EE16B
Designing Information Devices and Systems II

Lecture 12A
Sampling
Aliasing
Discrete Signals

Intro

- Last time:
  - Interpolation
  - Started the sampling theorem
- Today:
  - Sampling theorem
  - Aliasing
  - Discrete signals

Sampling and Recovery

- Can we perfectly recover an analog signal from its samples?

Analog signal:

$$y(x) = f(x)$$

Sample:

$$y[n] = f(n\Delta)$$

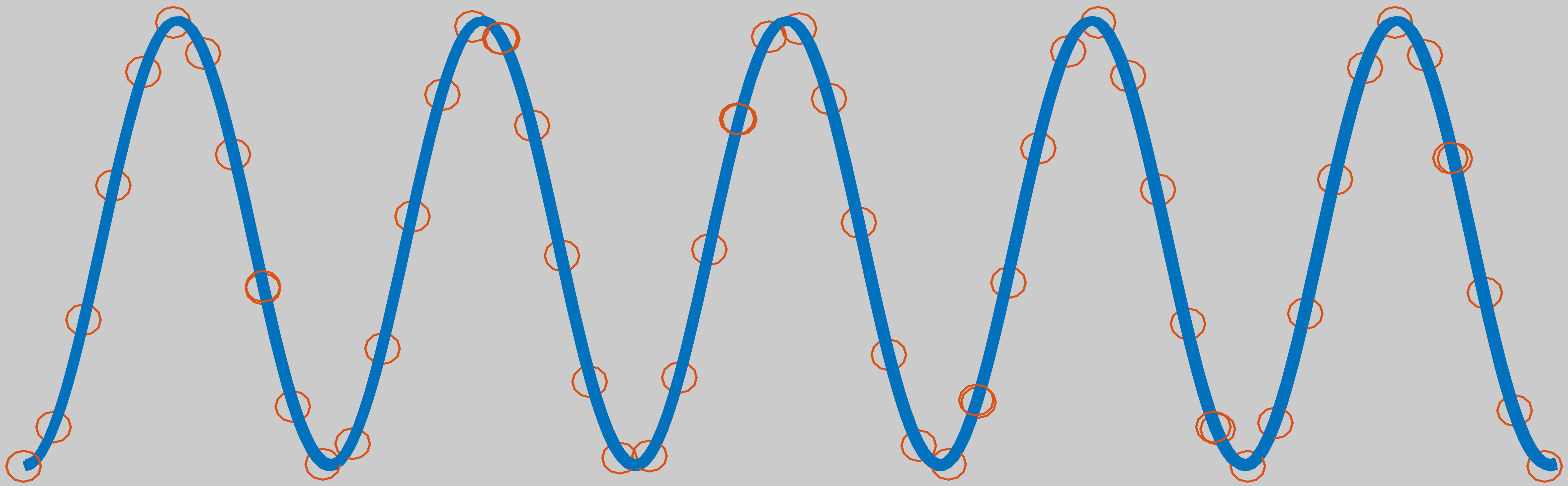
Interpolate:

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n]\Phi(x - n\Delta)$$

$$=? f(x)$$

Sampling a sinusoid

- What rate should you be sampling a sinusoid?



Bandlimitedness

- The sinc function does not contain frequencies beyond a certain bandwidth

$$\text{sinc}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(\omega x) d\omega$$

$$\left.\frac{\sin(\omega x)}{\pi x}\right|_0^{\pi} = \frac{\sin \pi x}{\pi x} \quad x \neq 0$$

$$\text{sinc}\left(\frac{x}{\Delta}\right) = \frac{1}{\pi}\int_0^{\pi} \cos\left(\frac{\omega}{\Delta}x\right) d\omega \qquad \Rightarrow \omega_{\text{max}} = \frac{\pi}{\Delta}$$

Sampling Theorem

- If f(x) is bandlimited by frequency w_{max}, then

$$f(x) = \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n]\Phi(x - n\Delta) \qquad \Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$$

As long as,

$$\omega_{\text{max}} < \frac{\pi}{\Delta}$$

$$\frac{\omega_{\text{max}}}{\pi} < \frac{1}{\Delta}$$

$$2\frac{\omega_{\text{max}}}{2\pi} < \frac{1}{\Delta}$$

$$2f_{\text{max}} < f_s \qquad \omega_s > 2\omega_{\text{max}}$$

Proof: EE120, EE123

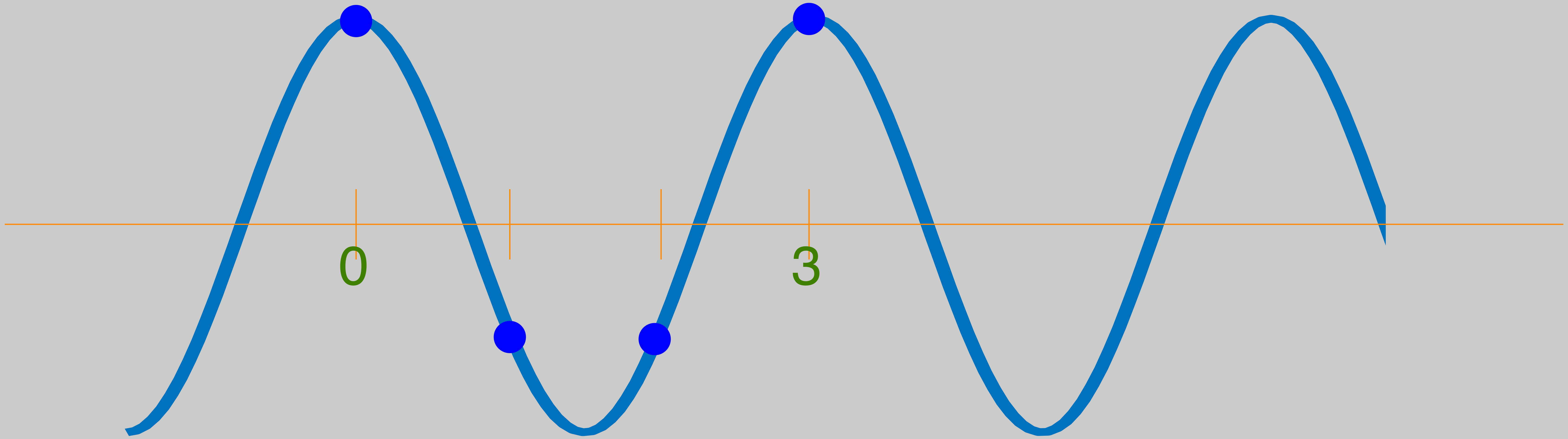
Examples

- Audio Signals:
  - – Can hear up to 18-20KHz
  - – Sampling 44.1KHz, or 48KHz

- Speech: 500Hz – 3.5KHz
  - – MSP430 samples at about 2.8KHz. Is that enough?

Example 1

$$f(x) = \cos(\frac{2\pi}{3}x) \qquad \Delta = 1 \qquad \omega_{\max} <? \frac{\pi}{\Delta}$$



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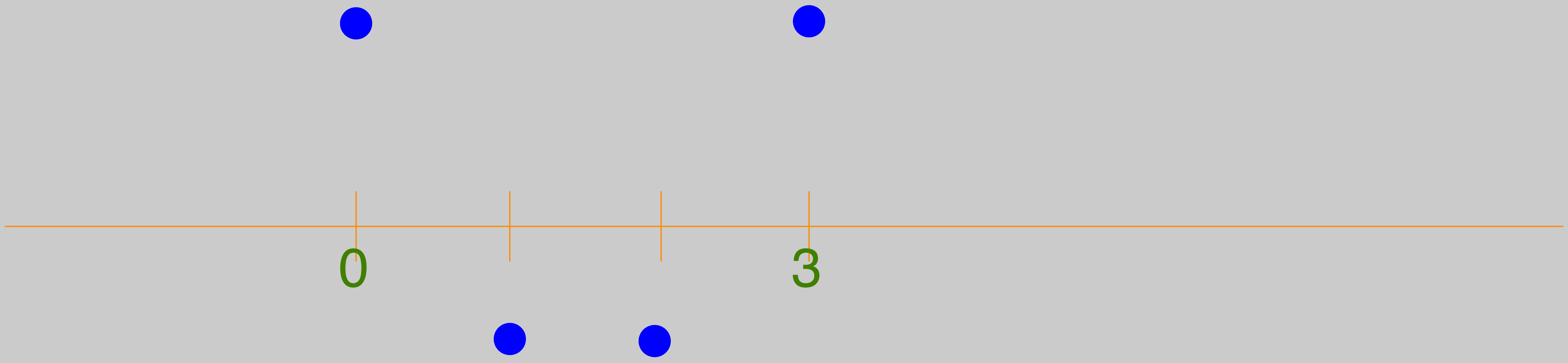

