EE16B
Designing Information Devices and Systems II

Lecture 11A
Moore Penrose Inverse
Sampling and Interpolation
(Polynomial Interpolation)

Intro

- Last time:
  - Examples of PCA
  - Labeled vs non-labeled clustering
- Today
  - Finish the PCA/SVD Module
    Moore Penrose Pseudo-Inverse
  - New module: Sampling and interpolation
  - Polynomial interpolation

Labeled VS non labeled Classification

Word1

Word2

Word3

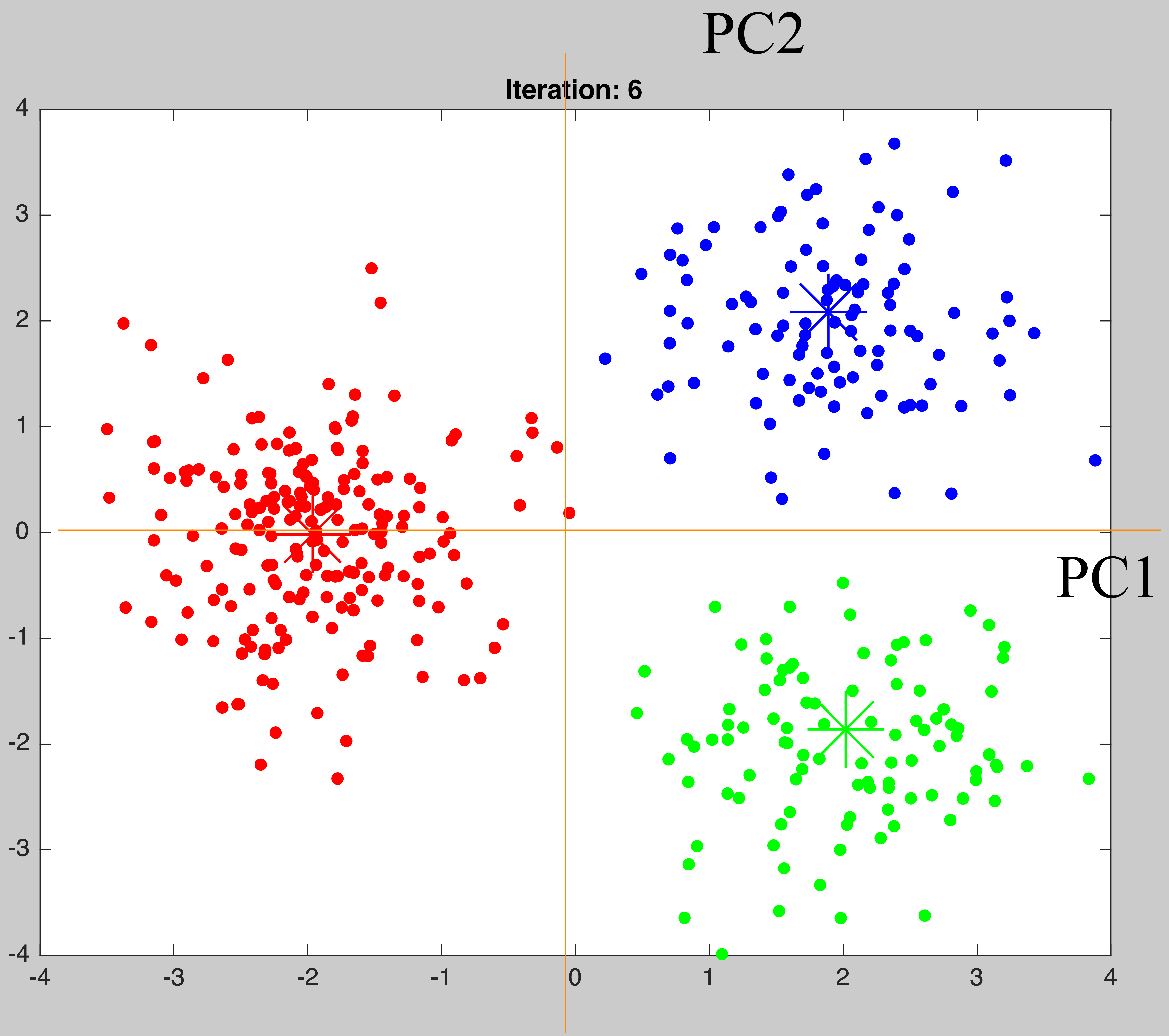
Word4

Word5

Word6

Word7

Word8



Labeled VS non labeled Classification

Word1

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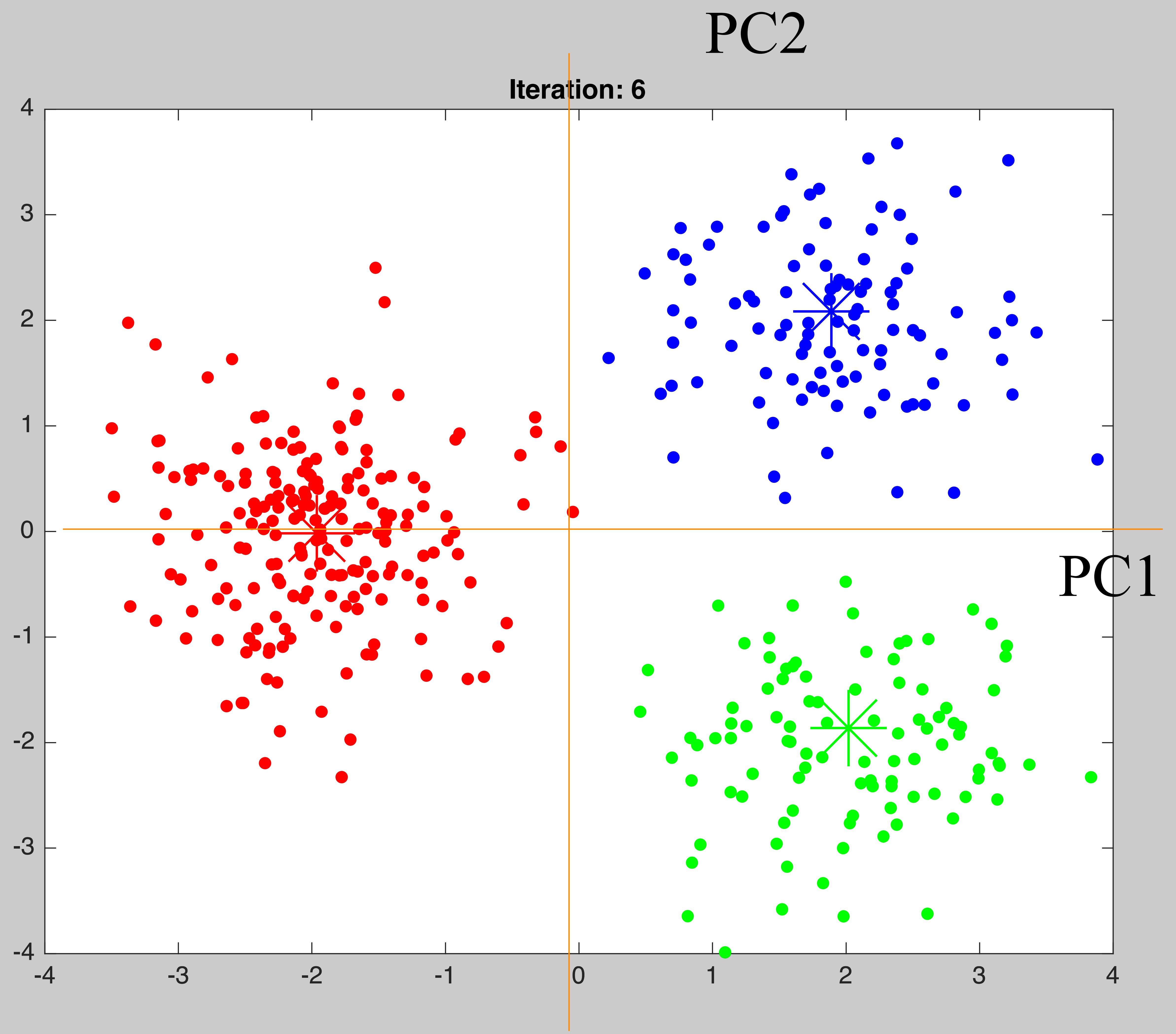


Labeled VS non labeled Classification

"Banana"

"Banana"

"Banana"

"Mango"

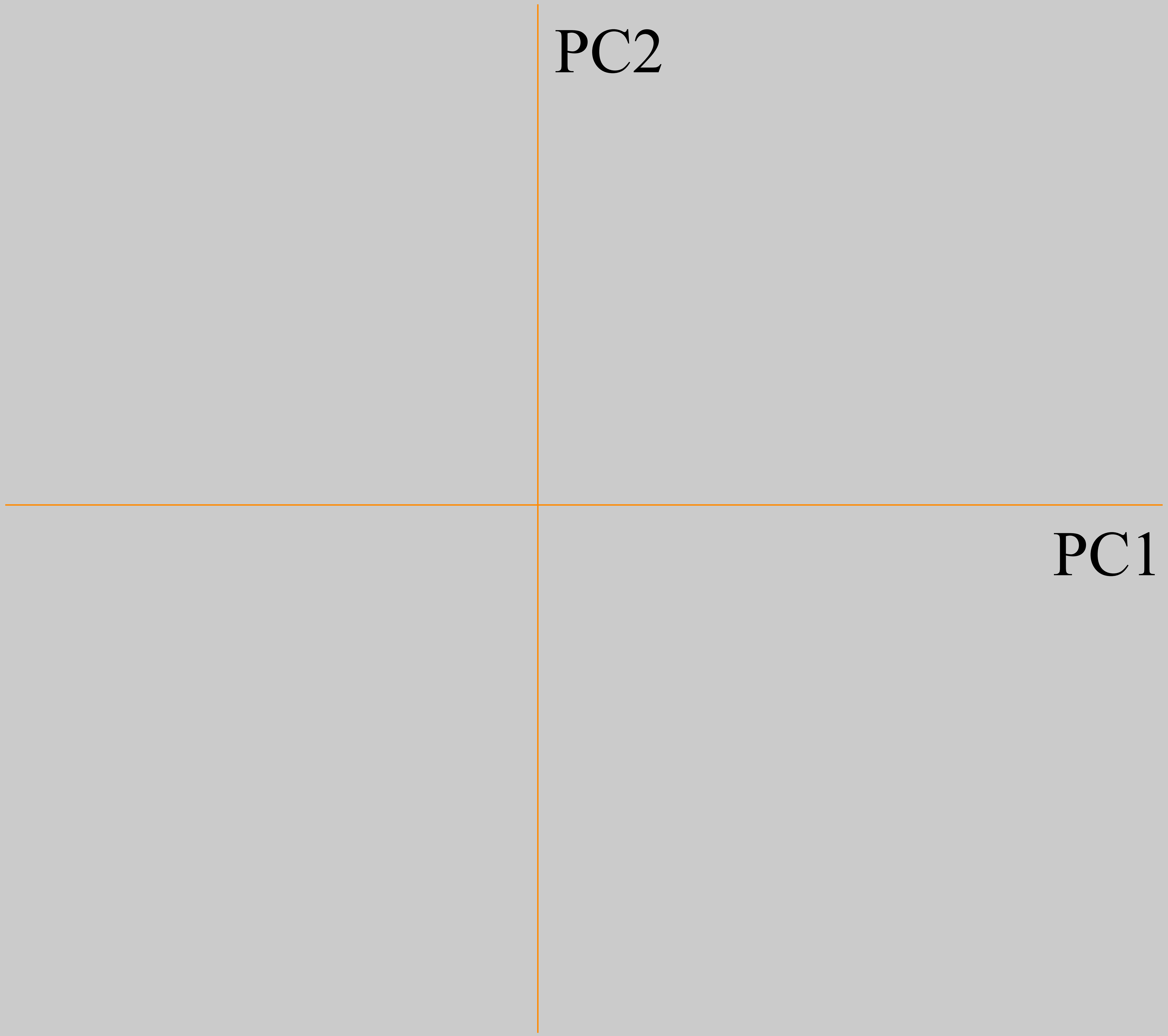
"Mango"

