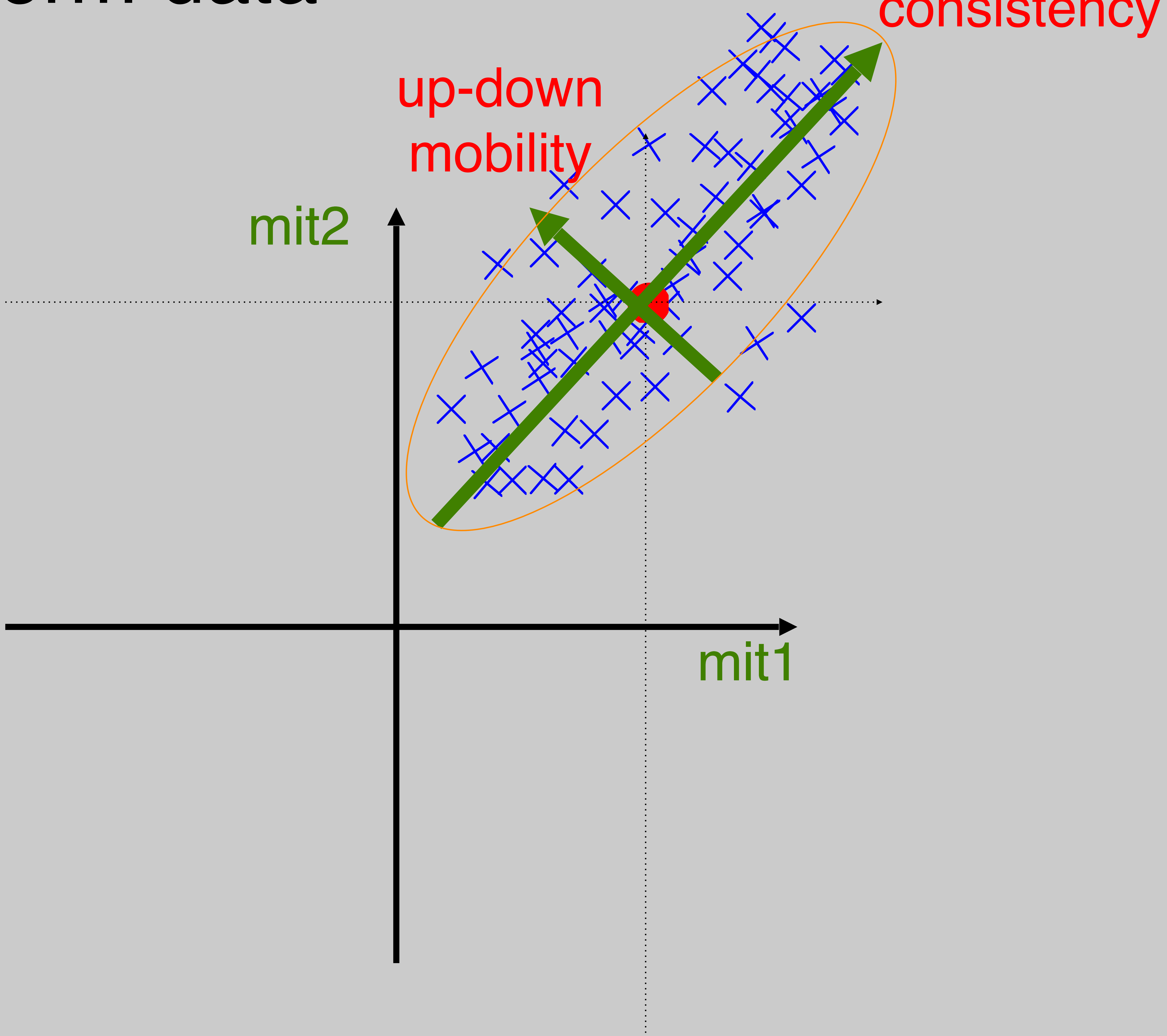

PCA Procedure

Remove averages from column of A

From $A^T A$, find $\sigma_i, \quad \vec{v_i}$

$\vec{v_i}$ are principal components!