EE16B
Designing Information Devices and Systems II

Lecture 12B
Discrete Signals and Systems
LTI Systems, Convolution sum

Intro

- Last time:
  - Sampling Theorem
  - Aliasing
  - Discrete Signals
- Today
  - Discrete systems

Complex Frequencies

- Sinusoids are sums of left and right rotating complex exponentials

$$2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$$

"Positive" and "Negative" frequencies

Discrete frequencies with period N:

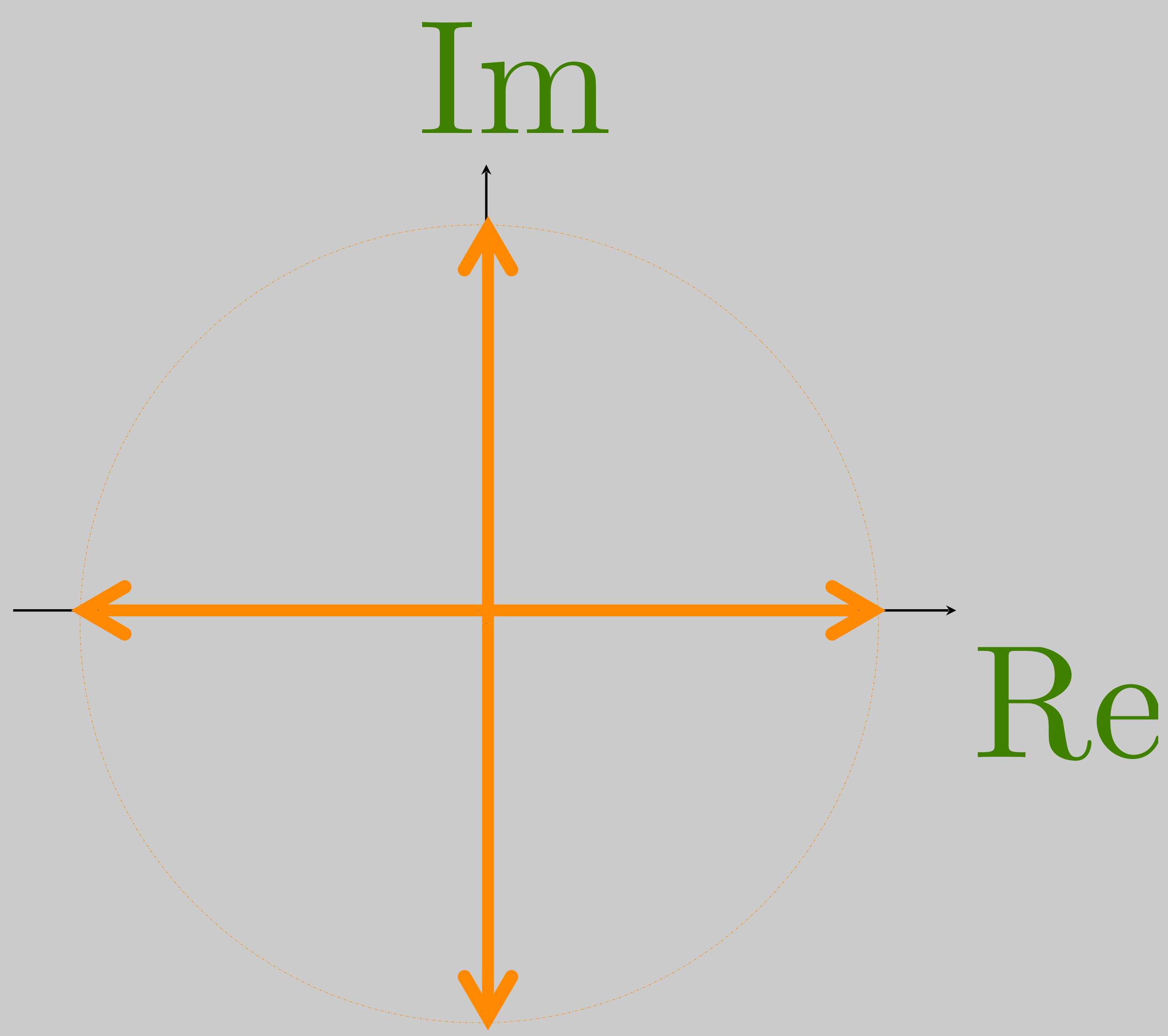
$$y[n] = e^{j2\pi n/N}$$

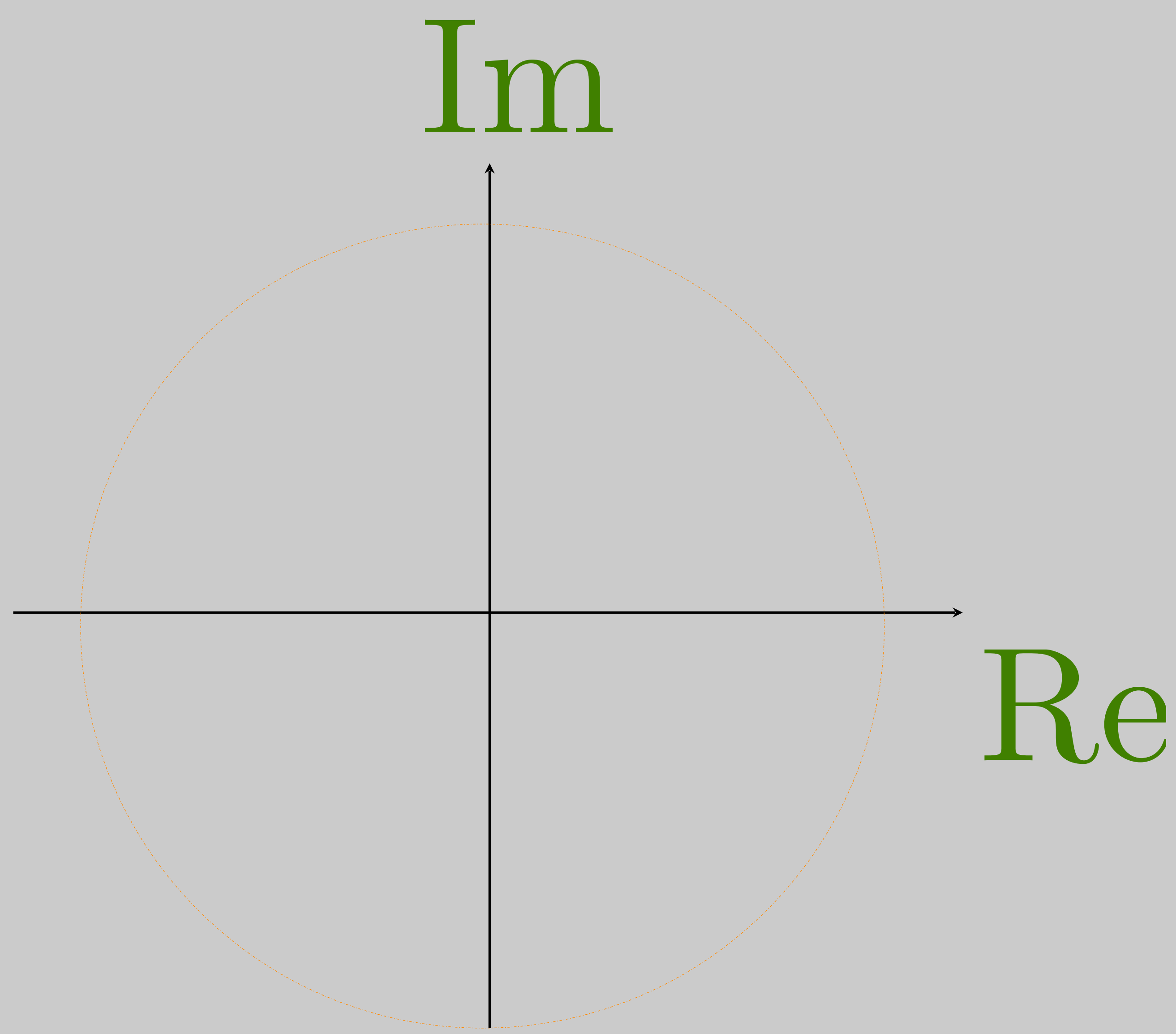
$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow y[n] = W_N^n$$