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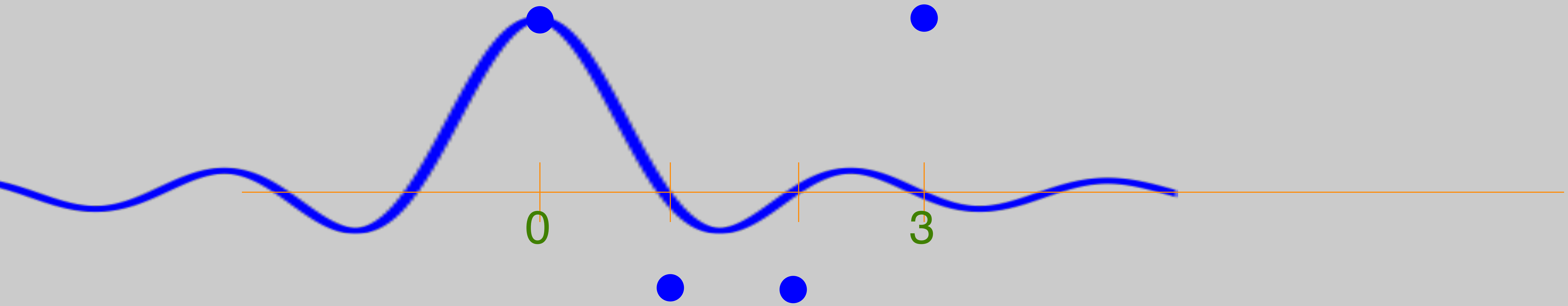

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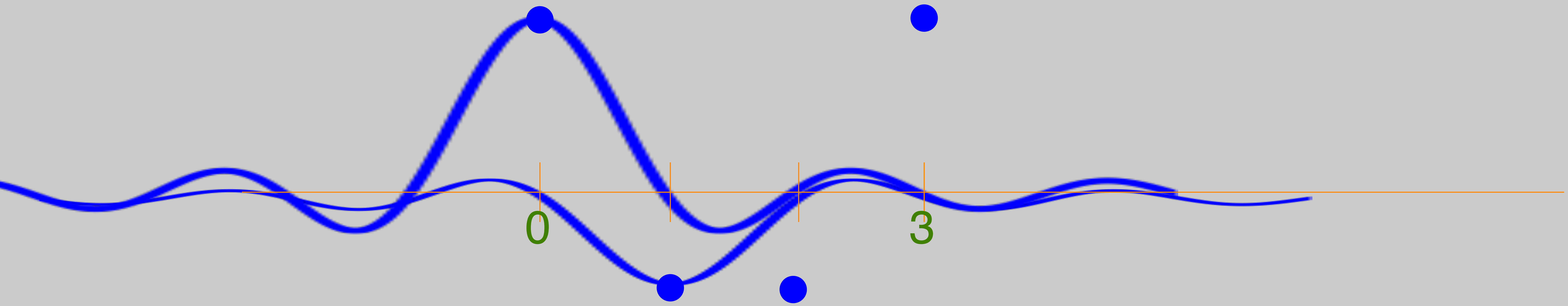

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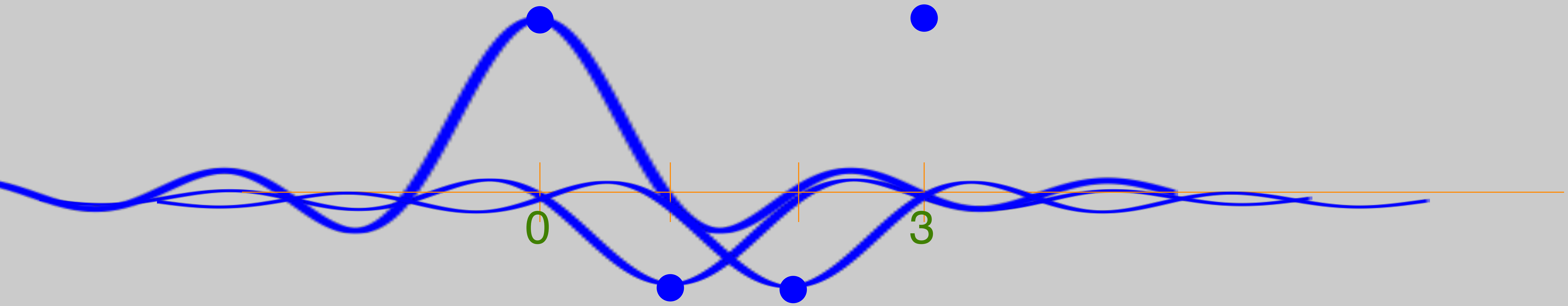

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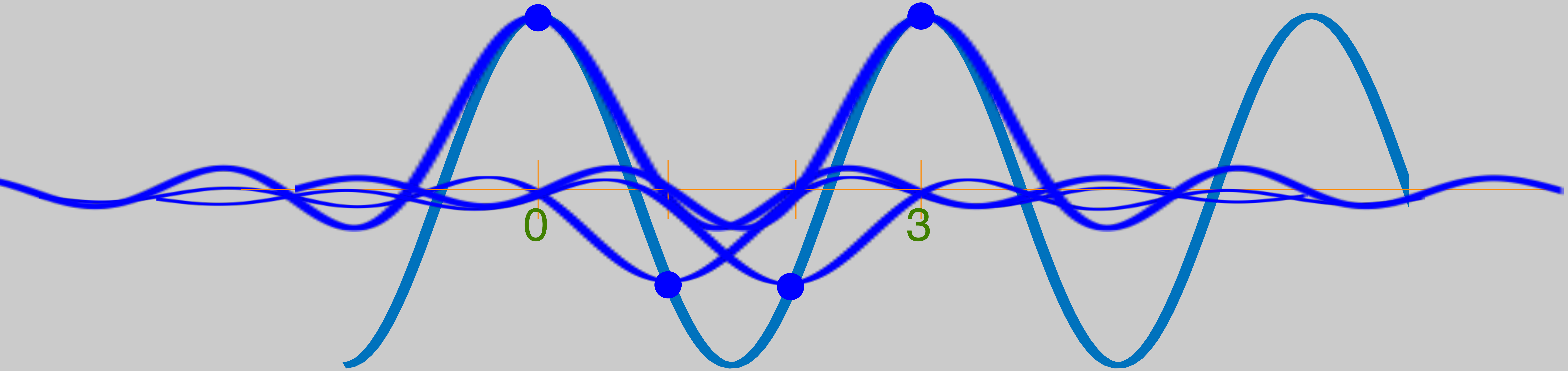