"Mango"

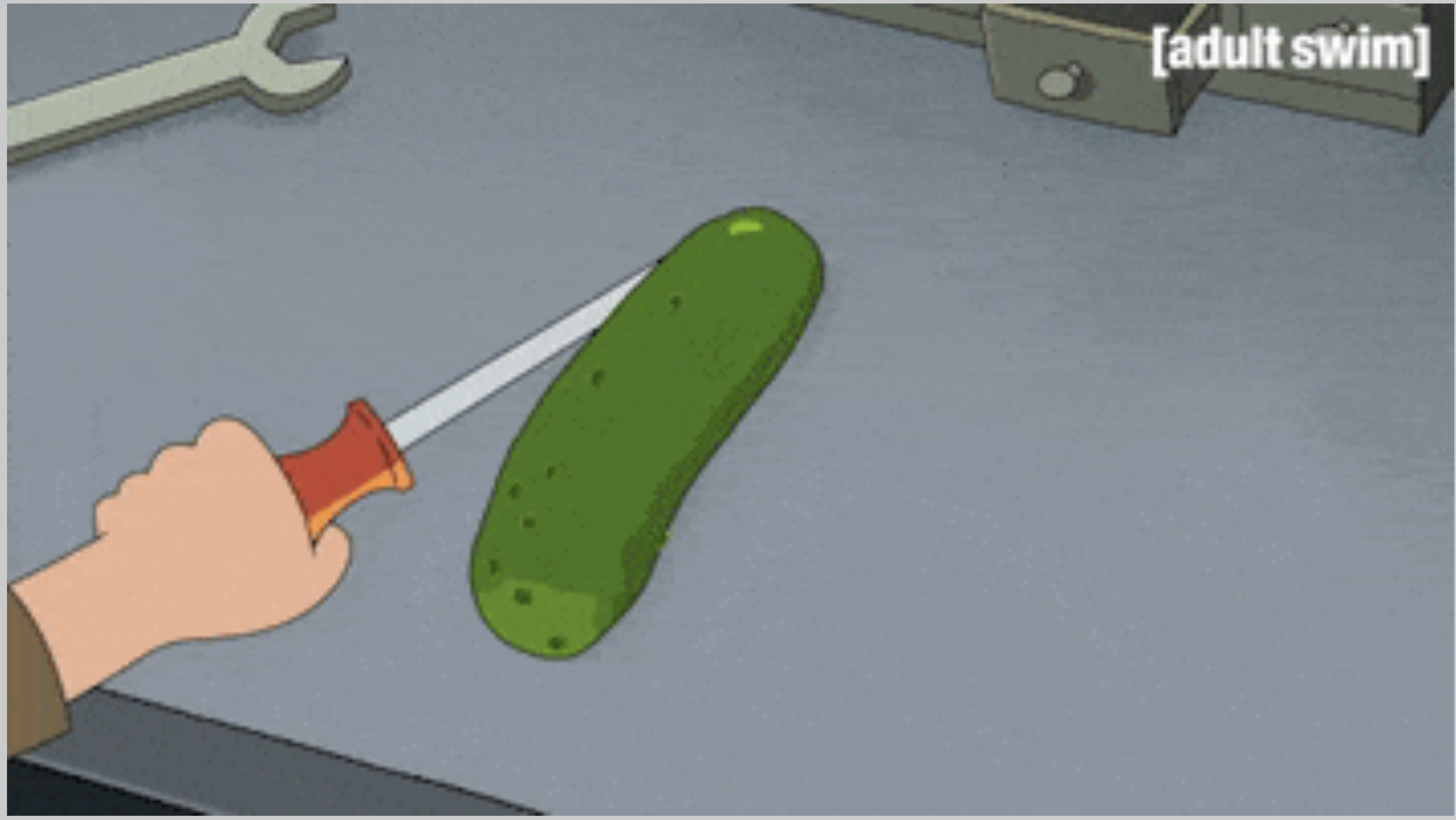
"Chop"

"Chop"

"Chop"



Matrix Inverse, and Pseudo-Inverse



- Square, full rank (N) A:

$$Ax = y$$

$$x = A^{-1}y$$

$$A^{-1}A = AA^{-1} = I_{N \times N}$$

- Tall, full rank (N) A (Least squares):

$$Ax = y$$

$$x = (A^TA)^{-1}A^Ty$$

$$(A^TA)^{-1}A^T\ A = I_{N \times N}$$

Also, left inverse

Pseudo Inverse and the SVD

$$Ax = y$$

$$x = (A^TA)^{-1}A^Ty$$

- SVD:

$$A = U_1 \; S \; V_1^T$$

$$(A^TA) = V_1SU_1 \;\; U_1SV_1^T = V_1S^2V_1^T$$

$$(A^TA)^{-1} = \; V_1S^{-2}V_1^T$$

$$(A^TA)^{-1}A^T = V_1S^{-2}V_1^T \;\; V_1^TSU_1^T = V_1S^{-2}SU_1^T =$$

$$= V_1S^{-1}U_1^T = A^\dagger$$

Moore-Penrose Pseudo-inverse

Uder-determined Linear Systems

- Fat, full rank (N) A:

$$Ax = y$$ Infinite solutions!

- Claim:

$$AA^{\dagger} = I_{N \times N}$$ Also, right inverse

- SVD:

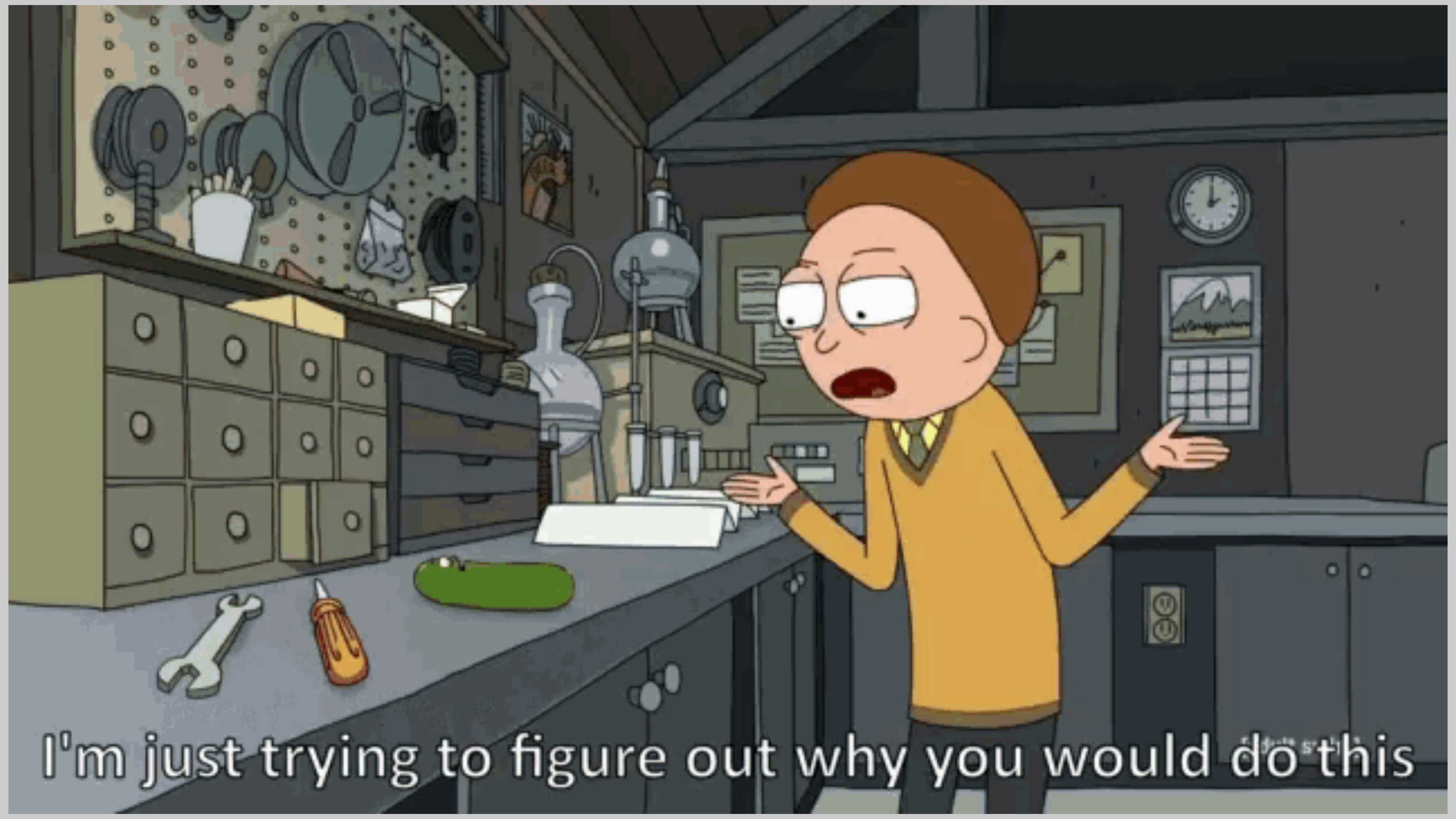
$$A = U_1 \; S \; V_1^T$$

$$AA^{\dagger} = U_1 S V_1^T \; V_1 S^{-1} U_1^T = U_1 S S^{-1} U_1^T = U_1 U_1^T = I_{N \times N}$$

Uder-determined Linear Systems

- Fat, full rank (N) A:

$$Ax = y$$ Infinite solutions!