Complex Frequencies

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow y[n] = W_N^n$$

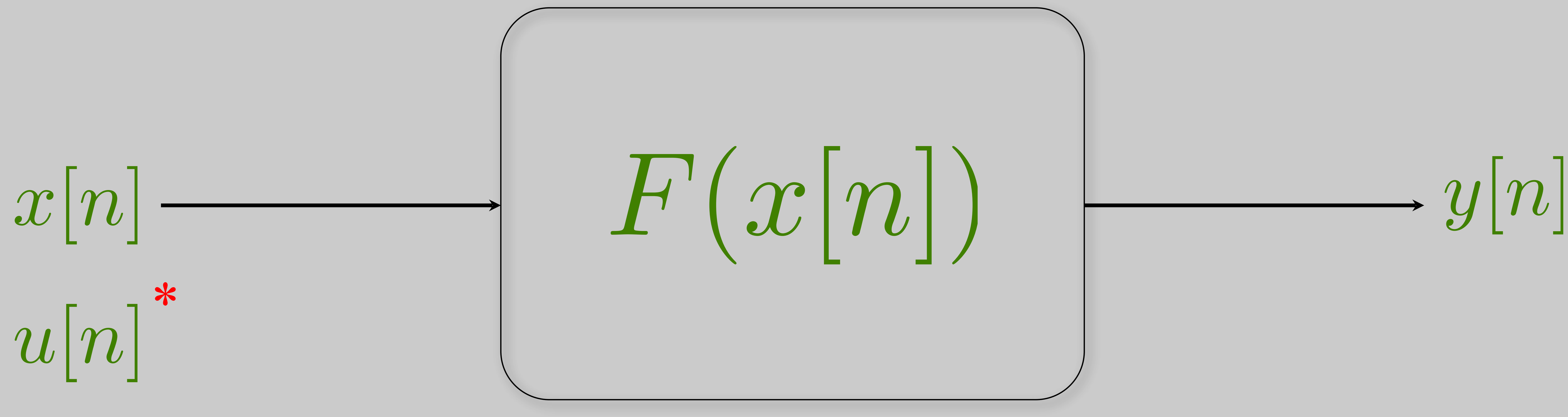
- N = 4 $\quad y[n] = W_4^n$





- N = 6, neg. freq. $\quad y[n] = W_6^{-n}$

Discrete Time Systems



- What Properties?
  - Causality
  - Linearity
  - Stability
  - Time/shift invariance

*WARNING: Going to interchange x[n] and u[n] as inputs

$\vec{x}[n]$ will be a state, not input

U[n] is unit step, not to be confuse with u[n]

Properties of D.T. Systems

- Causality:
  - $y[n_0]$ depends only on $x[n]$ for $\infty \leq n \leq n_0$

Causal?

$$\vec{x}[n+1] = A\vec{x}[n] + Bu[n]$$
$$y[n] = C\vec{x}[n]$$

$$\vec{x}[n] = A^n\vec{x}[0] + \sum_{k=0}^{n-1} A^{n-1-k}Bu[k]$$
$$y[n] = CA^n\vec{x}[0] + \sum_{k=0}^{n-1} CA^{n-1-k}Bu[k]$$

Properties of D.T. Systems

$$y[n] = F\{x[n]\}$$

- Linearity
  - Homogeneity: scaling the input, scales the output

$$F\{ax[n]\} = aF\{x[n]\} = ay[n]$$

Properties of D.T. Systems

$$y[n] = F\{x[n]\}$$

- Linearity
  - Homogeneity / scaling: scaling the input, scales the output

$$F\{ax[n]\} = aF\{x[n]\} = ay[n]$$

  - Additivity: sum of inputs ⇒ sum of outputs

$$F\{x_1[n] + x_2[n]\} = F\{x_1[n]\} + F\{x_2[n]\} = y_1[n] + y_2[n]$$

  - Super position: Both additivity and homogeneity

Example:

$$\vec{x}[n+1] = A\vec{x}[n] + Bu[n]$$
$$y[n] = C\vec{x}[n]$$

Linear?

$$y[n] = CA^n\vec{x}[0] + \sum_{k=0}^{n-1} CA^{n-1-k}Bu[k]$$

Properties of D.T. Systems

$$y[n] = F\{x[n]\}$$

- BIBO Stability
  - If x[n] is bounded, then y[n] is bounded

$$|x[n]| < M < \infty \quad \forall n \Rightarrow \quad |y[n]| < P < \infty \quad \forall n$$

BIBO stable?

$$y[n] = CA^n\vec{x}[0] + \sum_{k=0}^{n-1} CA^{n-1-k}Bu[k]$$

Properties of D.T. Systems

$$y[n] = F\{x[n]\}$$

- Time Invariance: Shifted input ⇒ shifted output

$$y[n-n_0] = F\{x[n-n_0]\}$$

Time Invariant?

$$\vec{x}[n+1] = A\vec{x}[n] + Bu[n]$$

$$y[n] = C\vec{x}[n]$$

$$y[n] = CA^n\vec{x}[0] + CBu[n-1] + CABu[n-2] + \cdots + CA^{n-1}Bu[0]$$

Linear Time Invariant Systems

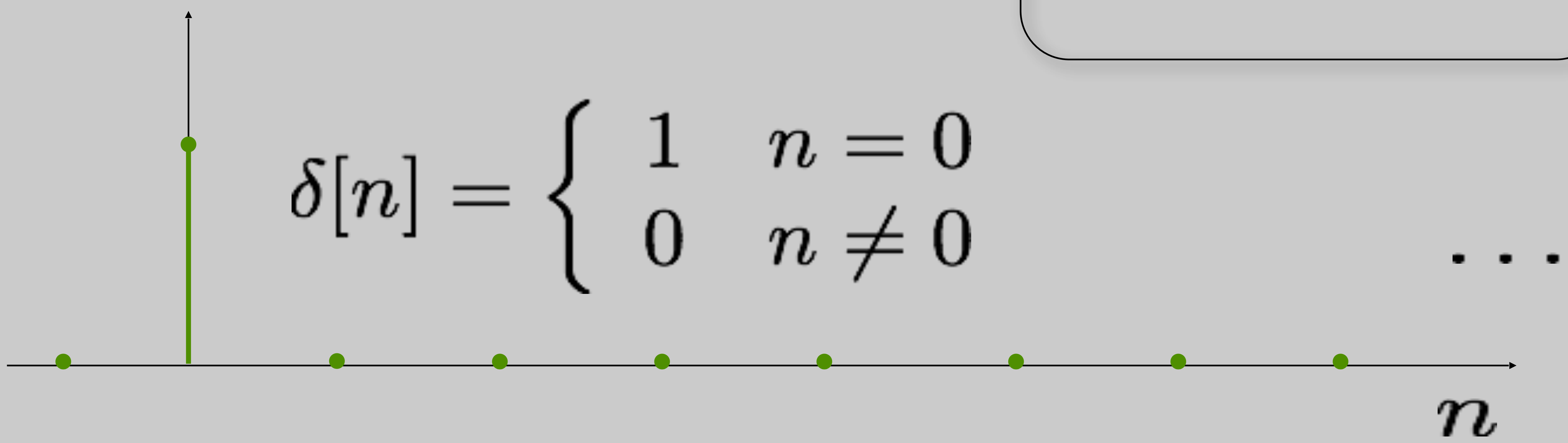
- Linear Time/Shift Invariant (LTI/LSI) systems are completely characterized by their impulse response h[n]

h[n] is the "DNA" of an LTI system

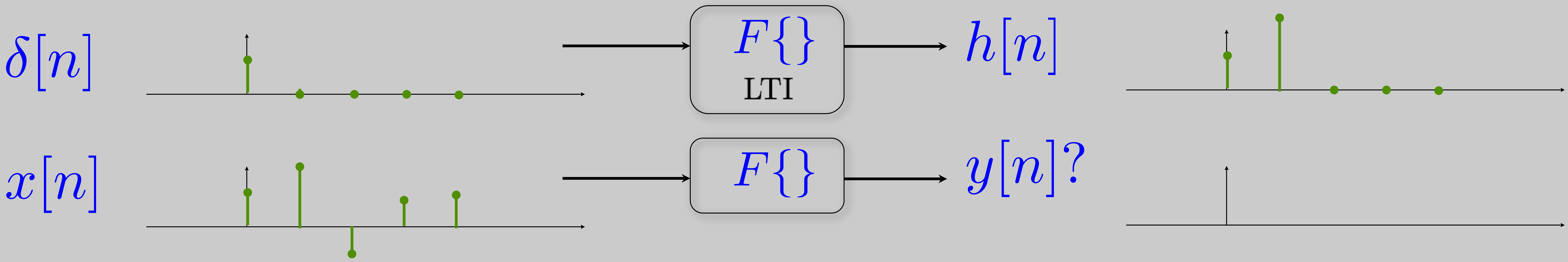
Knowing h[n] is enough to find y[n] for <u>ANY</u> x[n]!