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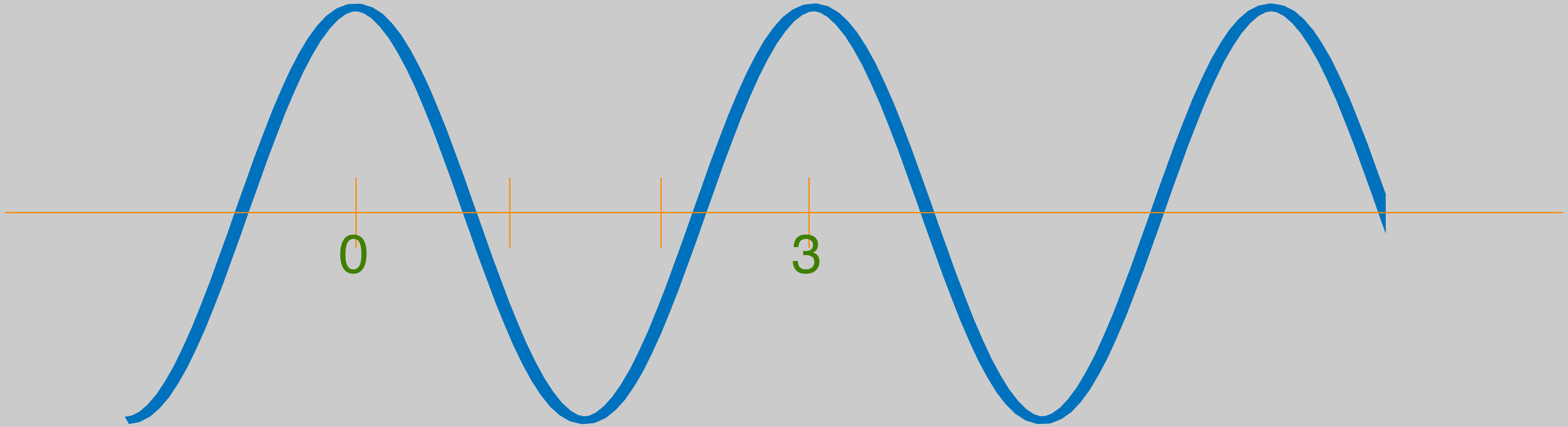
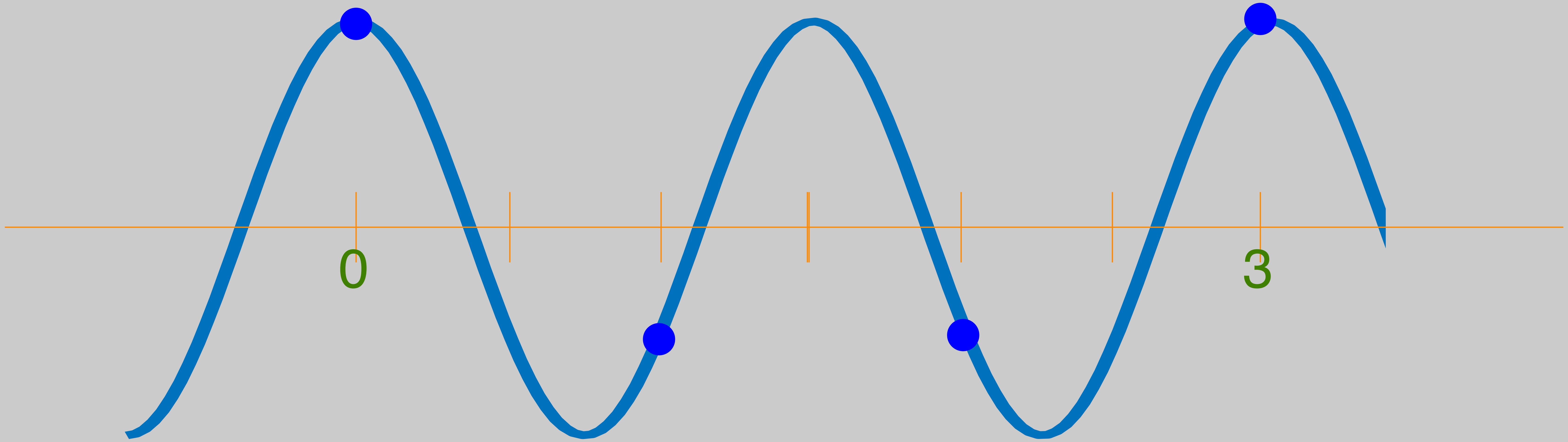


Example 2

$$f(x) = \cos(\frac{4\pi}{3}x) \qquad \Delta = 1 \qquad \omega_{\max} <? \frac{\pi}{\Delta}$$



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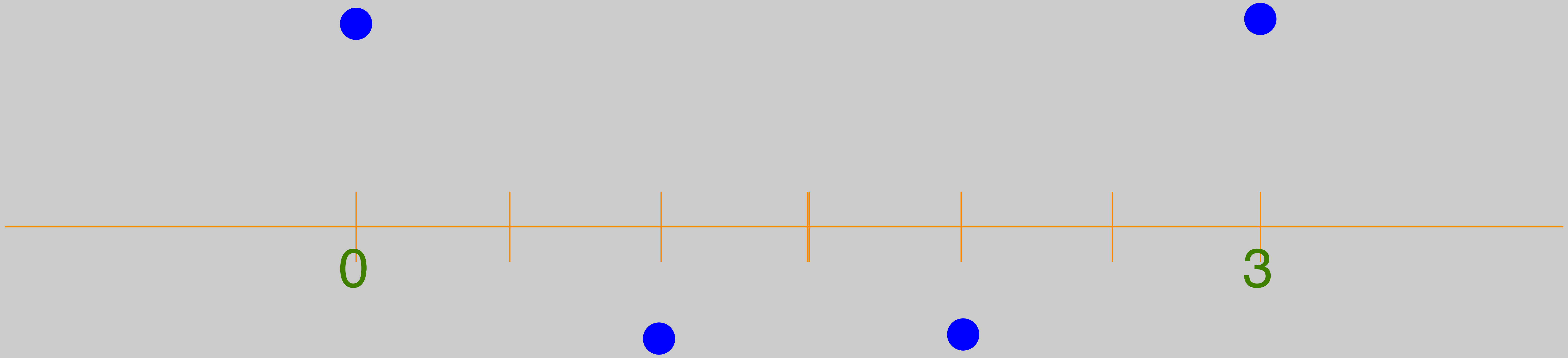

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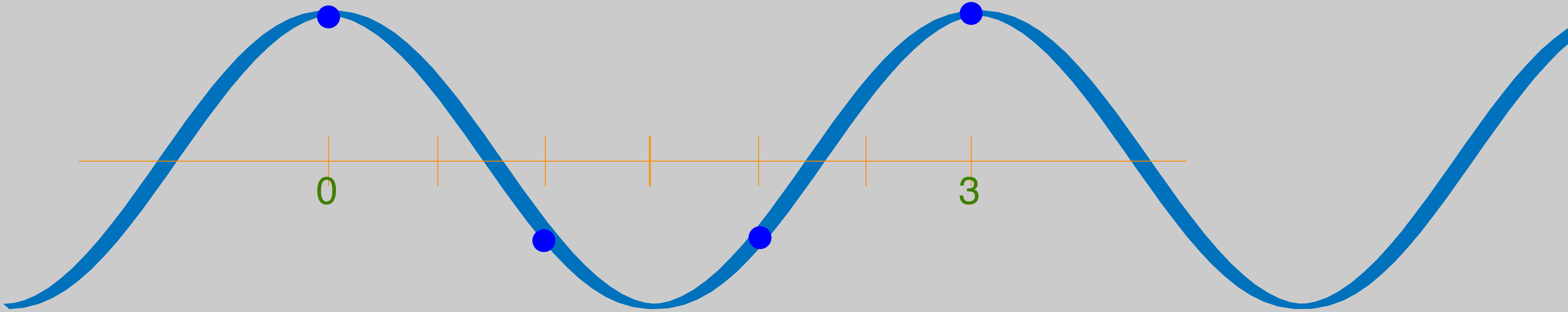


- Sinc interpolation gives: $\hat{f}(x) = \cos(\frac{2\pi}{3}x)$

Aliasing of high frequencies into lower ones!