- Fact:

$$AA^{\dagger} = I_{N\times N}$$

- A Solution:

$$\hat{x} = A^{\dagger}y$$

$$A\hat{x} = AA^{\dagger}y = Iy = y$$

- A minimum-norm solution!

$$||\hat{x}|| < ||\vec{x}|| \qquad \forall A\vec{x} = y$$

Minimum - Norm

- A minimum-norm solution! $\hat{x} = A^{\dagger} y$

$$||\hat{x}|| < ||\vec{x}|| \qquad \forall A\vec{x} = y$$

- Proof outline:

$$\vec{x} = \hat{x} + \tilde{x}$$

  - Show that $A\tilde{x} = 0$
  - Show that $\hat{x}^T \tilde{x} = 0$
  - Show that $||\vec{x}||^2 = ||\hat{x} + \tilde{x}||^2 > ||\hat{x}||^2$

Back to Open Loop Control

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

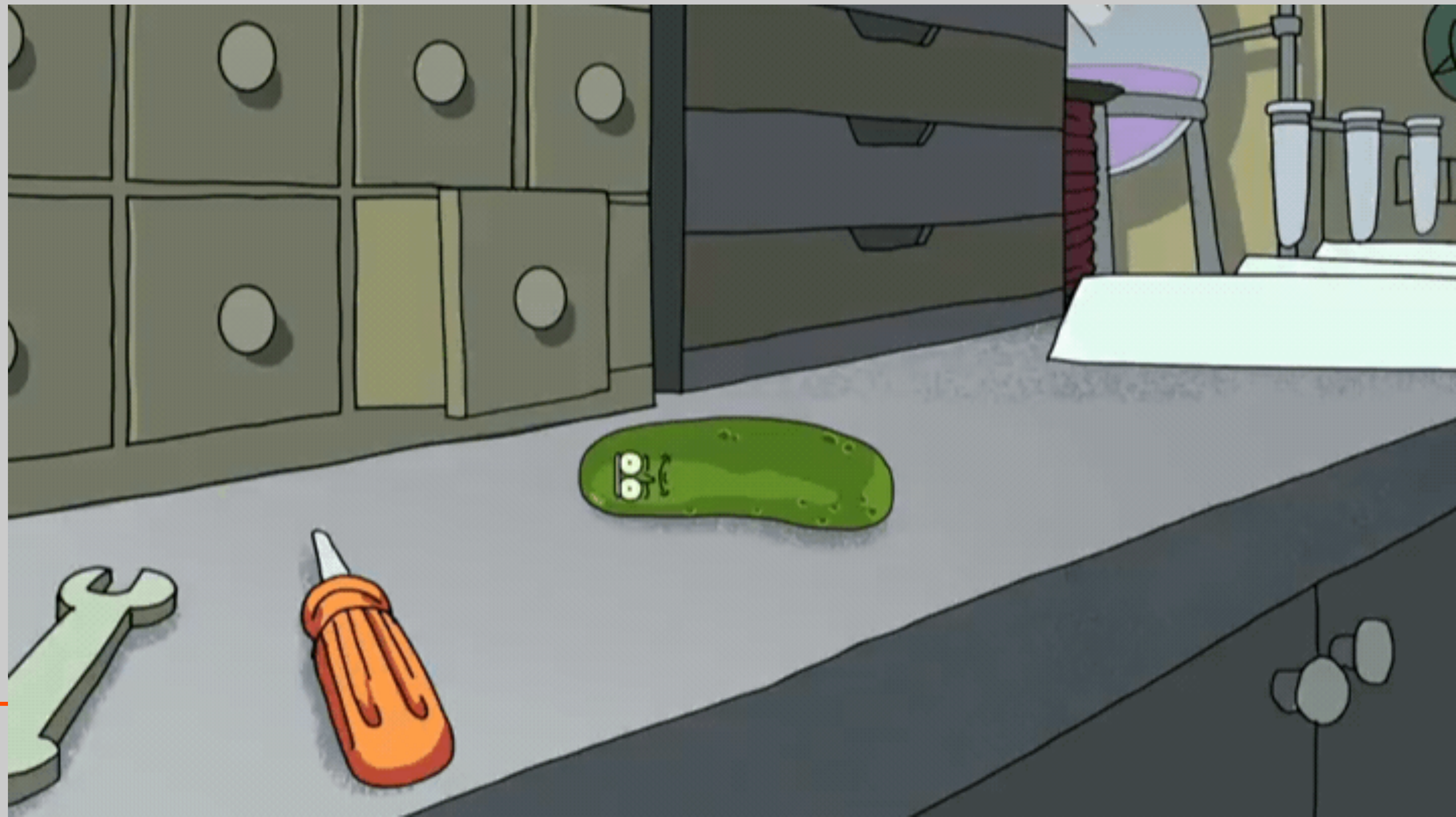
System is controllable. What if t >> N?
Whats a good sequence of u(t)?

Summary:

- Moore penrose Pseudo Inverse

$$A^{\dagger} = V_1 S^{-1} U_1^T$$

- For tall matrices is the least squares solution!
- For fat matrices is the least-norm solution!
- Computed via the SVD!

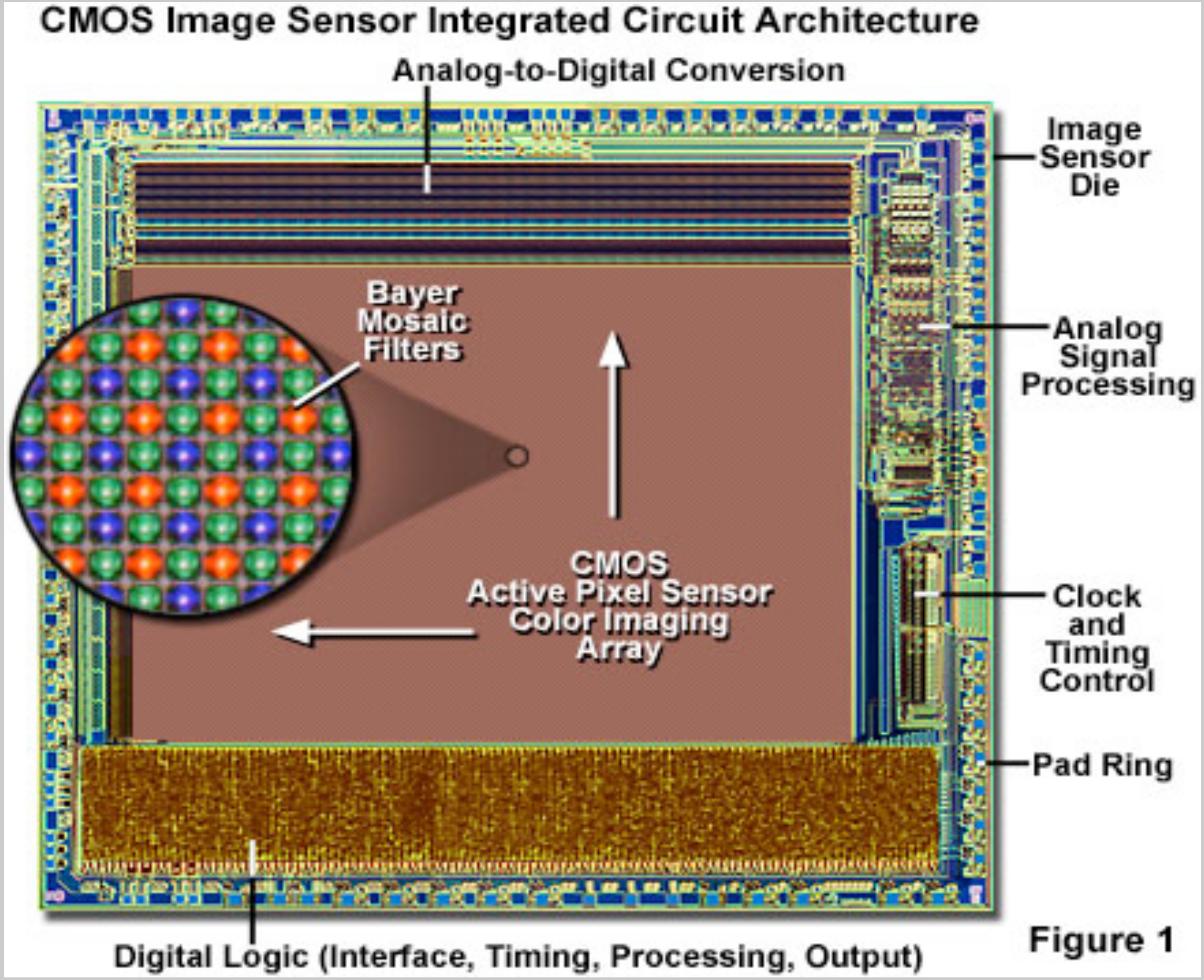


Sampling and Interpolation

- (Digital) Signal Processing - Only going to touch the surface
  – EE120, EE123, EE145A, EE121, EE225A, EE225B, CS 194-026, CS280

The Real World	Acquisition System (Physics/Physical)	Analog to Digital Conversion	Digital "magic": Processing, manupulation, interpretation Storage	Reconstruction & Digital to Analog Conversion
RF Audio Light Energy …	Conversion of X to electric Analog Filtering Amplification Modulation Microphone CCD/CMOS Accelerometers Antennas Radio-receivers …	Discrete values Discrete representation	Digital filtering Decoding Coding Compression Learning Rendering …	Audio Light RF Magnetic Display Communications …