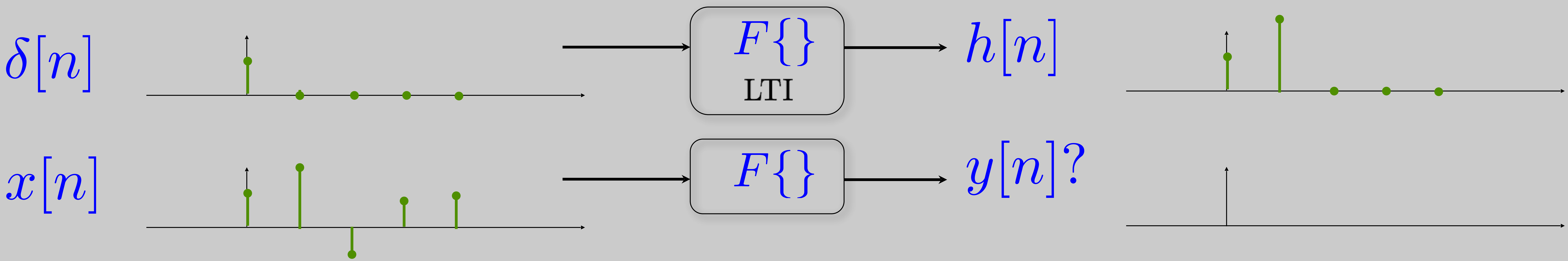
Linear Time Invariant Systems

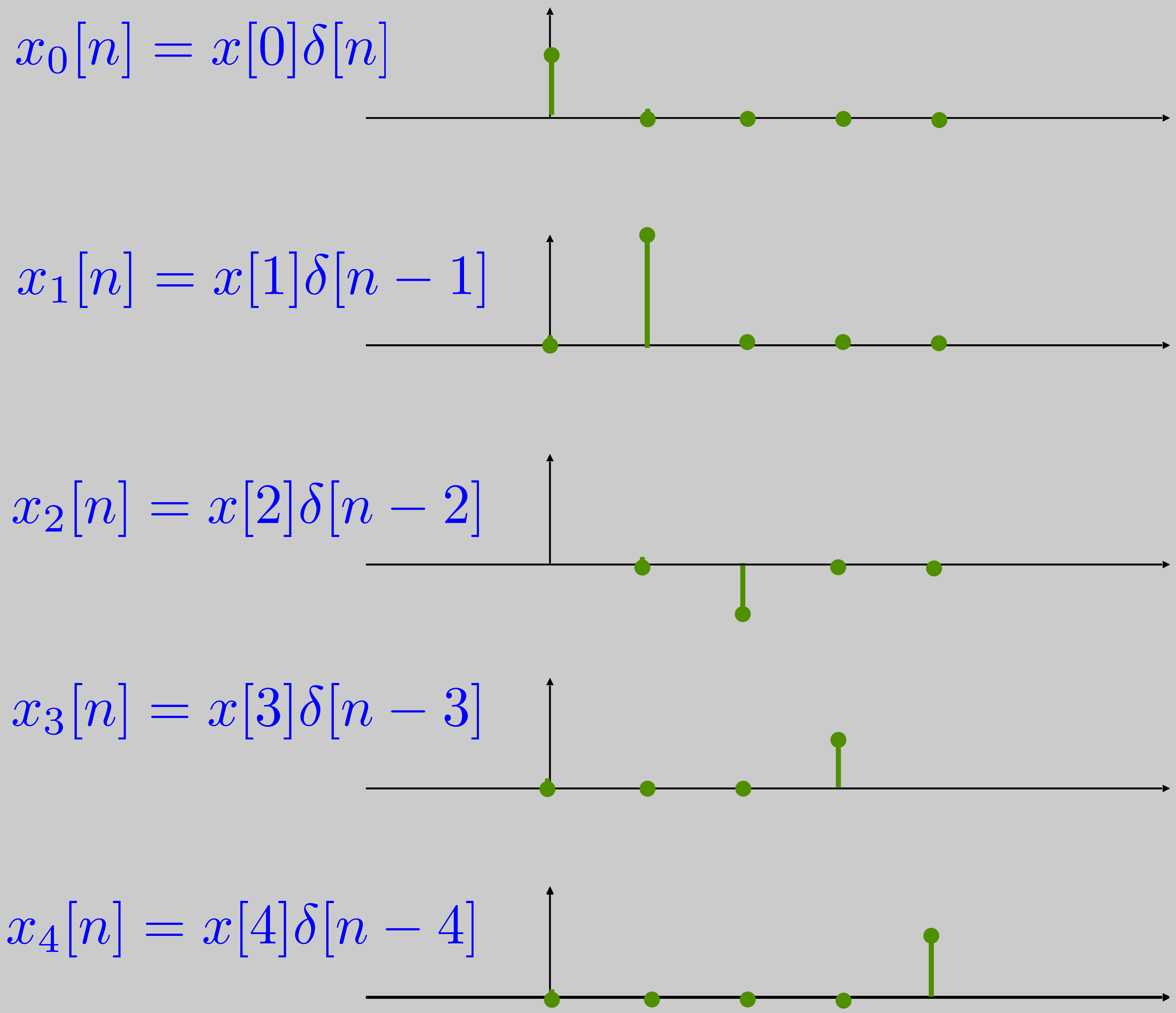
- Linear Time/Shift Invariant (LTI/LSI) systems are completely characterized by their impulse response h[n]