Aliasing and Phase Reversal

$$f(x) = \cos(\omega x + \phi) \qquad \Delta = 1$$

$$y[n] = \cos(\omega n + \phi)$$

- Highest interpolated frequency will not be higher than π

$$y[n] = \cos(\omega n + \phi) = \cos(2\pi n - (\omega n + \phi)) = \cos((2\pi - \omega)n - \phi)$$

$$\cos(2\pi n - \theta) = \cos(\theta)$$

If π < w < 2π and Δ=1, there's an equivalent lower frequency signal with the same samples!

$$\hat{f}(x) = \cos((2\pi - \omega)x - \phi)$$

Example 2

$$f(x) = \cos(\omega x + \phi) \qquad \Delta = 1 \quad \omega = \frac{4\pi}{3} \;\; \phi = 0$$

$$\hat{f}(x) = \cos((2\pi - \omega)x - \phi)$$

$$= \cos(\frac{2\pi}{3}x)$$

Example 3

$$f(x) = \sin(1.9\pi x) \qquad \Delta = 1$$

$$= \cos(1.9\pi x - \frac{\pi}{2})$$

$$\hat{f}(x) = \cos(0.1\pi x + \frac{\pi}{2})$$

$$= -\sin(0.1\pi x)$$

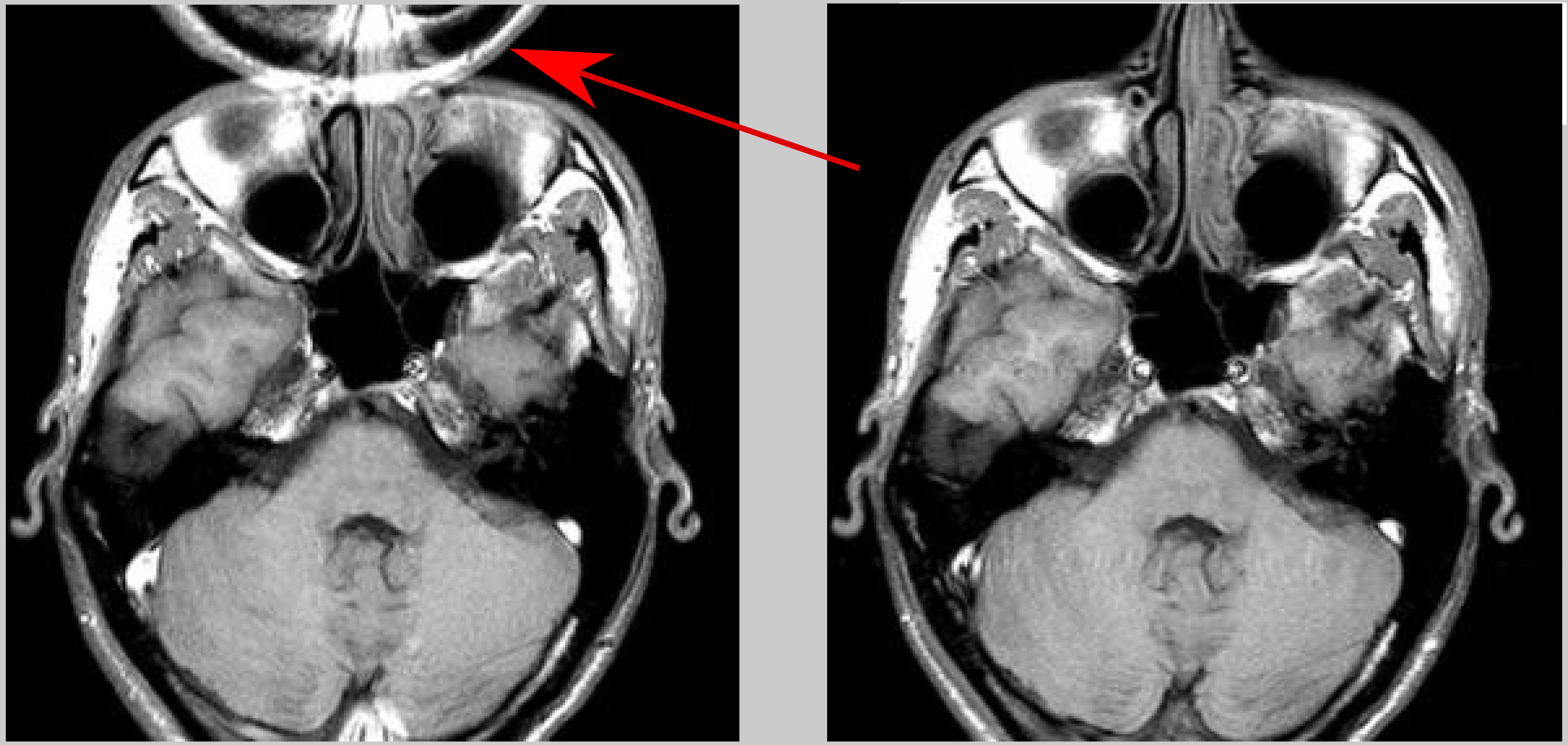
Aliasing Demo

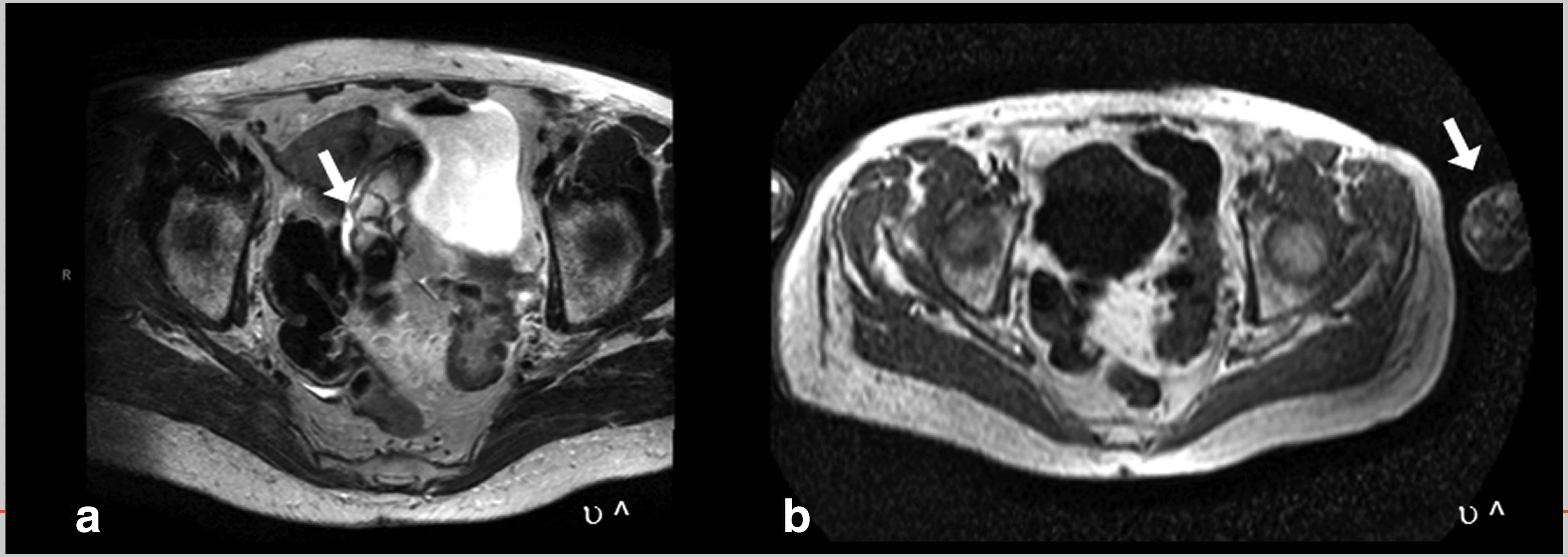
Aliasing in video

https://www.youtube.com/watch?v=cxddi8m_mzk

Aliasing in Images

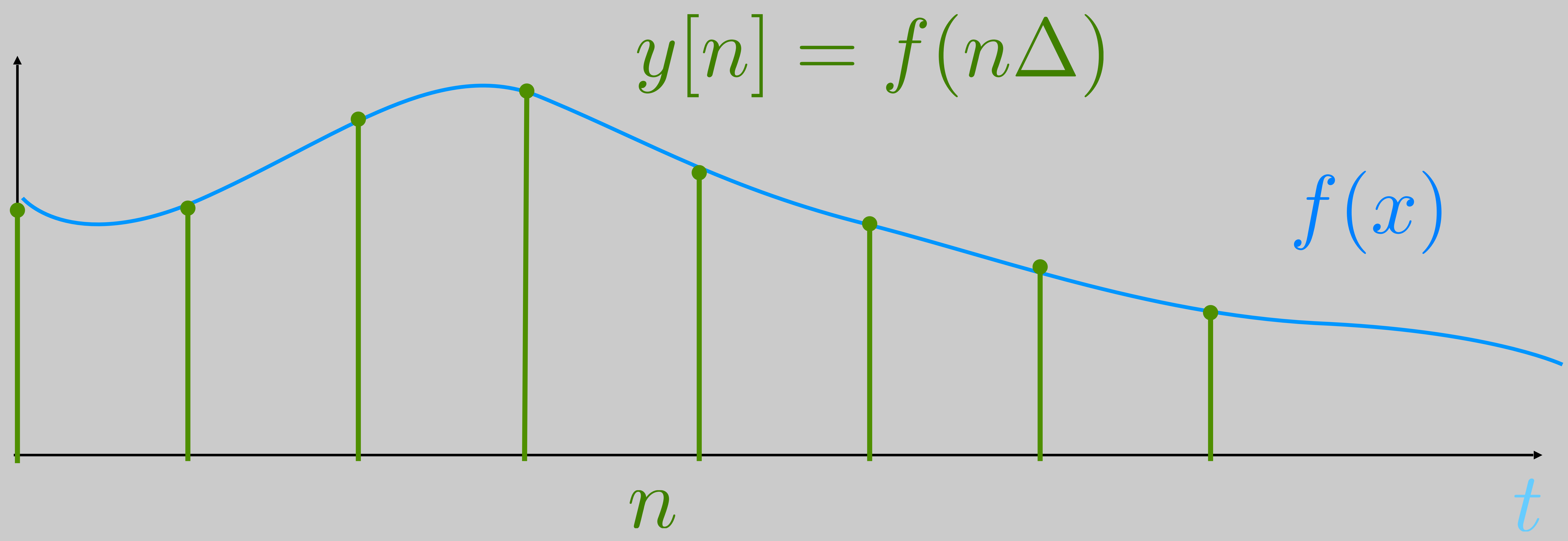
Aliasing in MRI





Discrete Time Signals

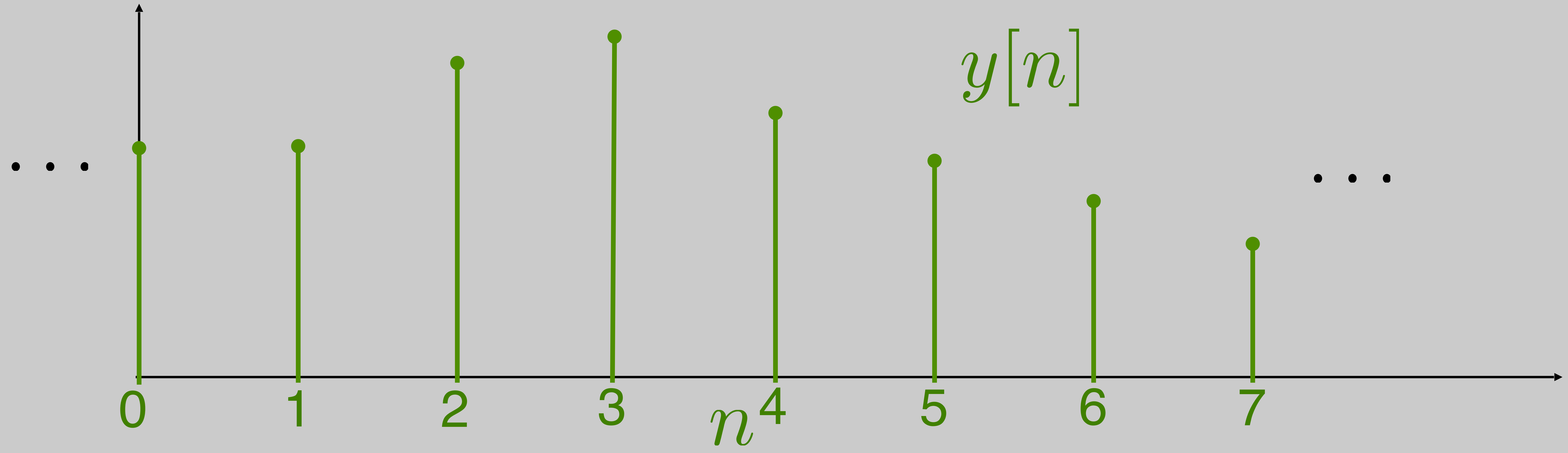
- Samples of a CT signal:



- Or, inherently discrete (Examples?)

Discrete Time Signals

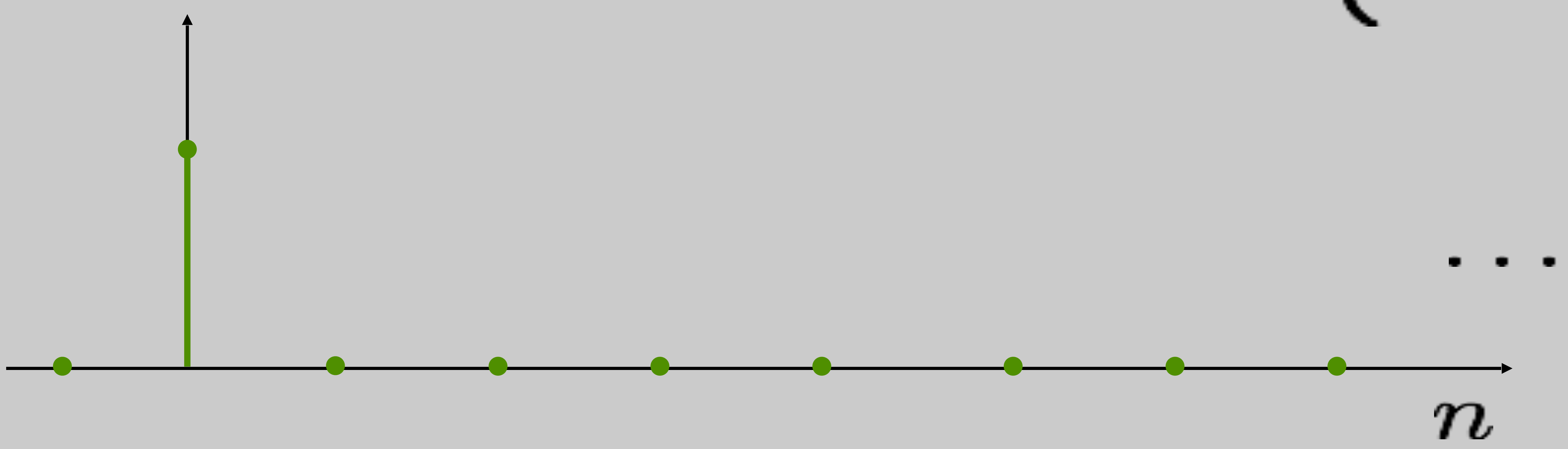
- At their core are “just samples”!



Basic Sequences

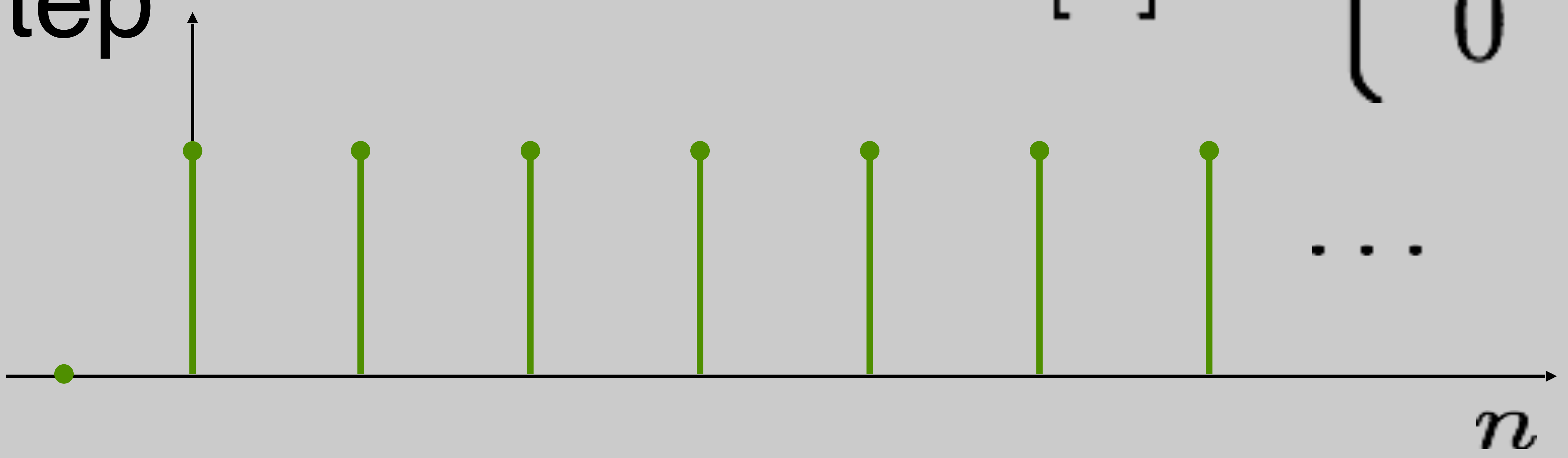
- Unit Impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- Unit Step