Example Digital Imaging Camera

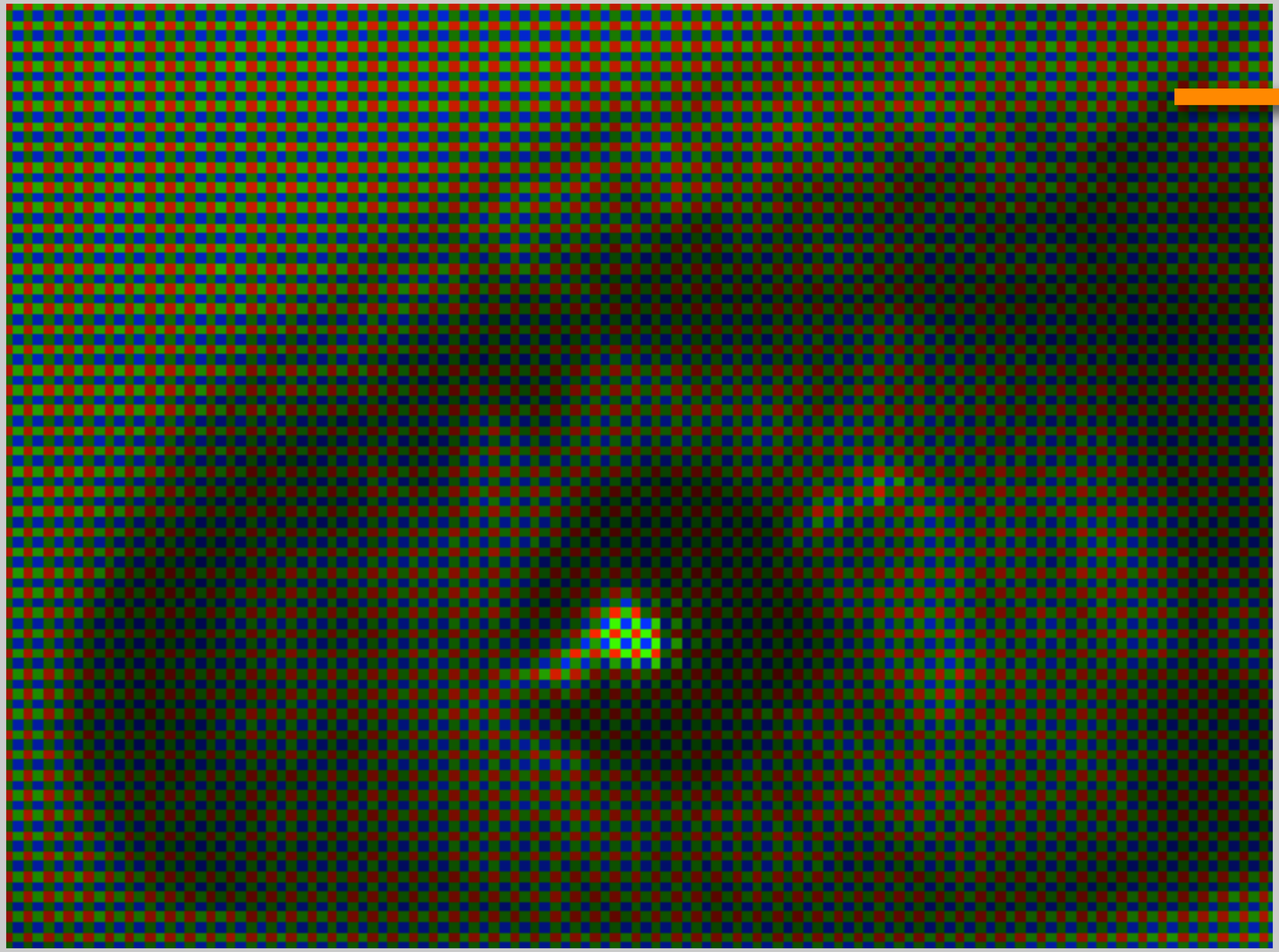


Focus/exposure Control → preprocessing → white-balancing → demosaic → Color transform → Post-processing → Compression

http://micro.magnet.fsu.edu/primer/digitalimaging/cmosimagesensors.html

Example: Digital Camera

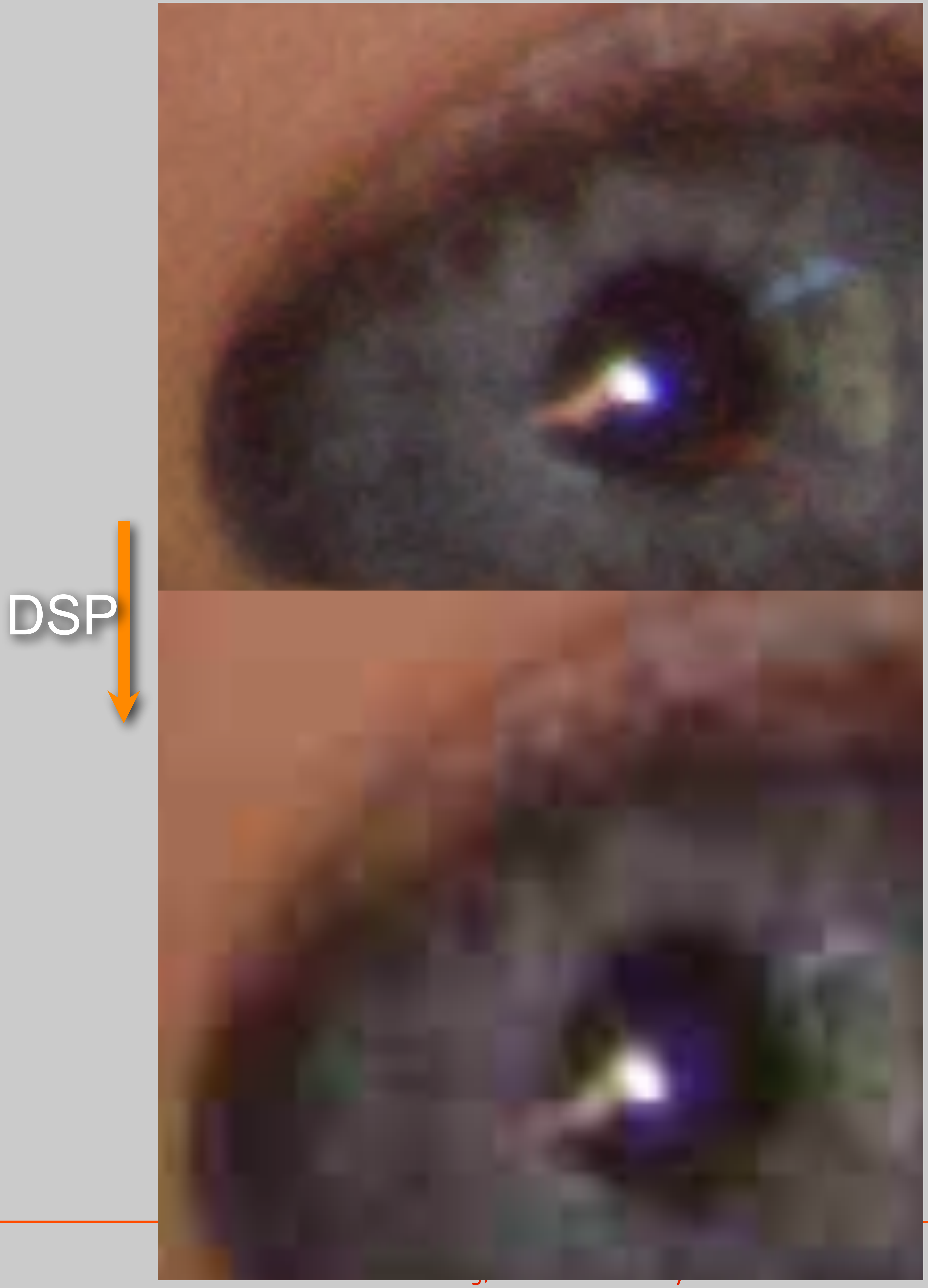
DSP



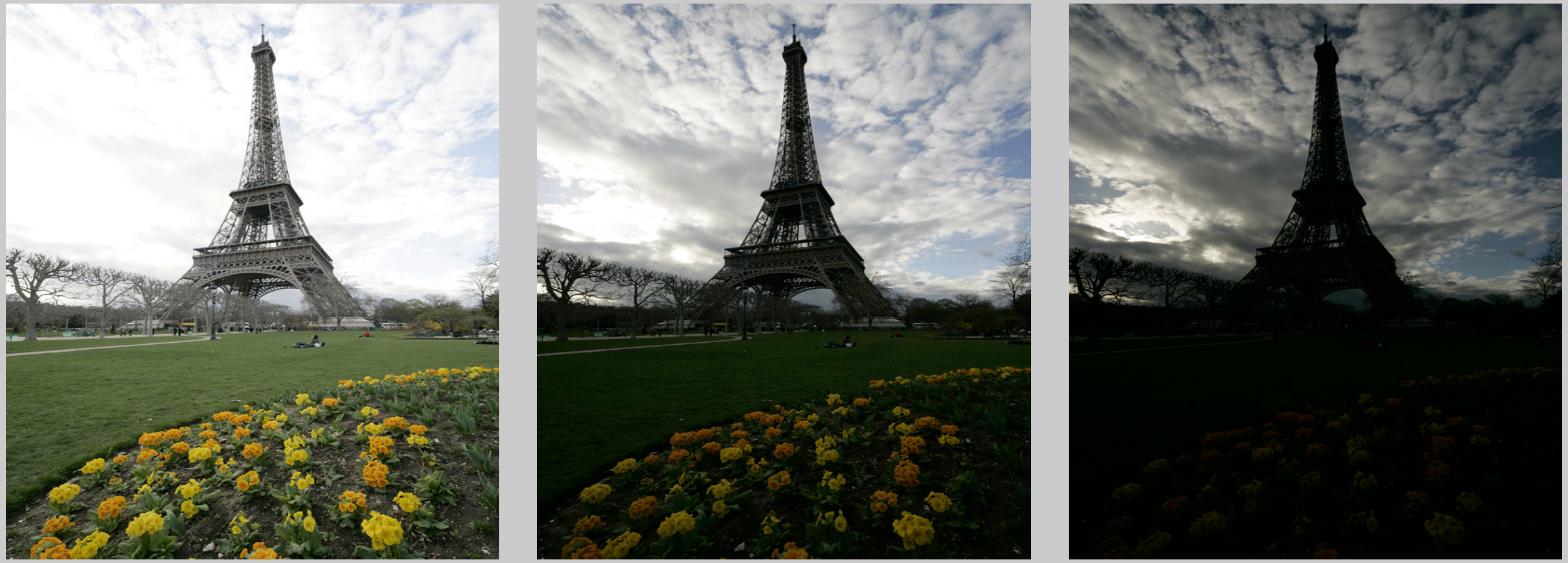
Example: Digital Camera

- Compression of 40x without perceptual loss of quality.

- Example of slight overcompression: difference enables x60 compression!