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

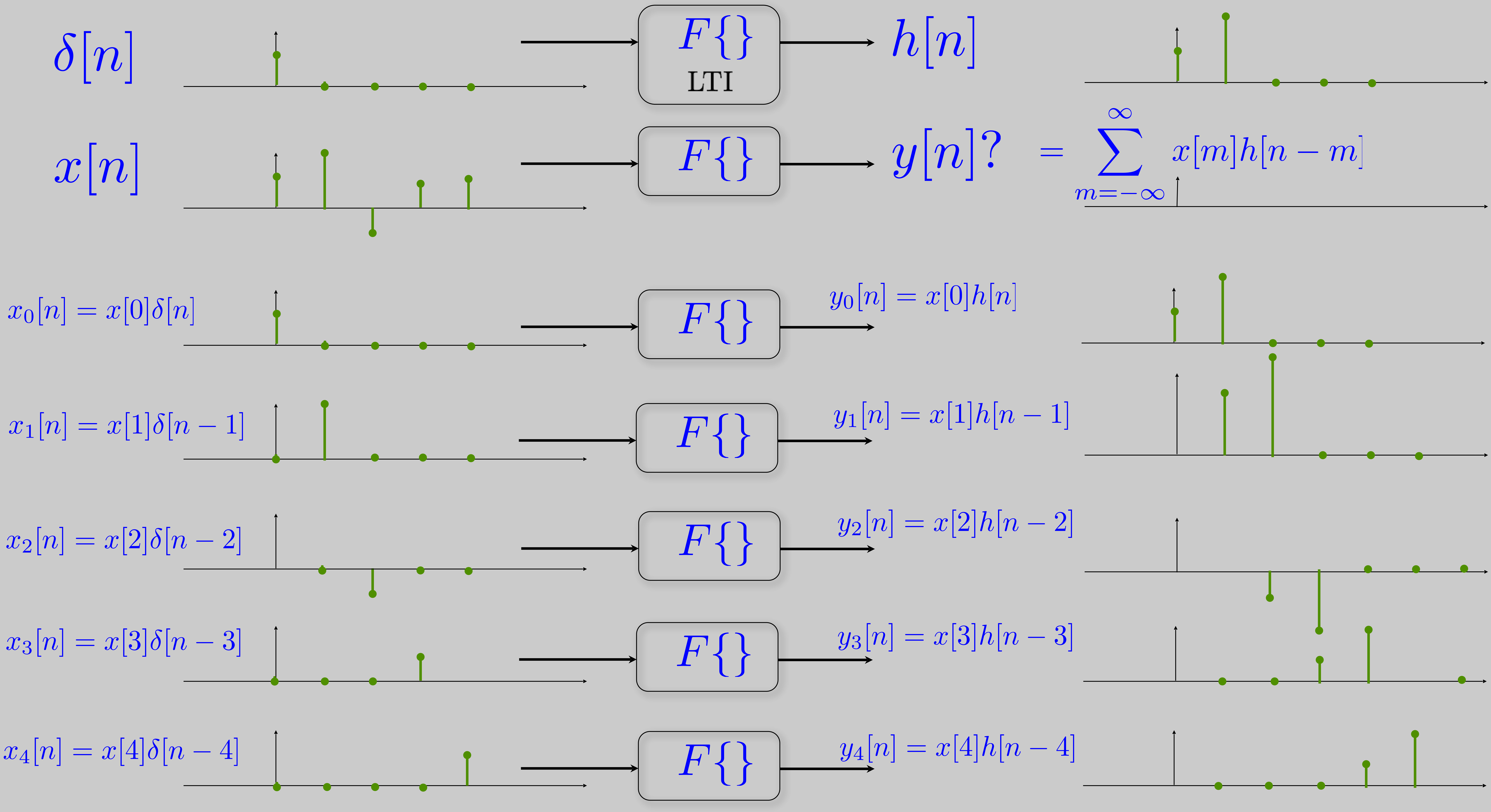

$\delta[n]$
$F\{\}$
LTI
$h[n]$
$x[n]$
$F\{\}$
$y[n]?$





$$x[n] = \sum_{m=-\infty}^{\infty} x_m[n]$$

$$= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$


$\delta[n]$
$F\{\}$
LTI
$h[n]$
$x[n]$
$F\{\}$
$y[n]? = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$
$x_0[n] = x[0]\delta[n]$
$F\{\}$
$y_0[n] = x[0]h[n]$
$x_1[n] = x[1]\delta[n-1]$
$F\{\}$
$y_1[n] = x[1]h[n-1]$
$x_2[n] = x[2]\delta[n-2]$
$F\{\}$
$y_2[n] = x[2]h[n-2]$
$x_3[n] = x[3]\delta[n-3]$
$F\{\}$
$y_3[n] = x[3]h[n-3]$
$x_4[n] = x[4]\delta[n-4]$
$F\{\}$
$y_4[n] = x[4]h[n-4]$

Linear Time Invariant Systems

- Decompose x[n]:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

$$= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

- Compute output:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

Convolution sum

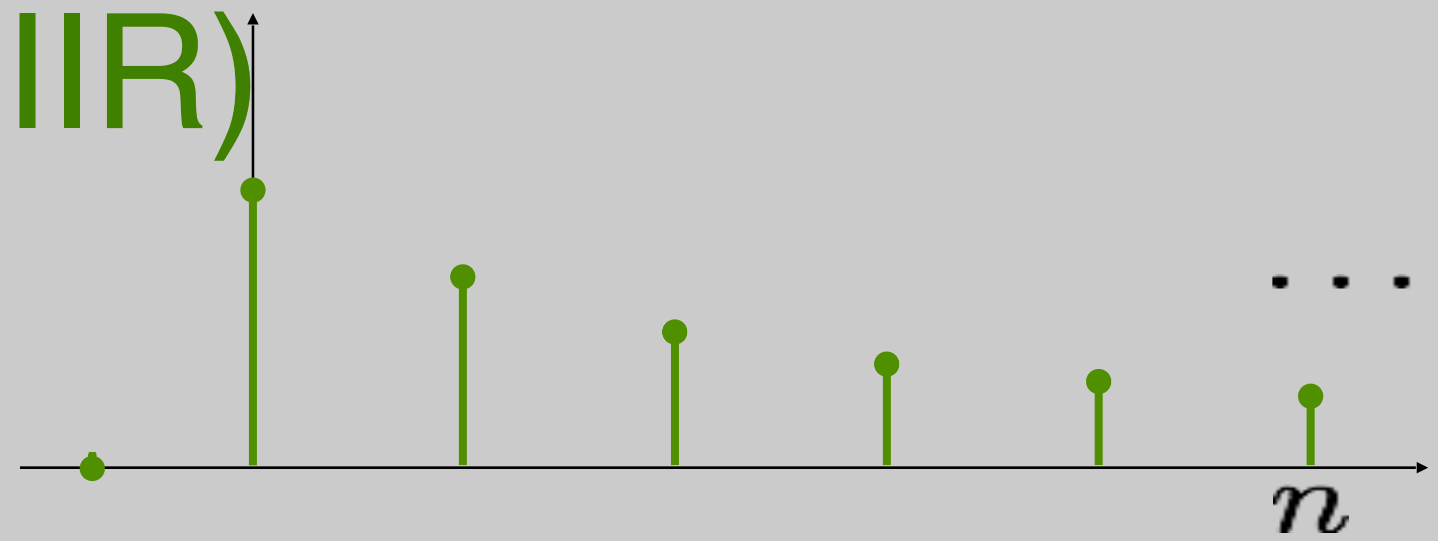
Sum of weighted, delayed impulse responses!

Example:

$$y[n] = ay[n-1] + x[n]$$

$$h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

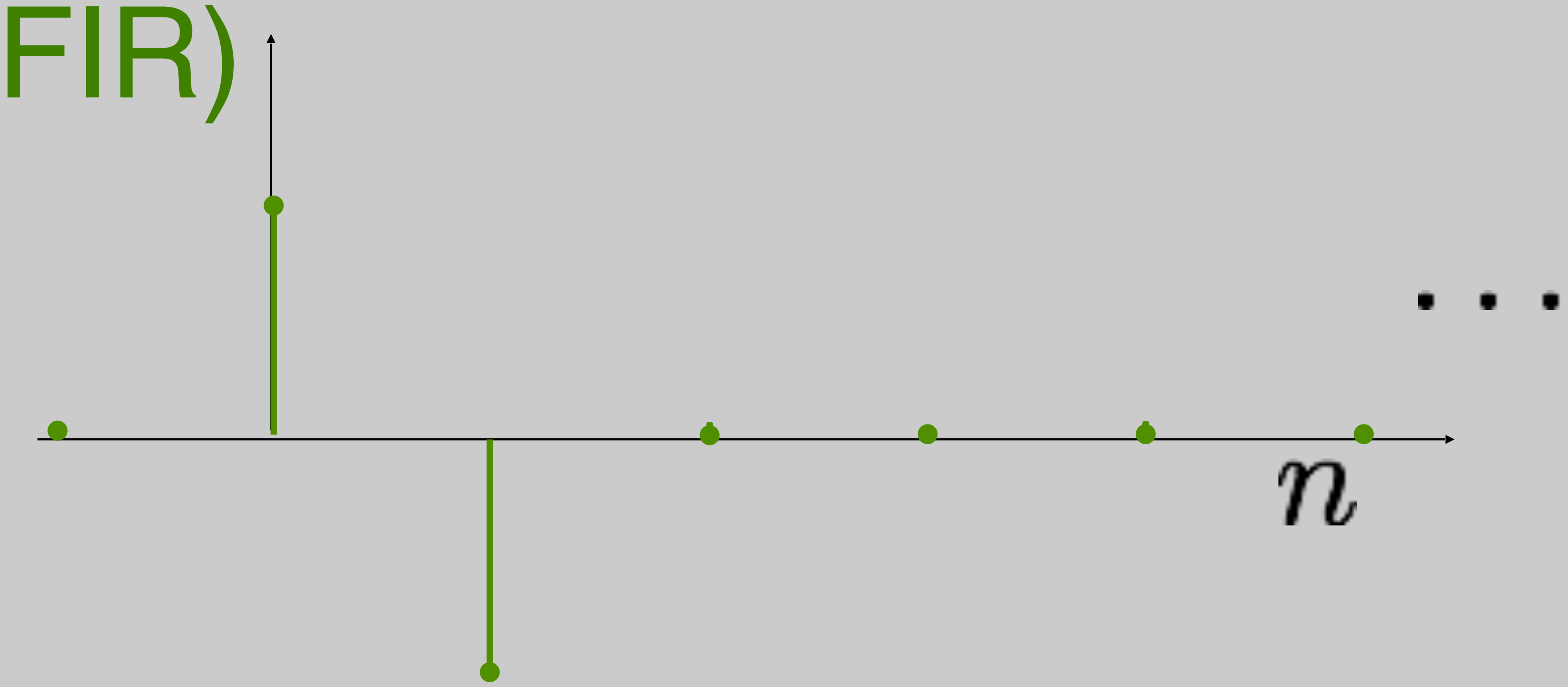
Infinite impulse response (IIR)