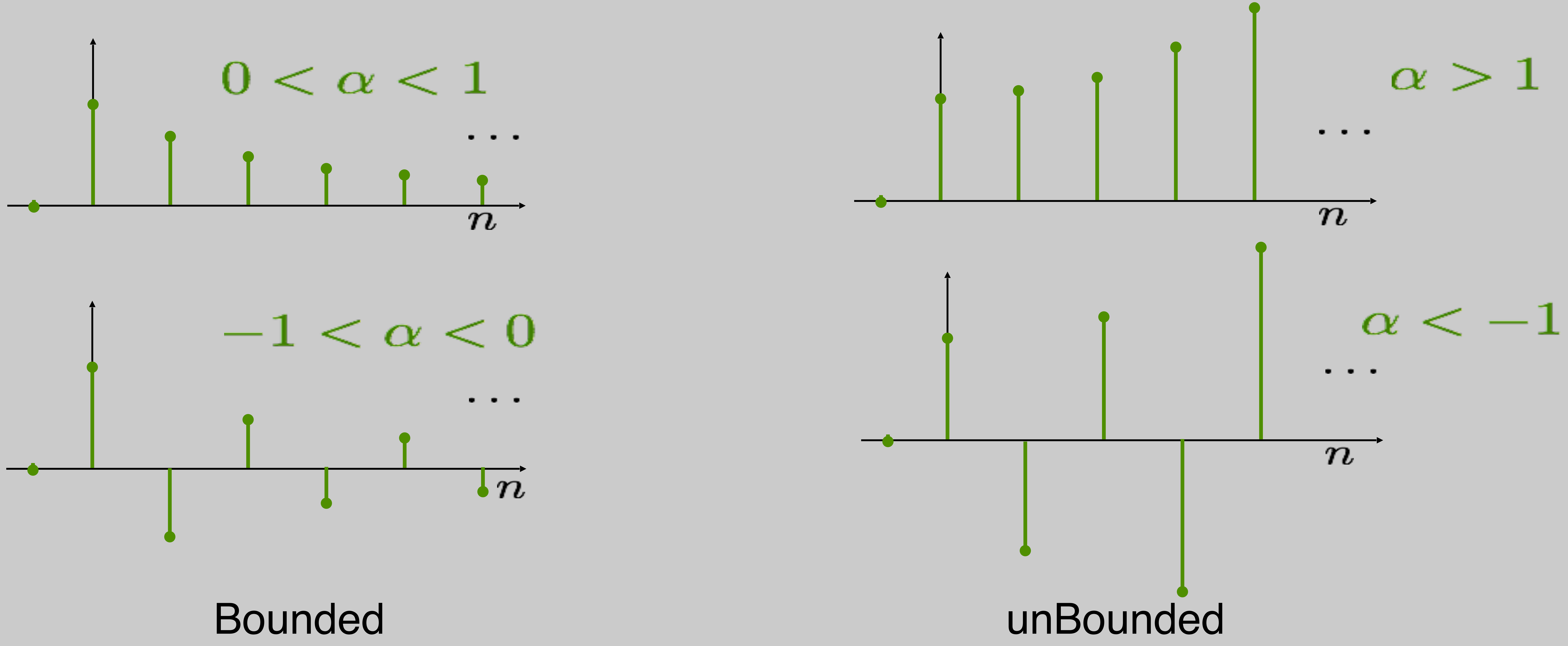
$$U[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Basic Sequences

- Exponential

$$y[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Discrete Sinusoids

$$y[n] = A\cos(\omega_0 n + \phi)$$

or, $$y[n] = Ae^{j\omega_0 n + j\phi}$$

Q: Is y[n] periodic? $y[n+N] = y[n] \quad | N \in \text{Integer}$

Q: Only if ω_0/π is rational

- To find fundamental period, N
  - Find the smallest integers N,K:

$$\omega_0 N = 2\pi K$$

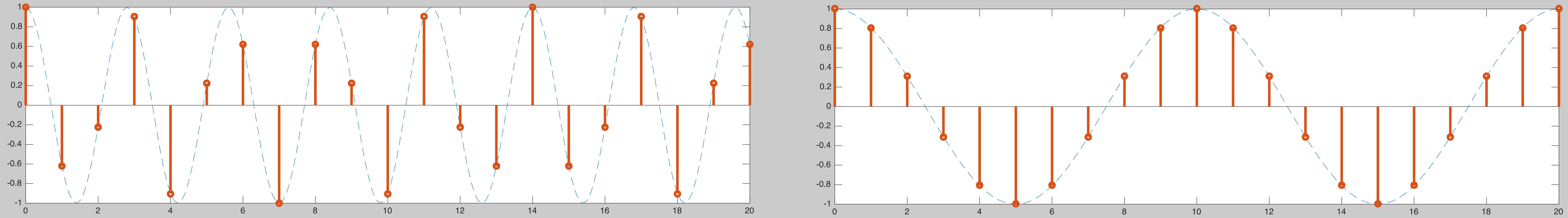
Discrete Sinusoids

$$\omega_0 N = 2\pi K$$

- Examples:

$\cos(\pi/5n)$ $\quad N = 10 \quad K = 1$

$\cos(\frac{5\pi}{7}n)$ $\quad N = 14 \quad K = 5$

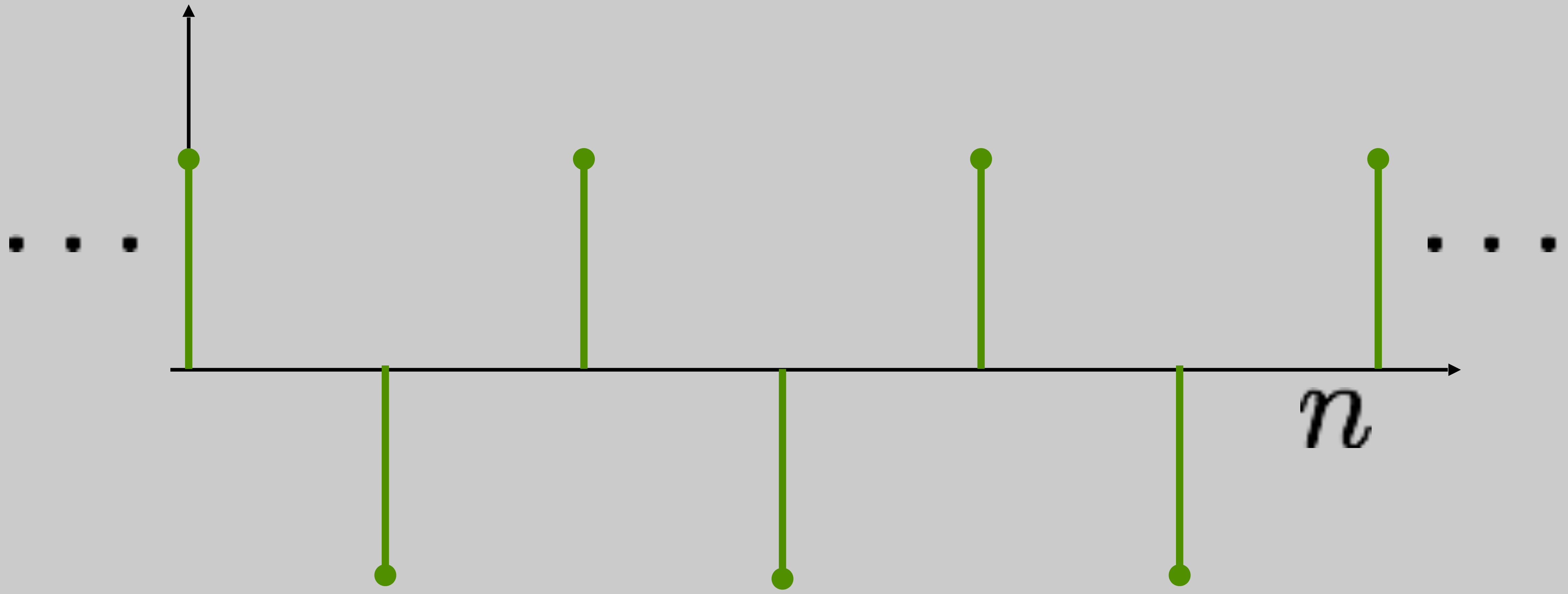


$$\cos(\frac{5\pi}{7}n) + \cos(\pi/5n) \quad \Rightarrow N = \text{S.C.M}\{10, 14\} = 70$$