Computational Photography



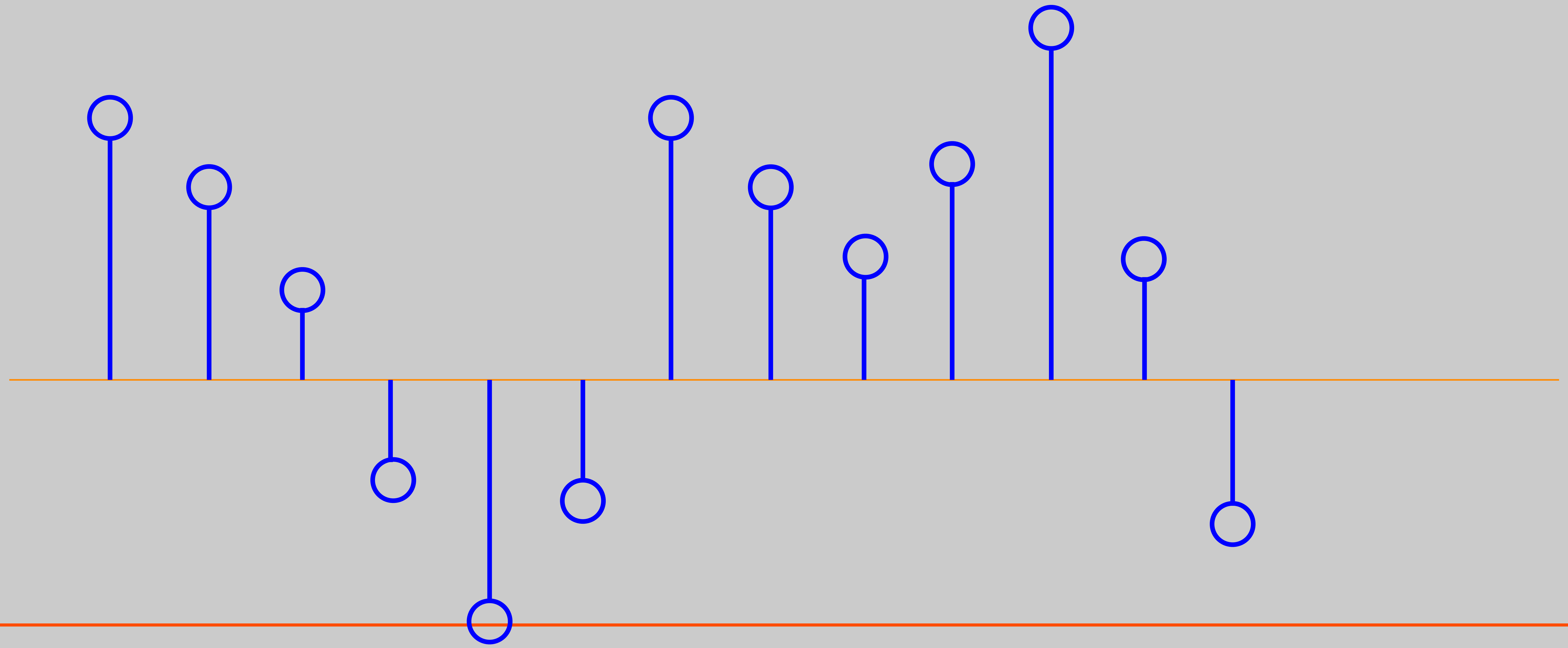
Implemented in all smart phones (HDR)

*www.hdrsoft.com

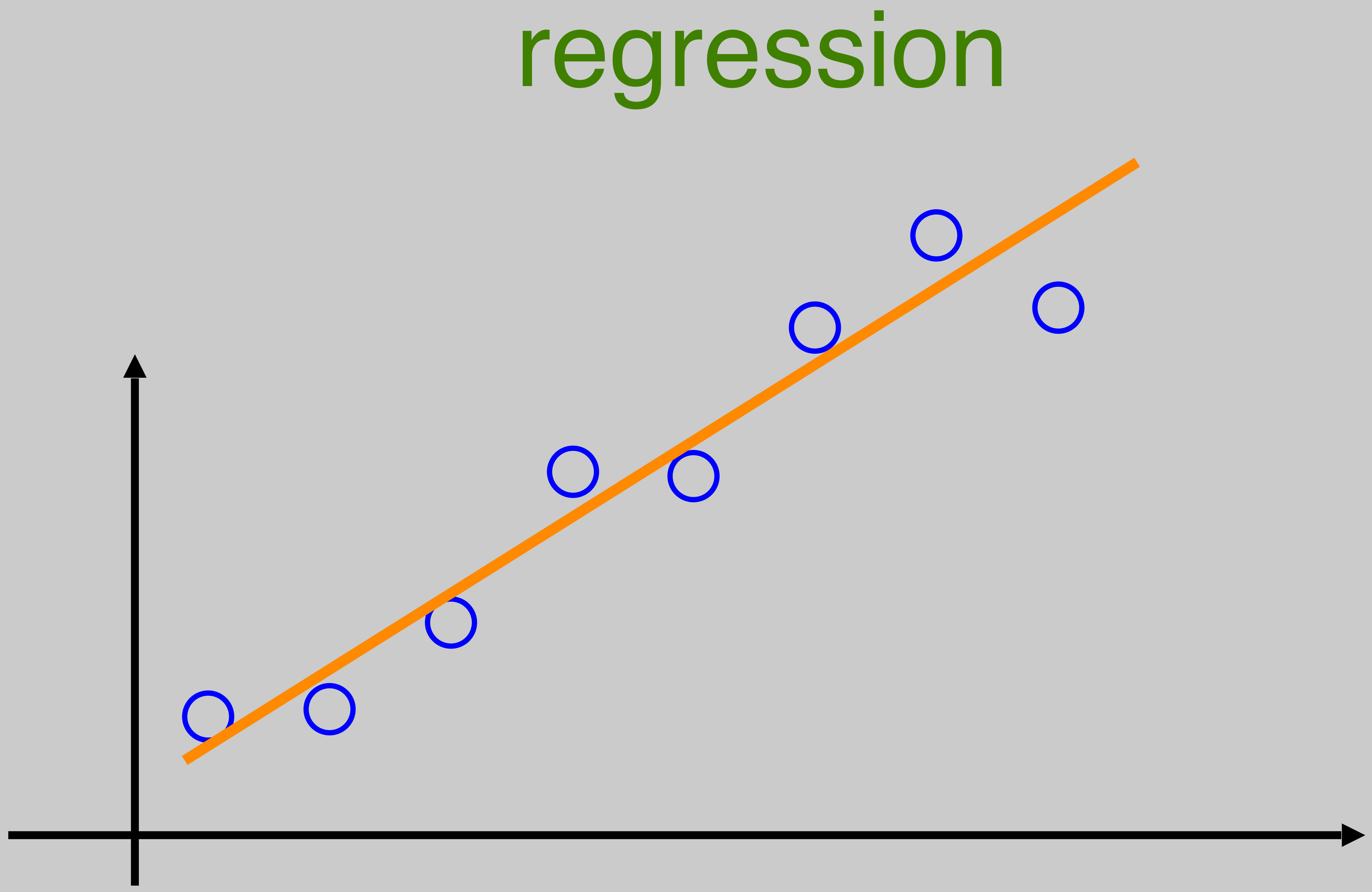
Interpolation

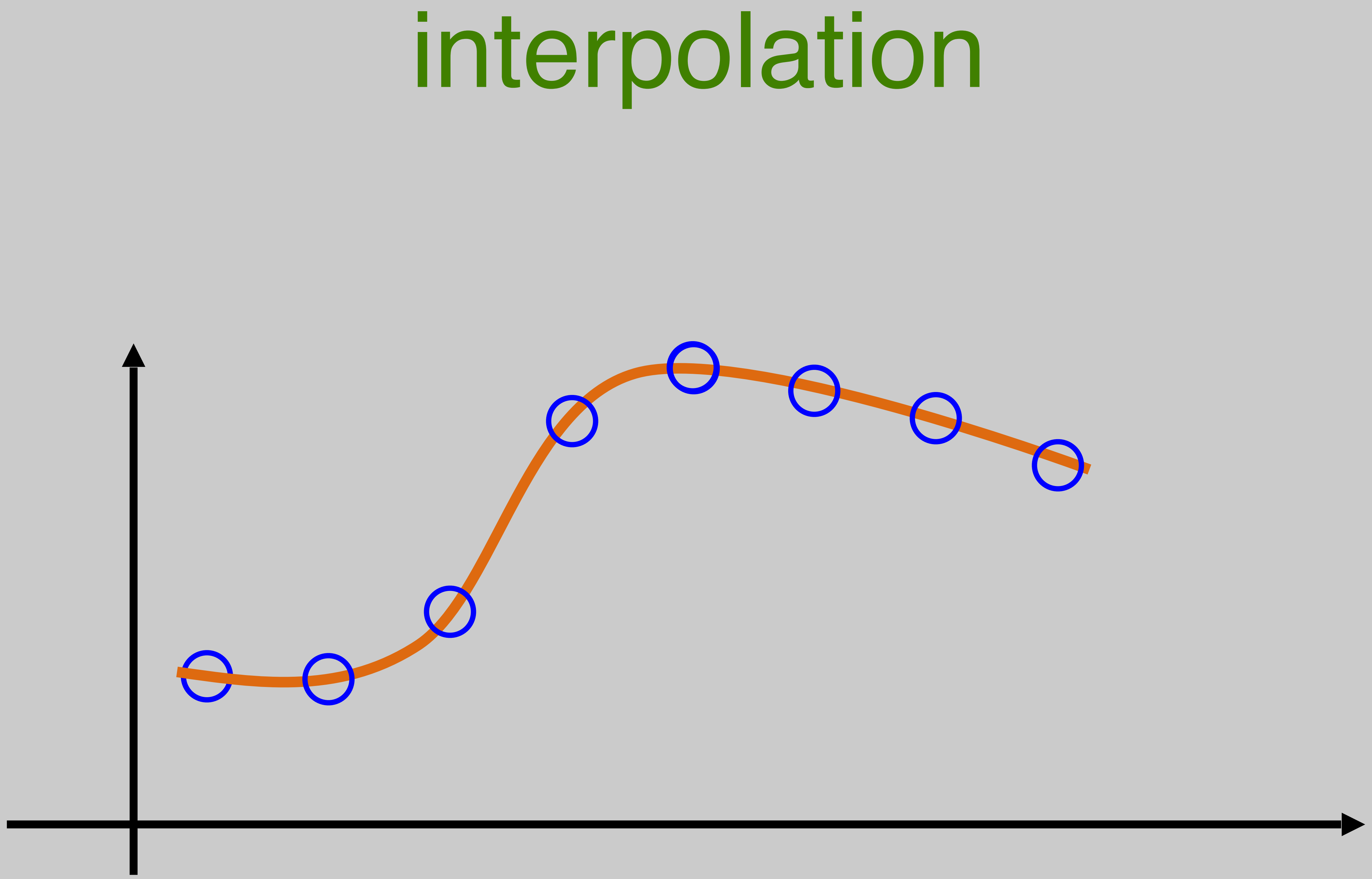
- Given data points (x_i, y_i) i=1,2,… n find a continuous function that exactly matches the points.