$$y[n] = x[n] - x[n-1]$$

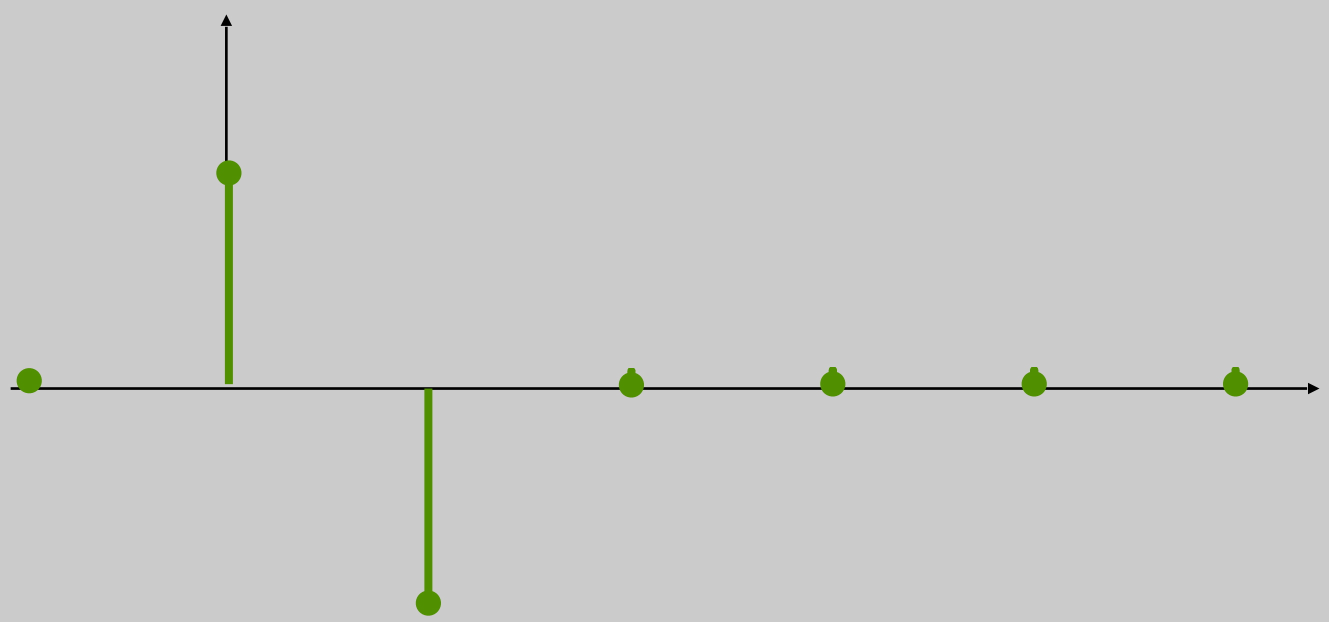
$$h[n] = \delta[n] - \delta[n-1]$$

finite impulse response (FIR)



Convolution Sum

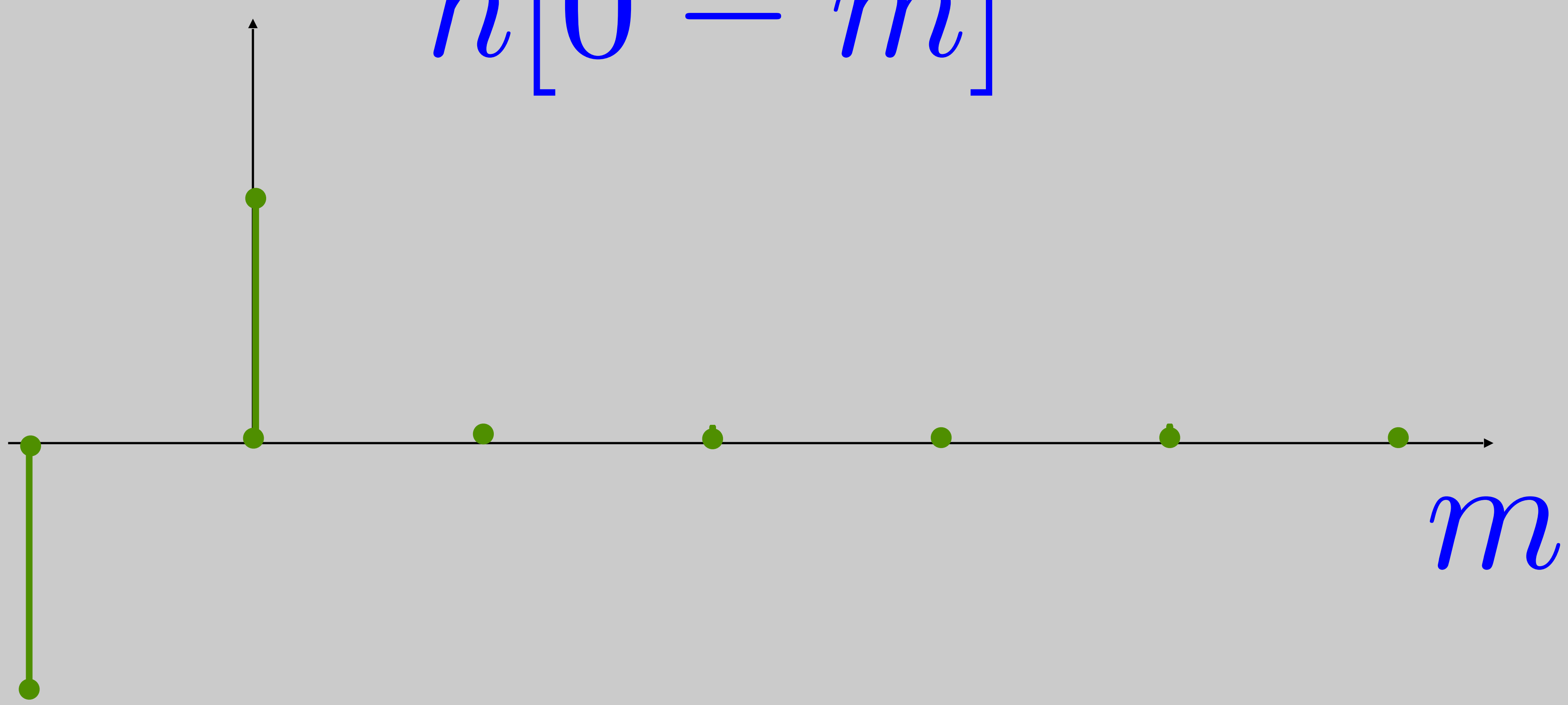
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$



$$h[n] = \delta[n] - \delta[n-1]$$

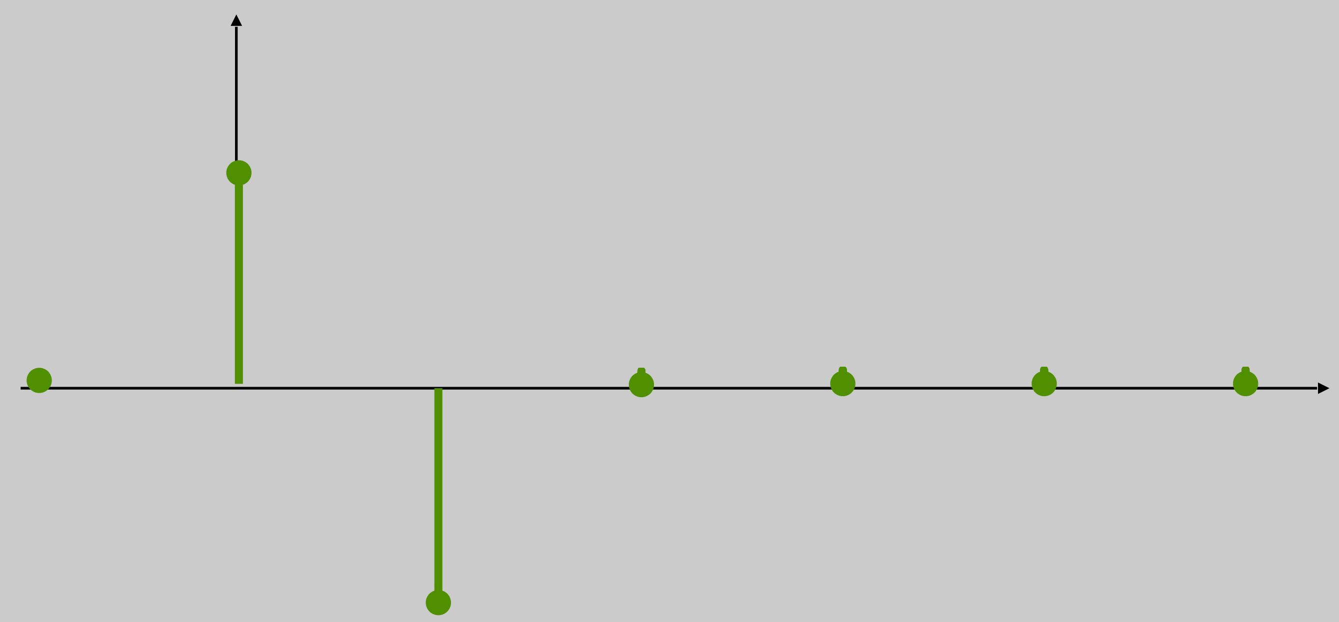
- What is h[n-m] for different n's?