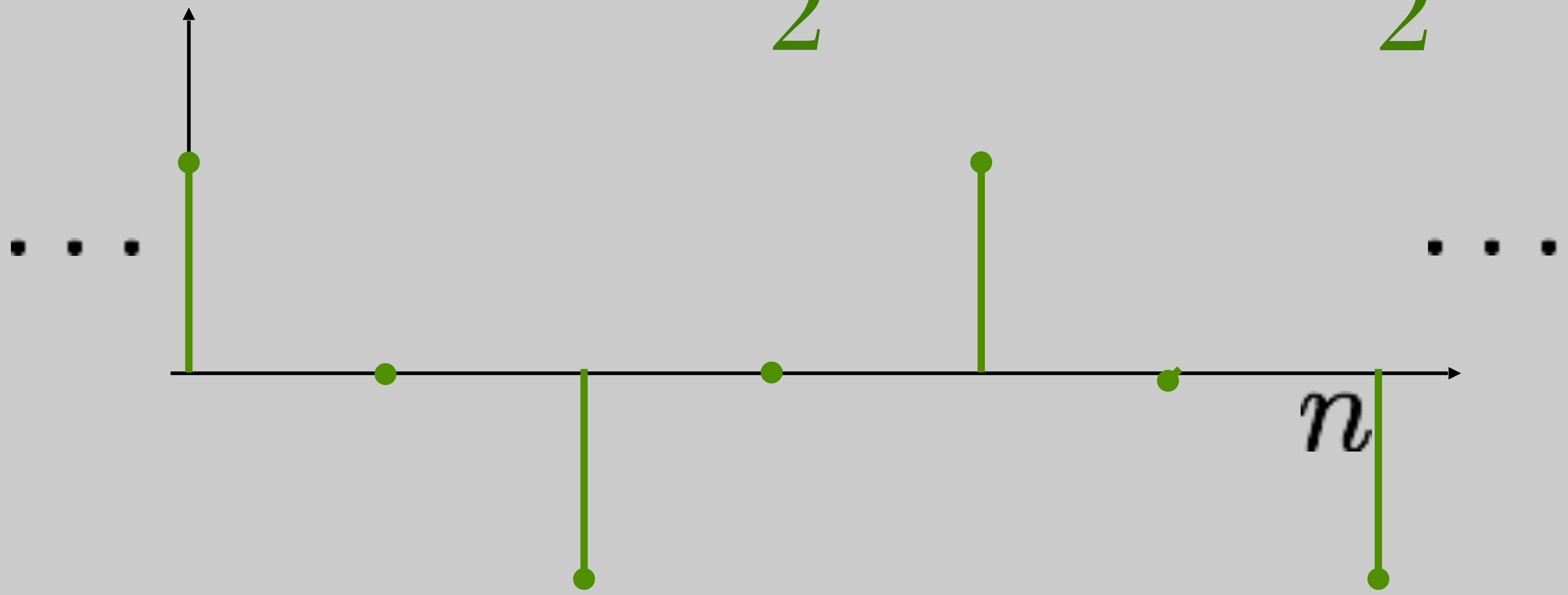
Discrete Sinusoids

Q: Which signal has a higher frequency?

$$\cos(\pi n)$$



$$\cos(\frac{3\pi}{2}n) = \cos(\frac{\pi}{2}n)$$



Discrete Sinusoids

- What's the lowest discrete frequency?

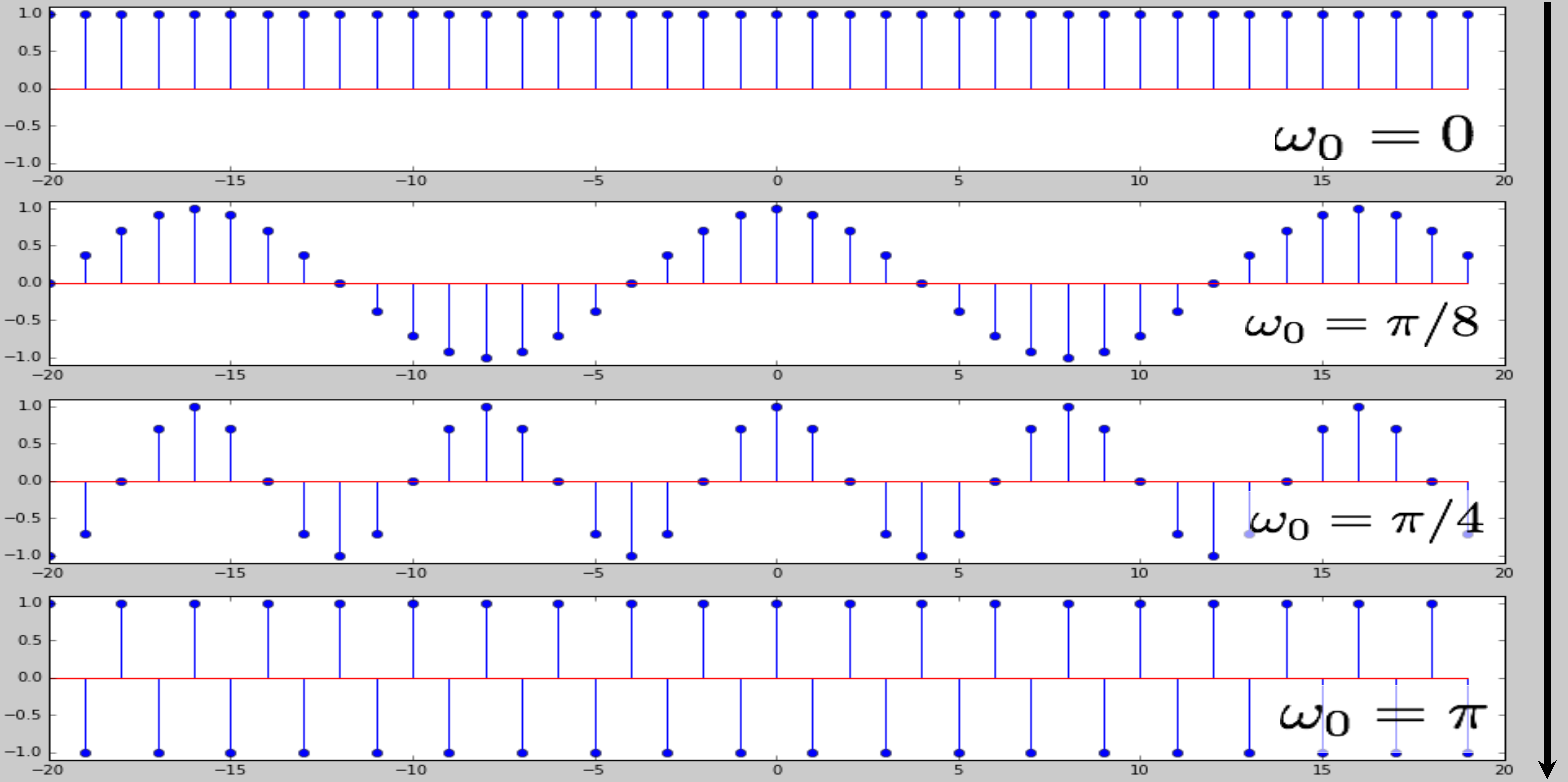
$$y[n] = \cos(0n) = 1$$

- What's the highest discrete frequency?