$$y = f(x) \qquad \Rightarrow f(x_i) = y_i, \qquad i = 1,\ 2,\ 3\ ,\cdots,\ n$$



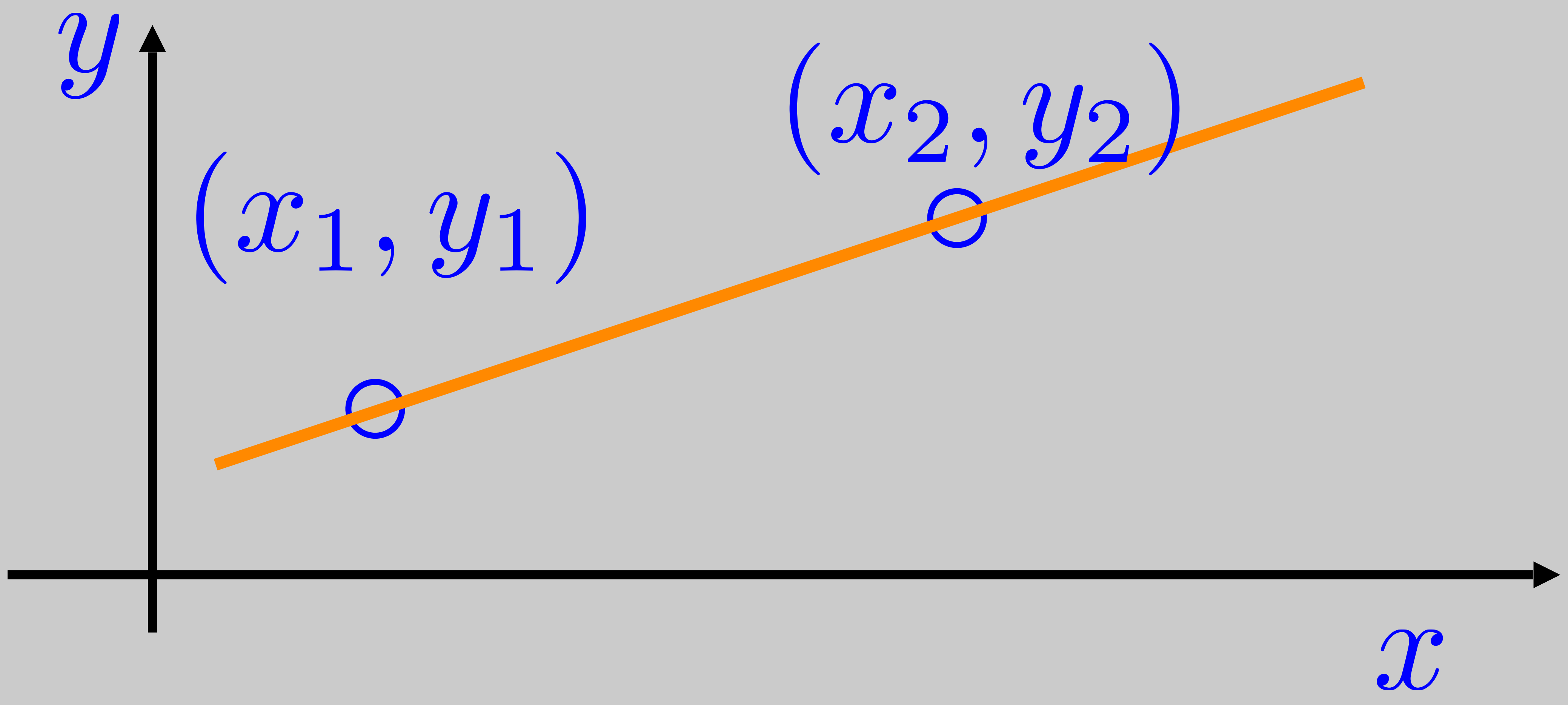
Regression Vs Interpolation





Polynomial Interpolation

- Assumption:
  - Data are samples of a polynomial function (smooth)
  - Lowest order polynomial that exactly fits points

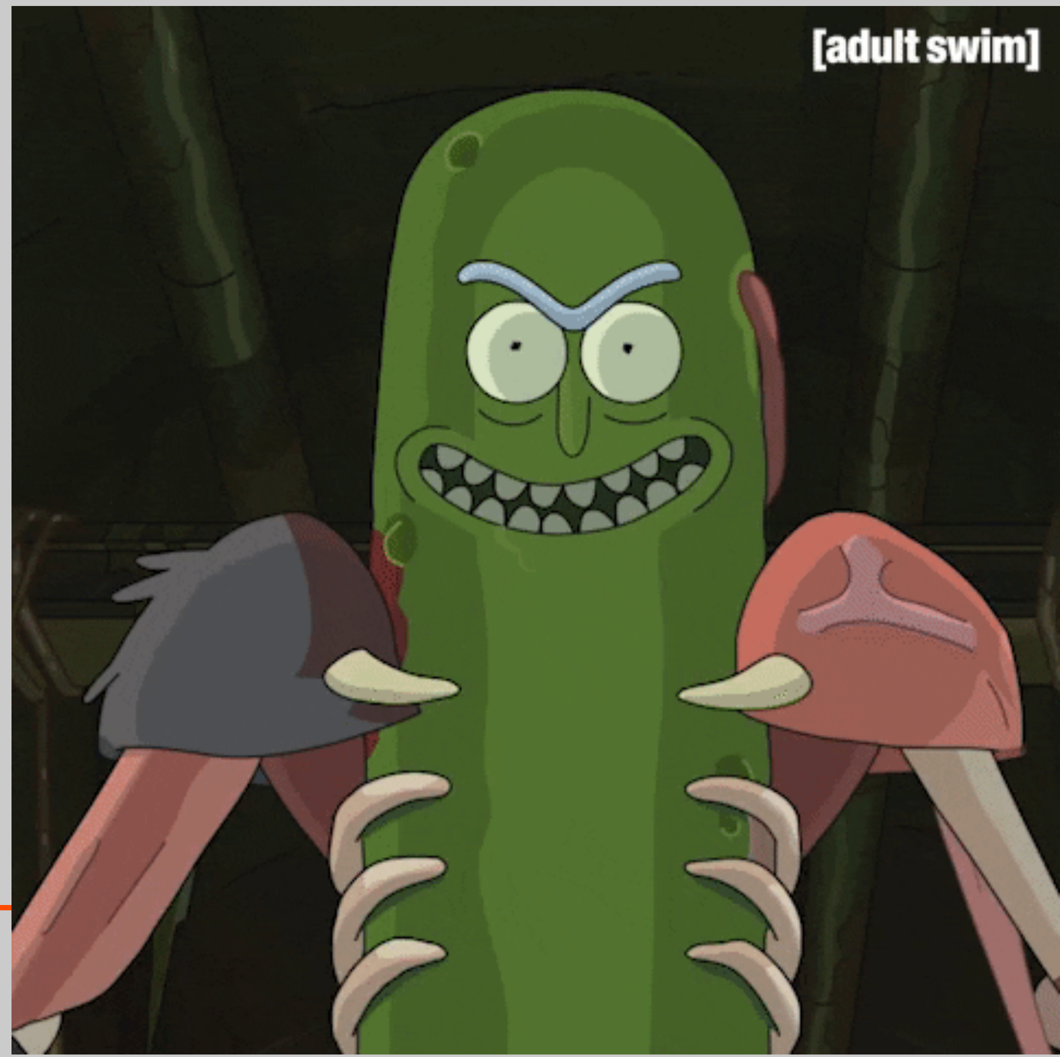


$$y = a_0 + a_1 x$$

$$a_0 + a_1 x_1 = y_1$$

$$a_0 + a_1 x_2 = y_2$$

$$\rightarrow \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$



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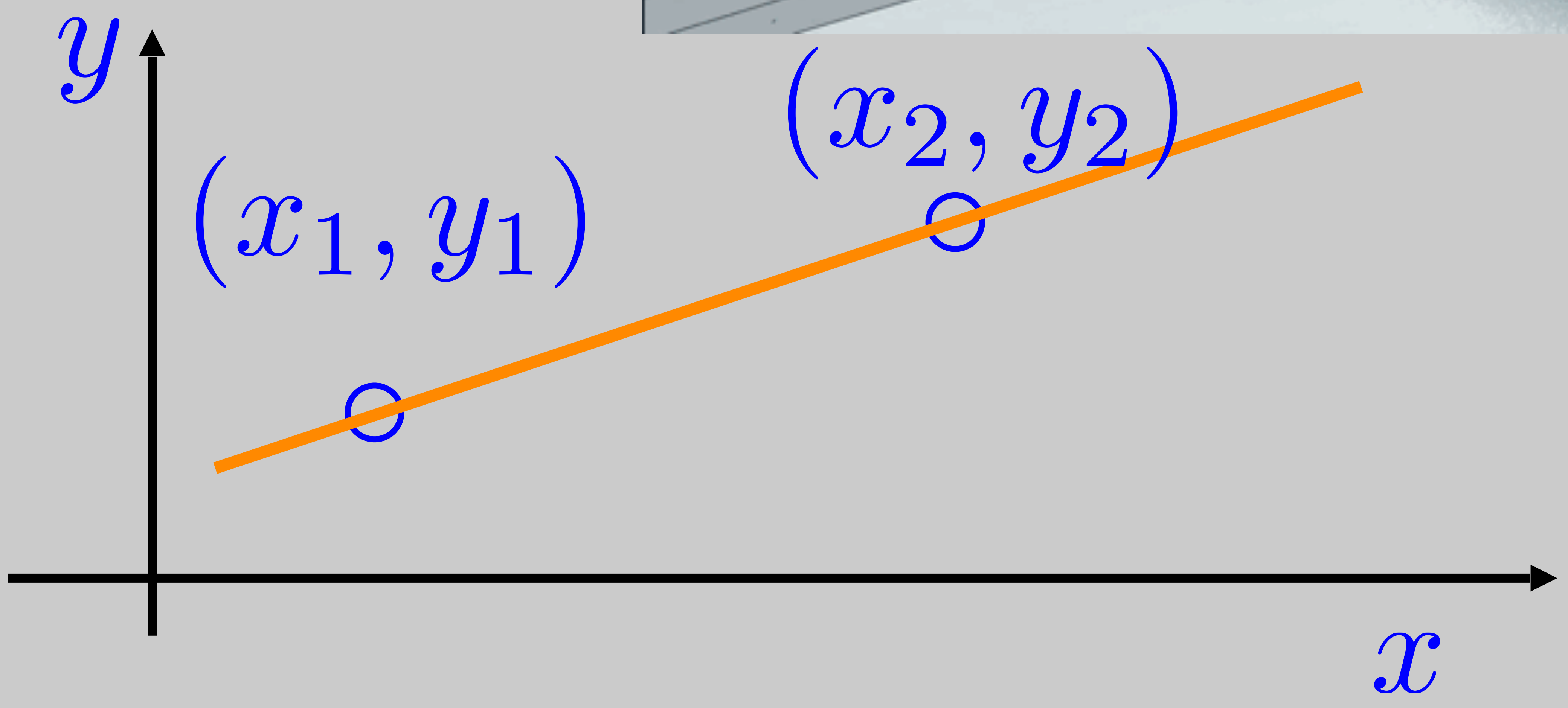


$$y = a_0 + a_1 x$$

$$\begin{aligned} a_0 + a_1 x_1 &= y_1 \\ a_0 + a_1 x_2 &= y_2 \end{aligned} \rightarrow \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