$$h[0-m]$$



Convolution Sum

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$



$$h[n] = \delta[n] - \delta[n-1]$$

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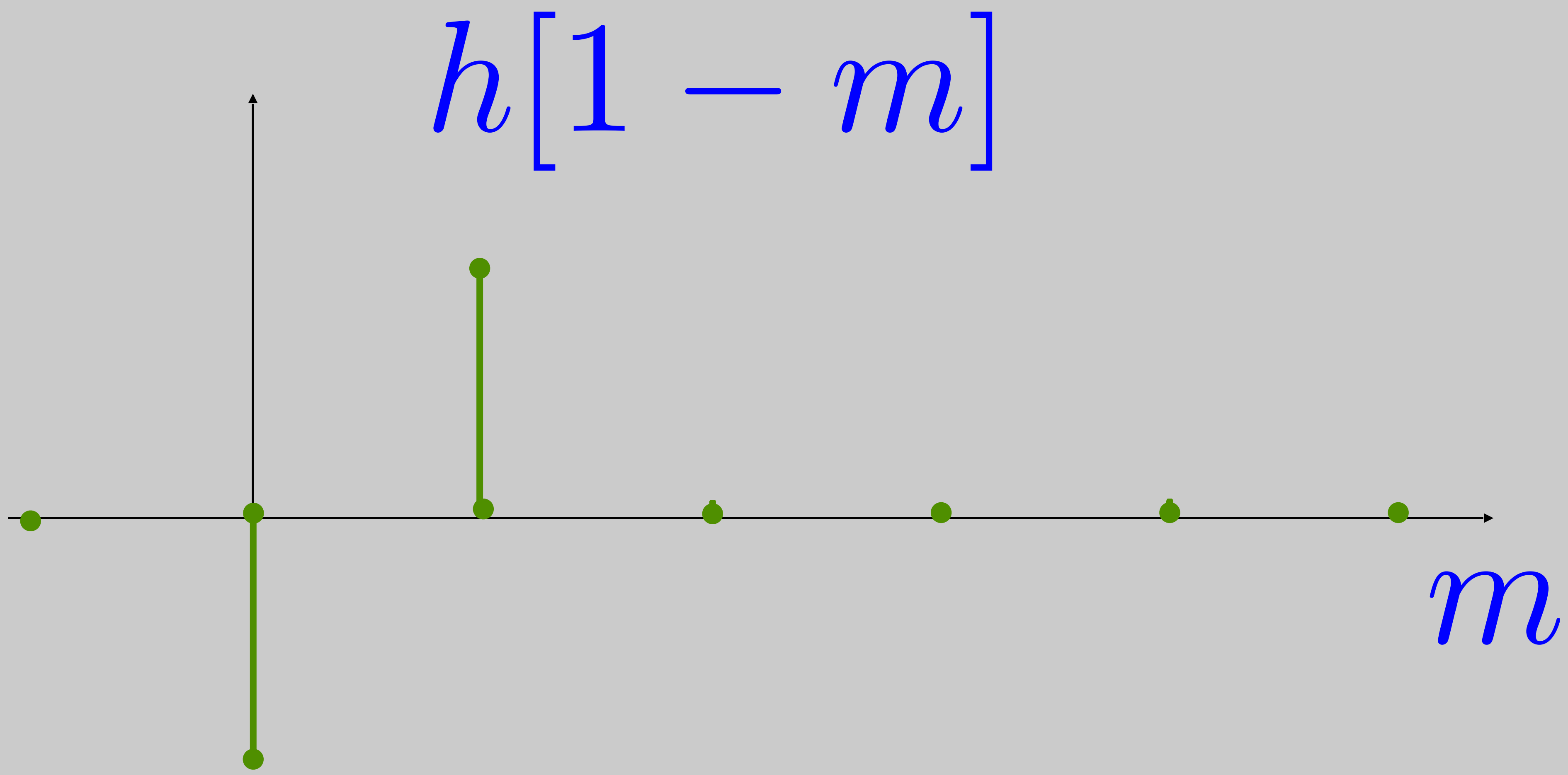
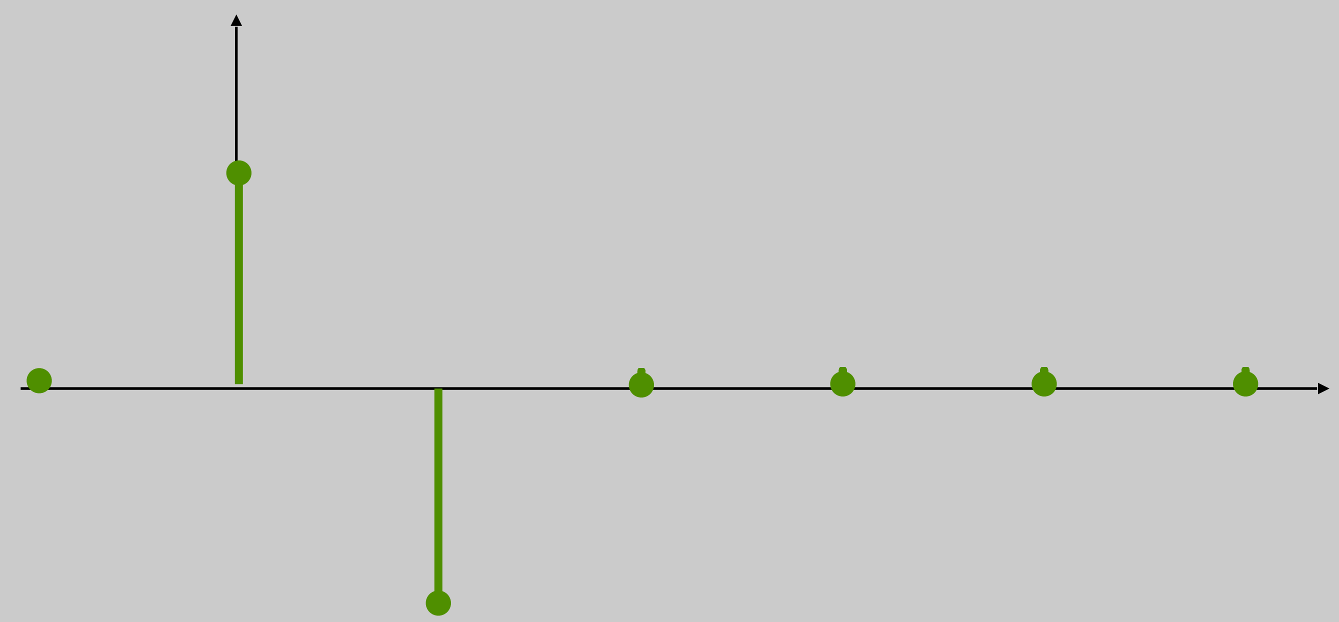


Convolution Sum

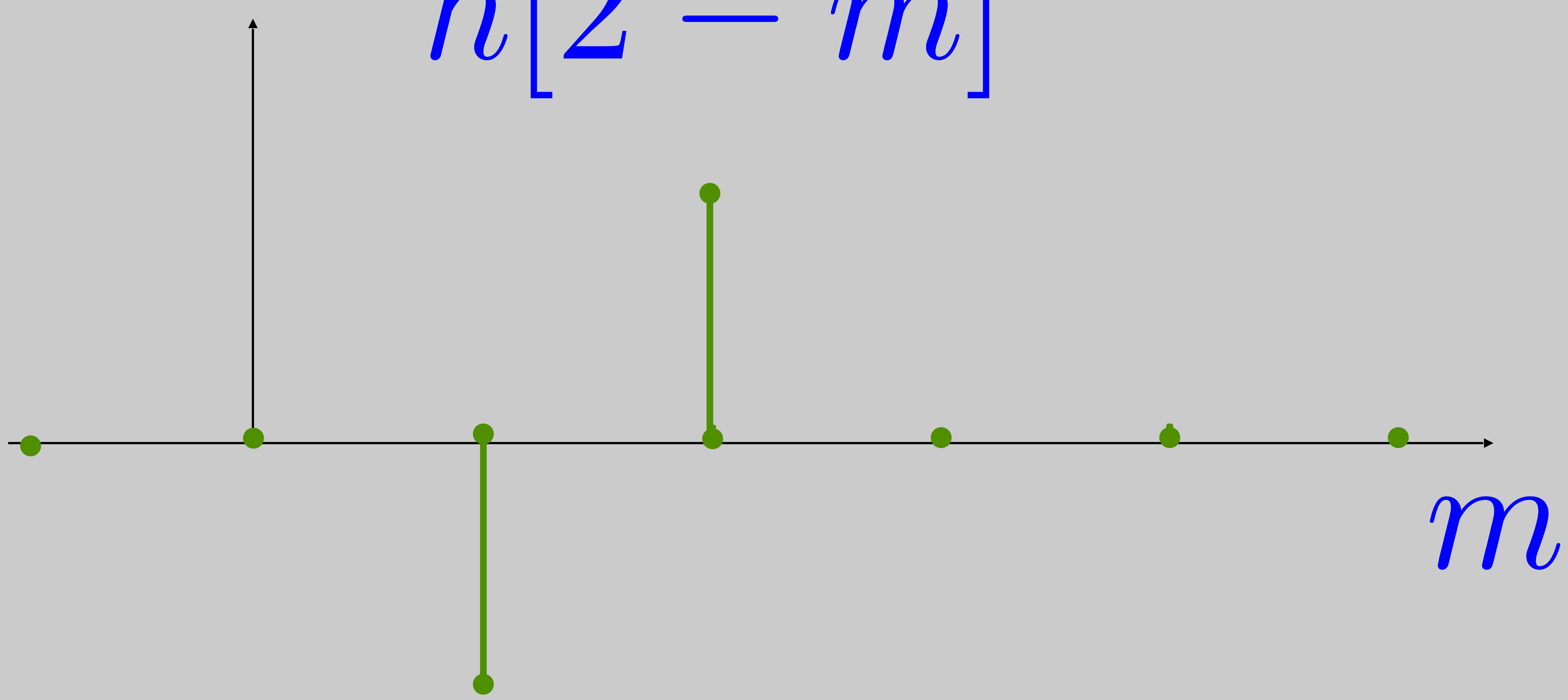
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