$$y[n] = \cos(\pi n) = (-1)^n$$

Discrete Sinusoids

$\cos(w_0 n)$



Discrete Sinusoids

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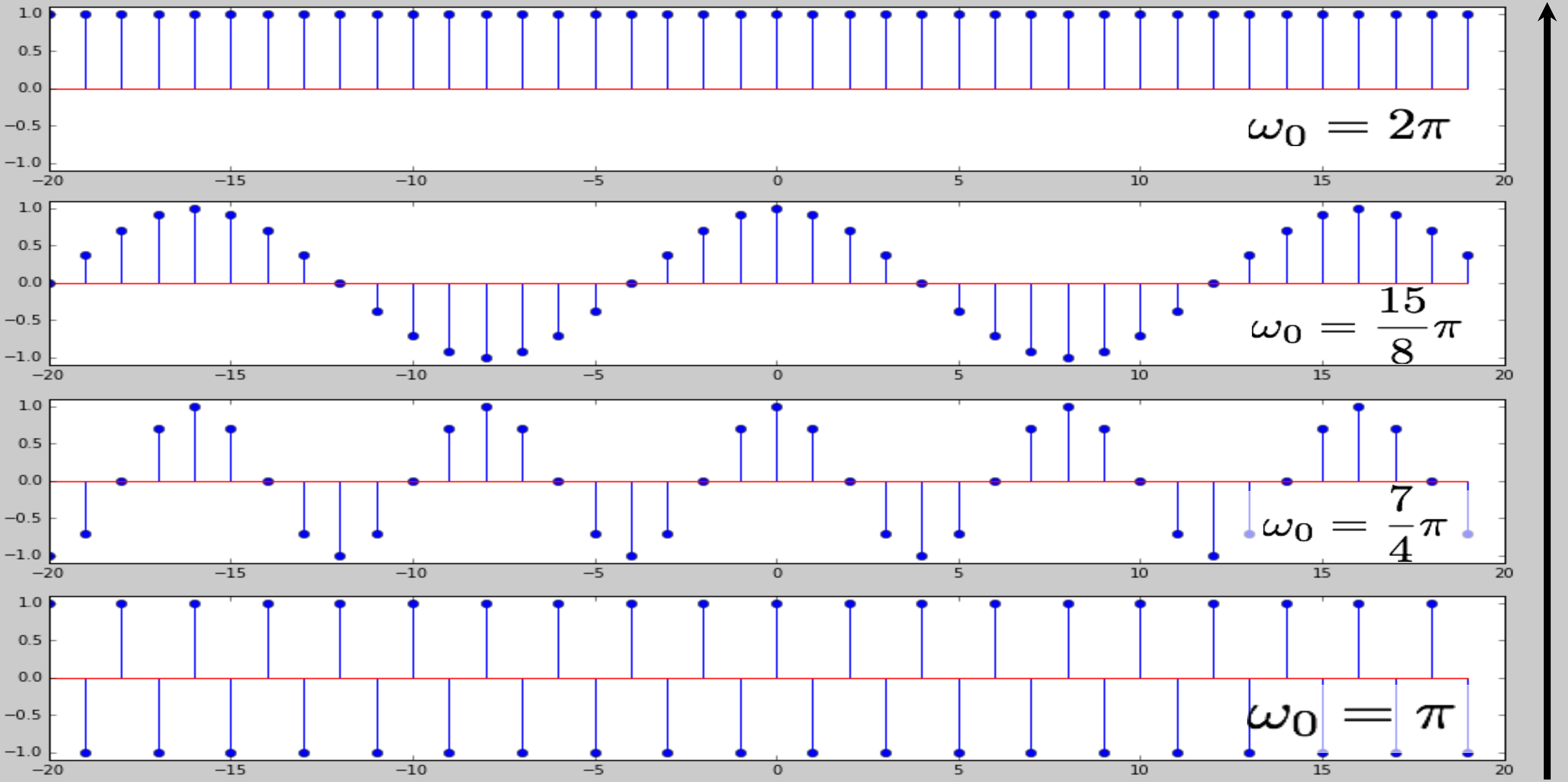


Complex Frequencies

- Sinusoids are sums of left and right rotating complex exponentials

$$2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$$

"Positive" and "Negative" frequencies

Discrete frequencies with period N:

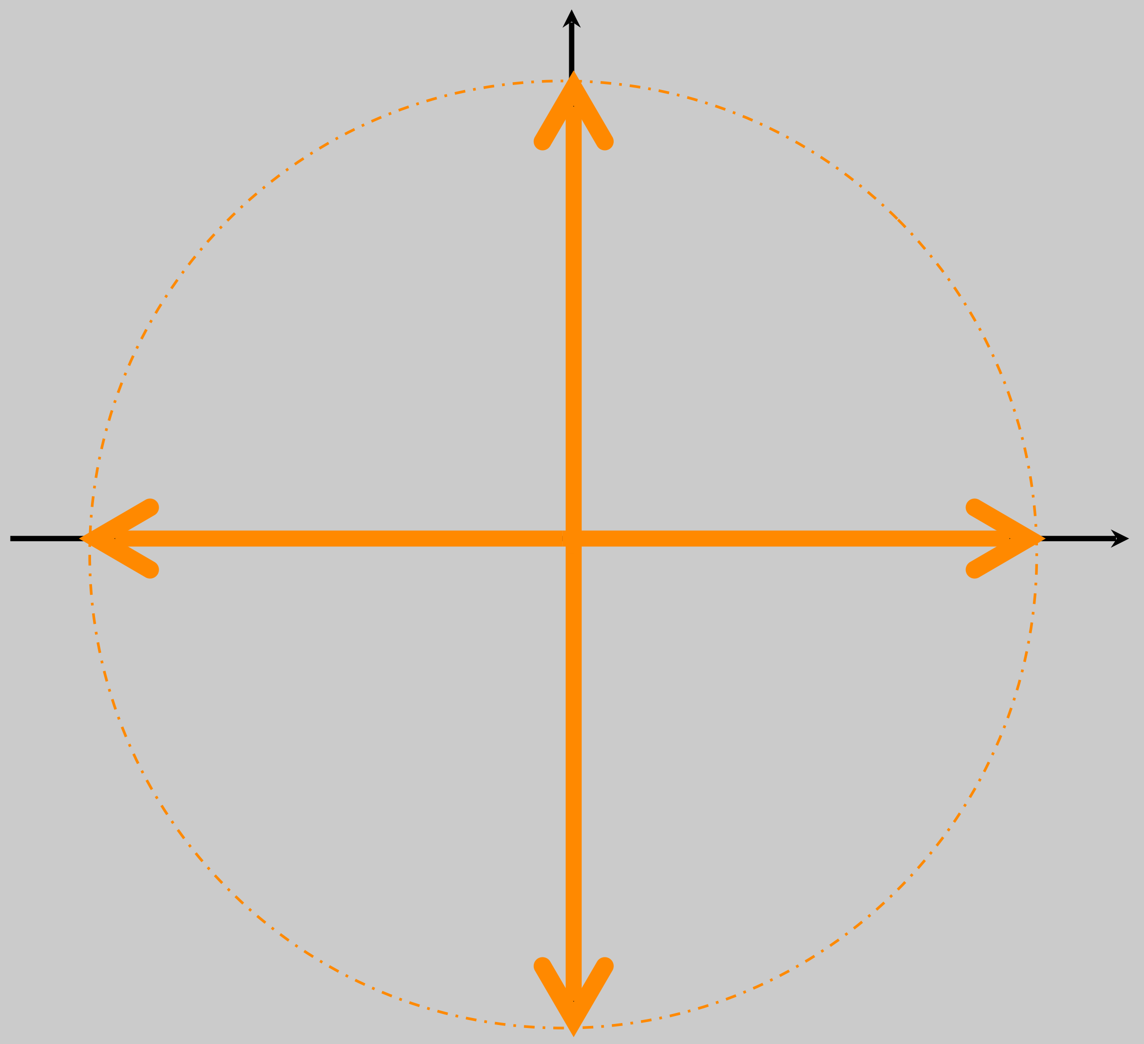
$$y[n] = e^{j2\pi n/N}$$

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow y[n] = W_N^n$$

Complex Frequencies

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow y[n] = W_N^n$$

- N = 4 $\quad y[n] = W_4^n$



- N = 6, neg. freq. $\quad y[n] = W_6^{-n}$