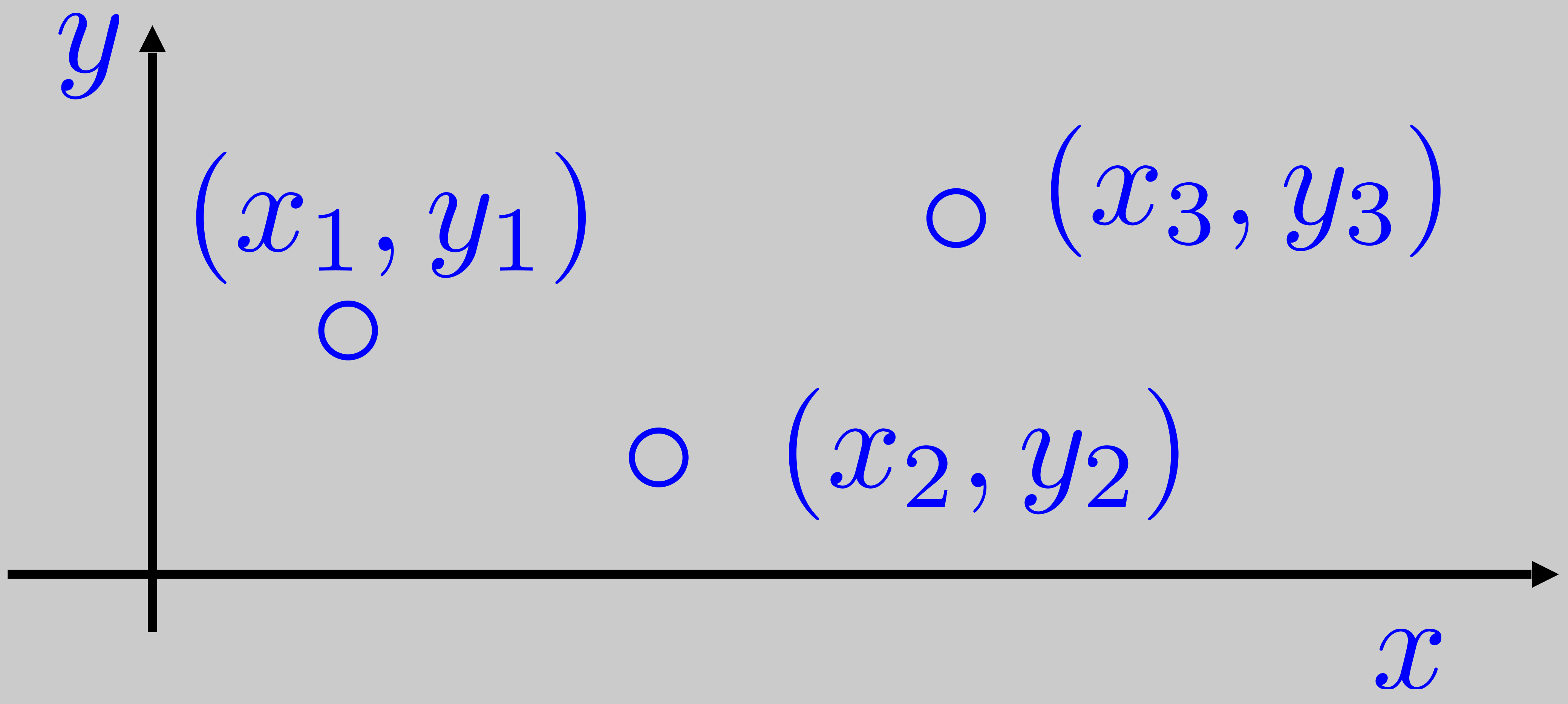
Invertible if $x_1 \neq x_2$

Polynomial Interpolation

$$(x_1, y_1)\ (x_2, y_2)\ (x_3, y_3)$$

$$y = a_0 + a_1 x + a_2 x^2 \rightarrow \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 \neq x_j$$



Polynomial Interpolation

- Given n distinct points, then there's exist a unique (n-1) order polynomial passing through them

$$y = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$$

$$\rightarrow \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Polynomial Interpolation

- Given n distinct points, then there's exist a unique (n-1) order polynomial passing through them

$$y = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$$

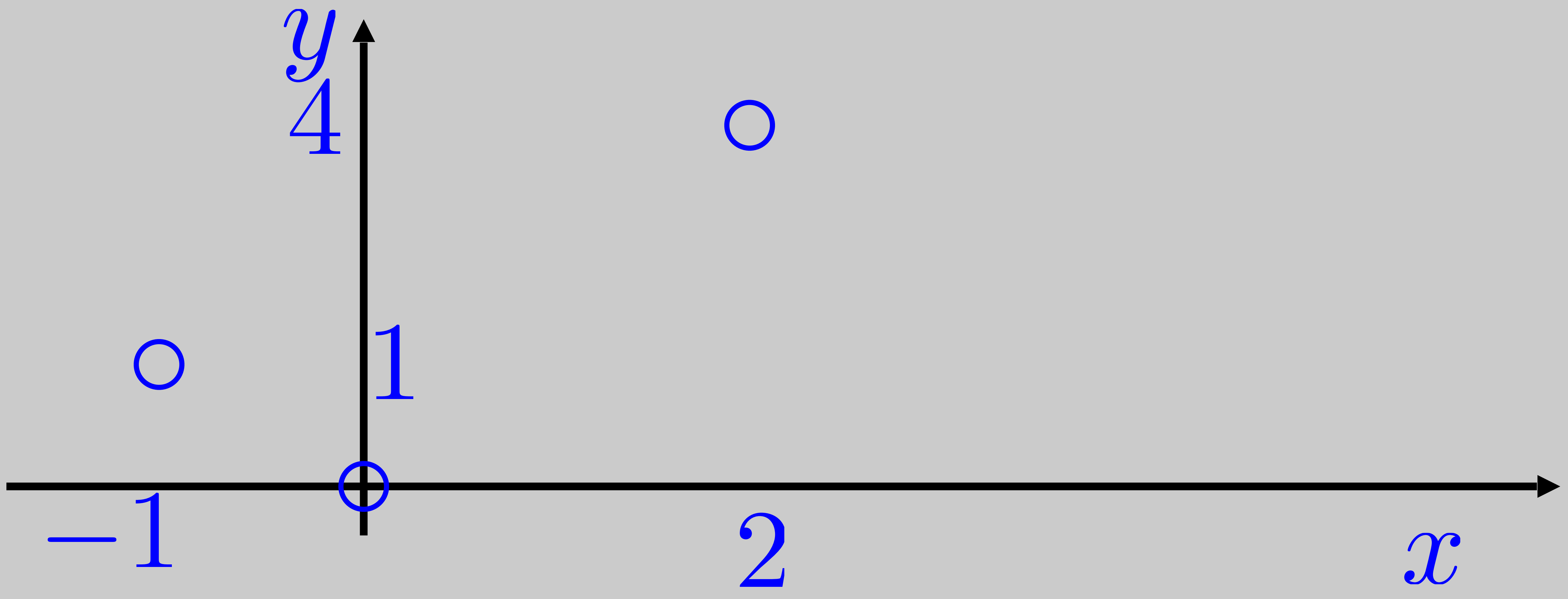
$$\rightarrow \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & \vdots & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

"Vandermonde" Matrix $\det(v) = \Pi_{1 \le i < j \le n}(x_j - x_i)$

Quiz

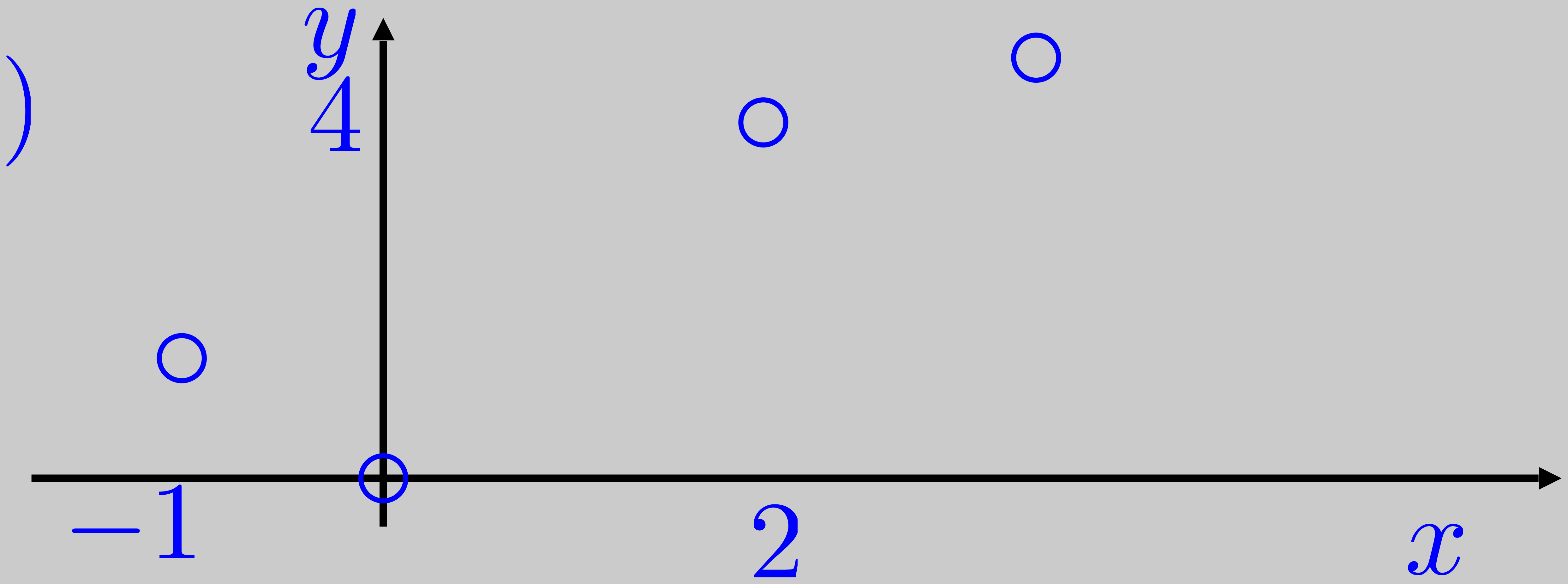
- What's the polynomial that passes through these points:

$$(-1,1),(0,0),(2,4)$$



Polynomial Regression

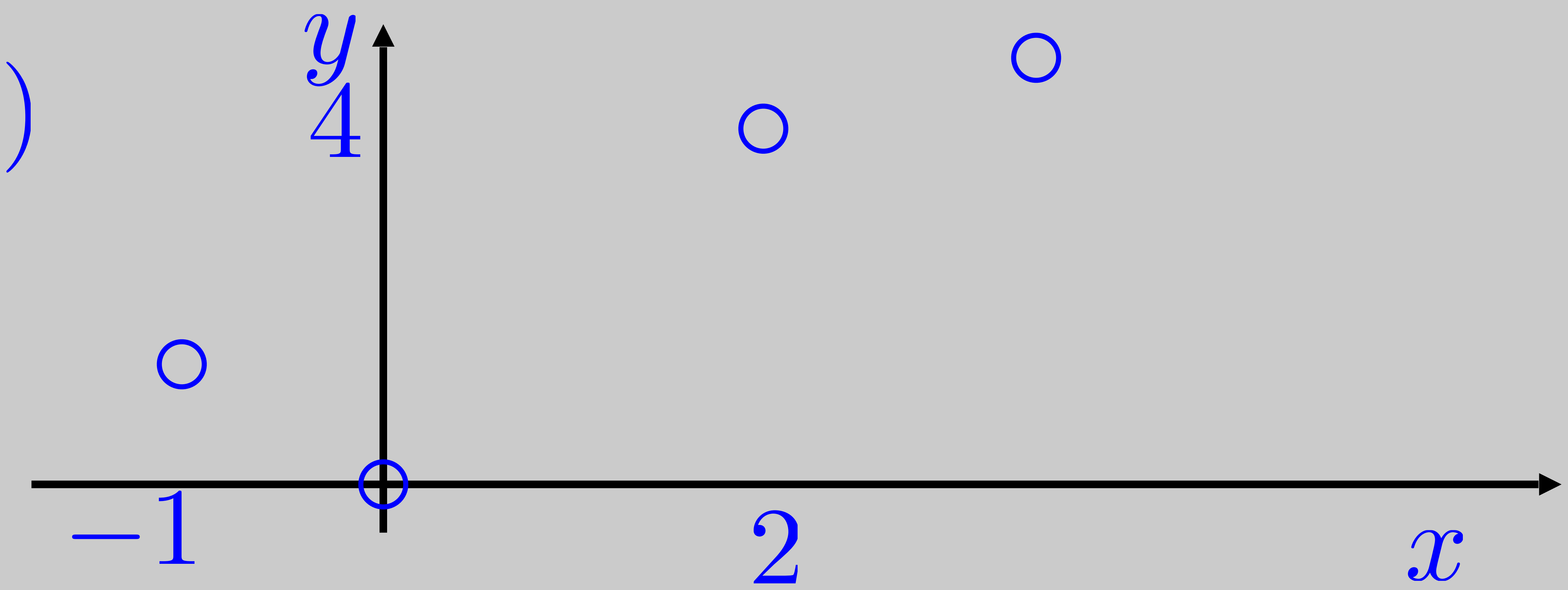
$$(-1, 1), (0, 0), (2, 4), (3, 5)$$



- What is the “best” quadratic polynomial that passes through the points?

Polynomial Regression

$$(-1,1),(0,0),(2,4),(3,5)$$



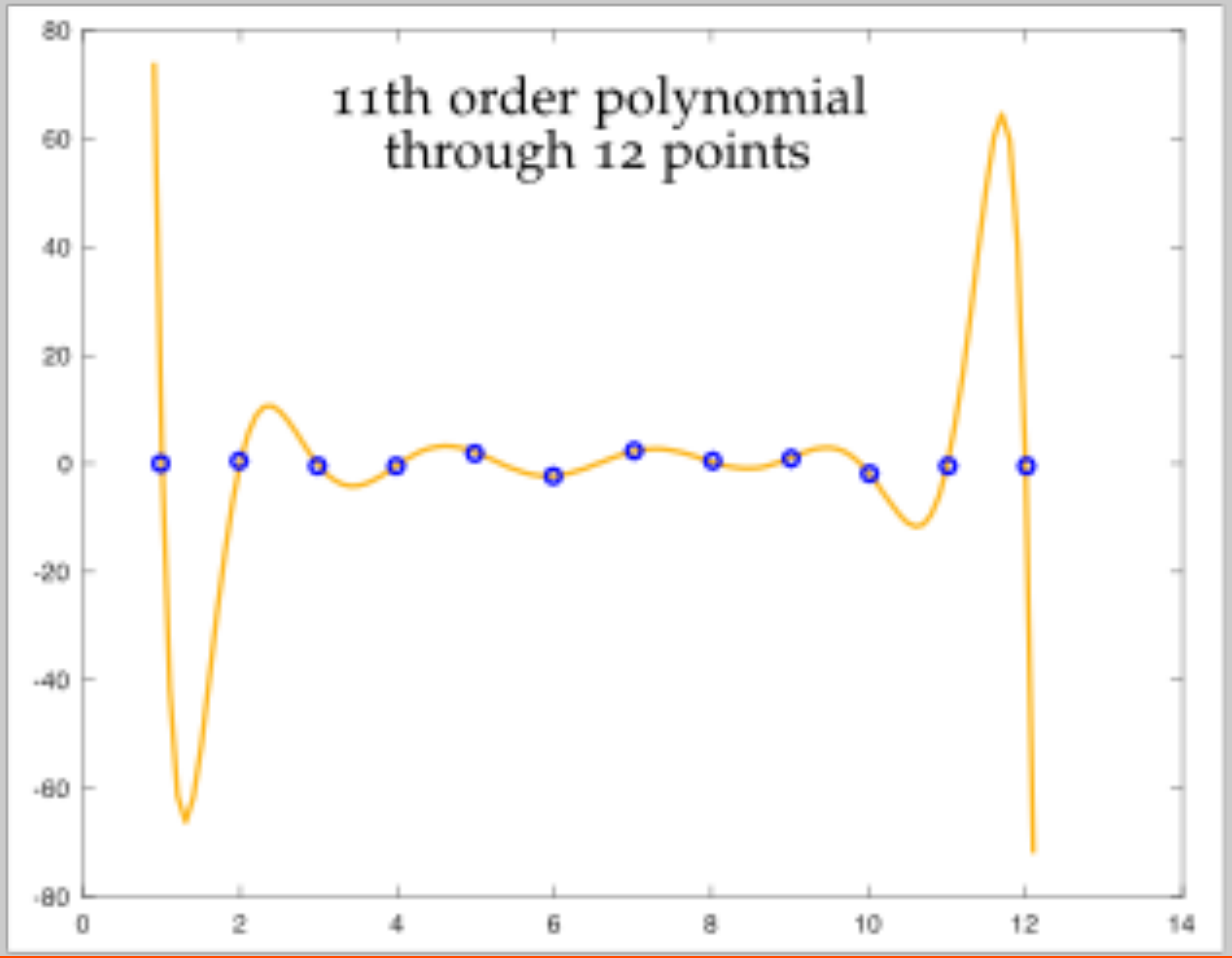
What is the “best” quadratic polynomial that passes through the points?

$$y = a_0 + a_1 x + a_2^2 + \cdots a_{n-1} x^{n-1}$$

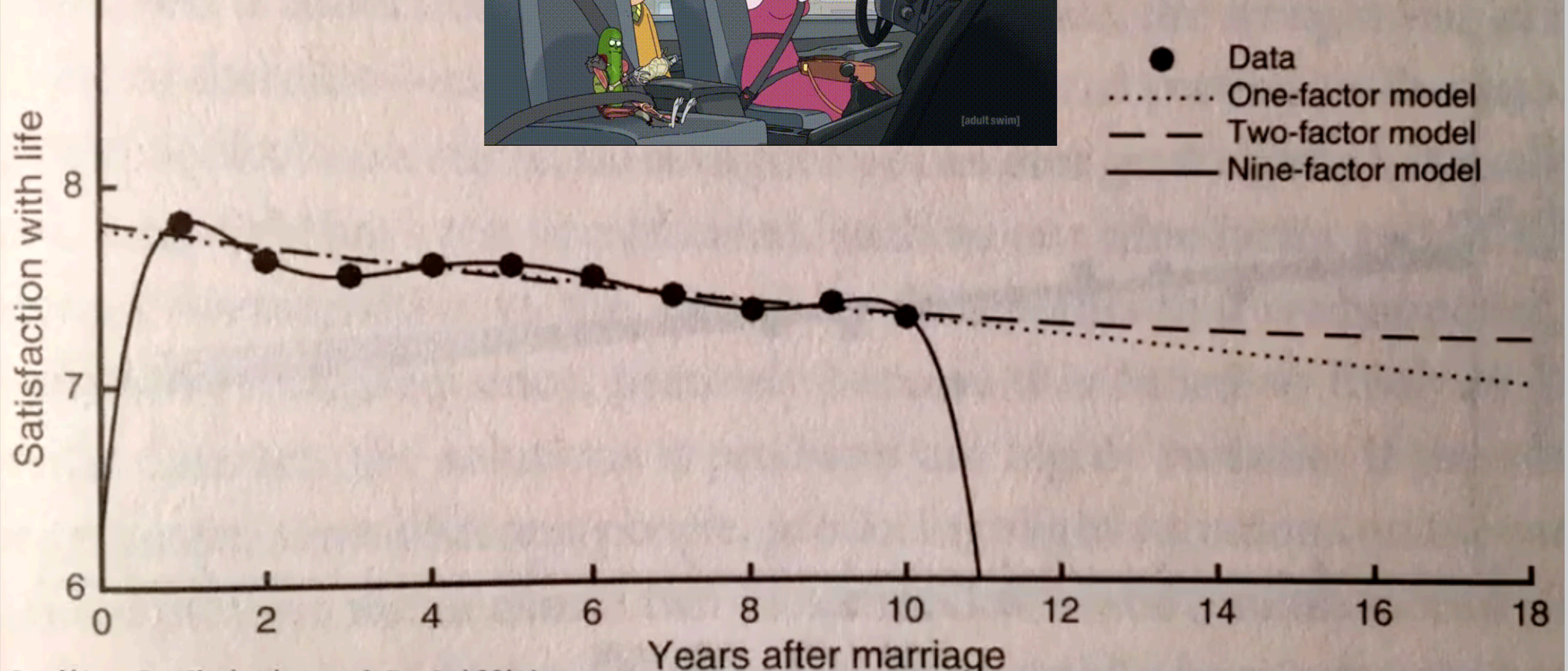
$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Issue with Polynomial Interpolation

- Tend to be oscilatory in high order interp/regression
- Not numerically stable! x^{n-1}



Example



Credit: B. Christian and T. Griffiths,