- What is h[n-m] for different n's?



$$h[n] = \delta[n] - \delta[n-1]$$

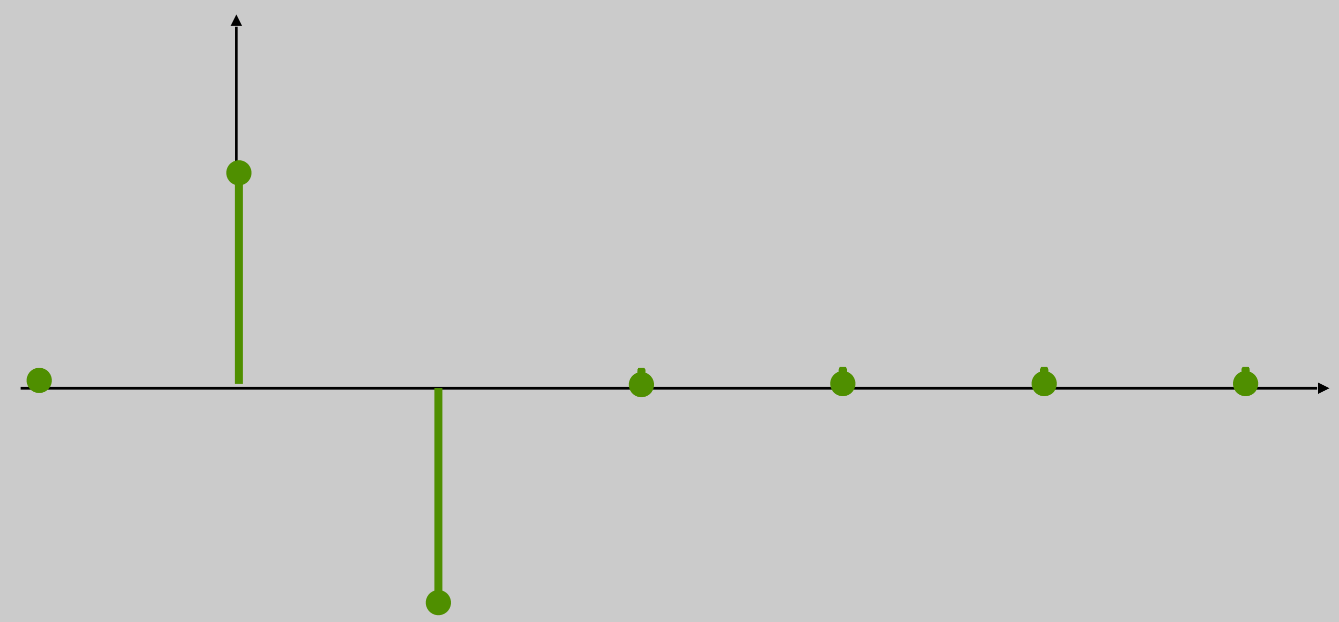
$$h[2-m]$$



Convolution Sum

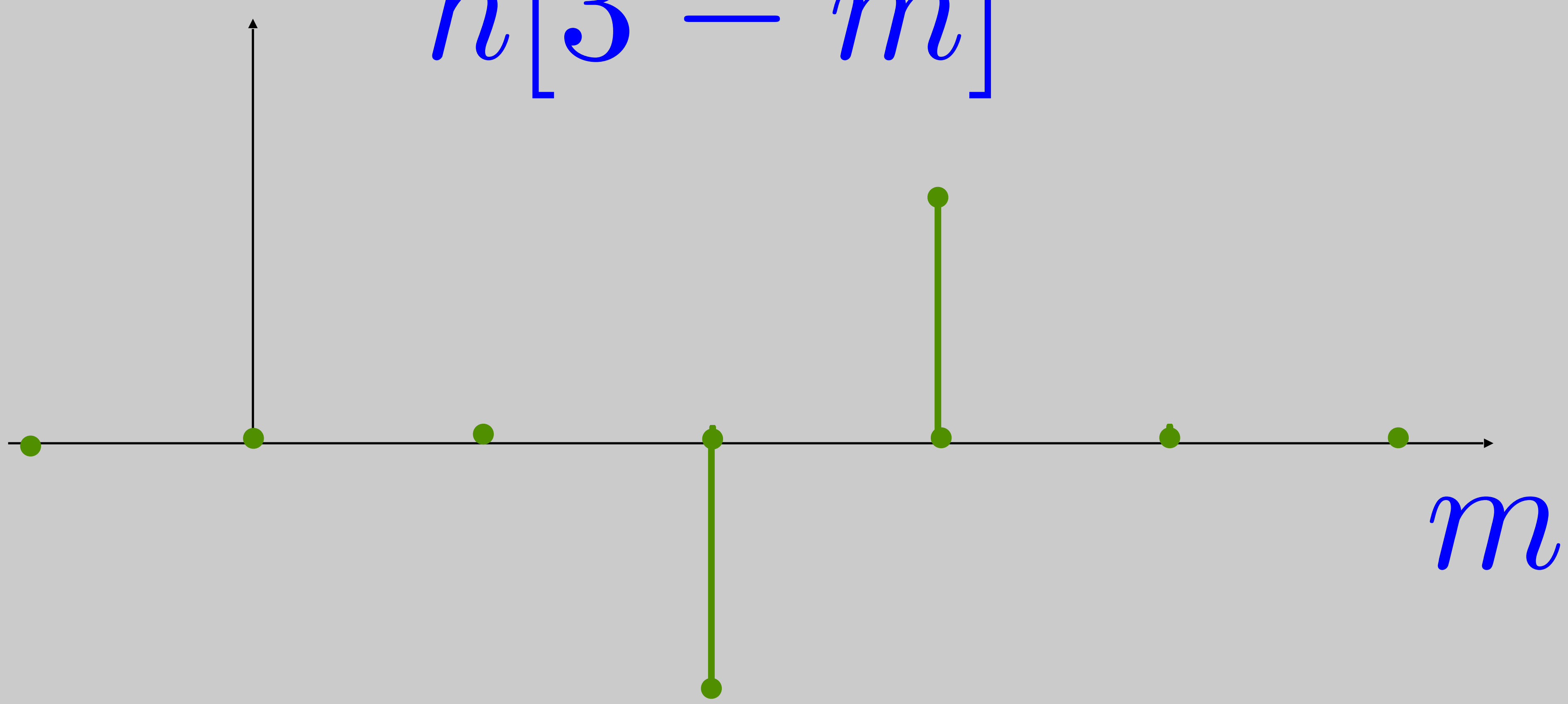
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

- What is h[n-m] for different n's?

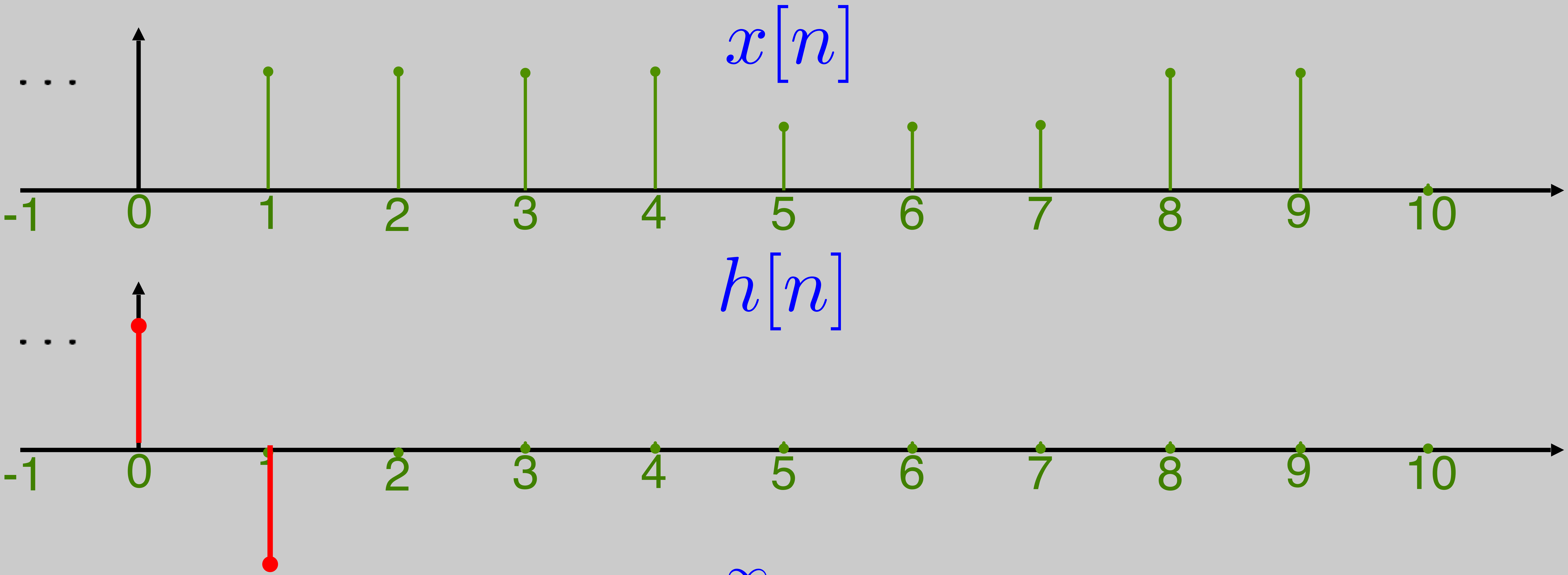


$$h[n] = \delta[n] - \delta[n-1]$$

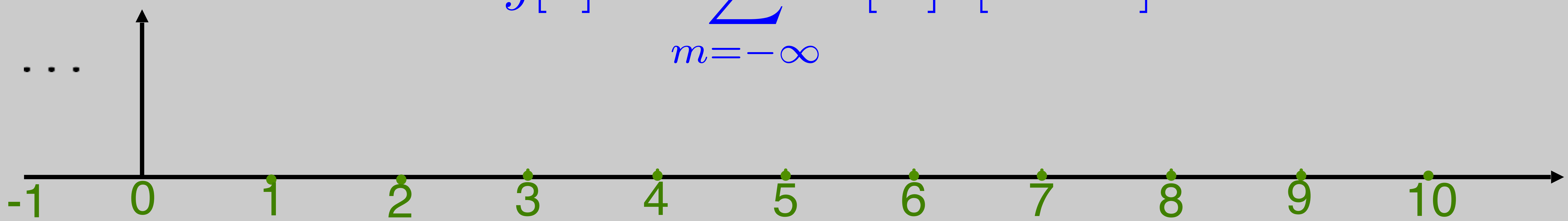
$$h[3-m]$$



Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



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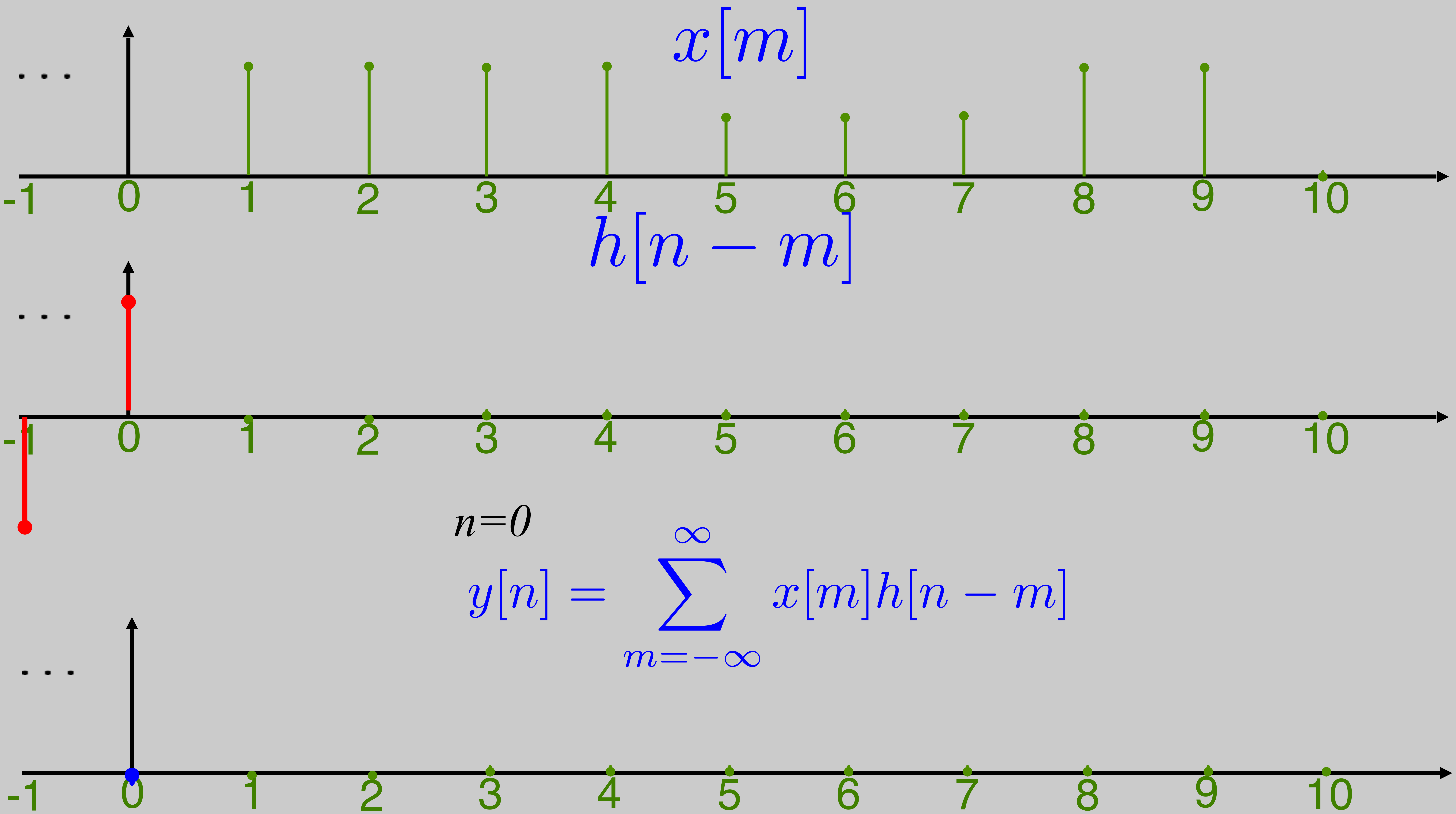
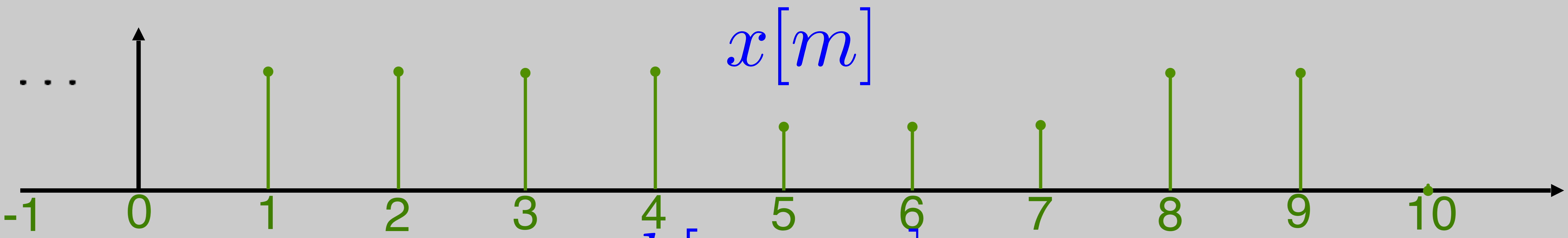
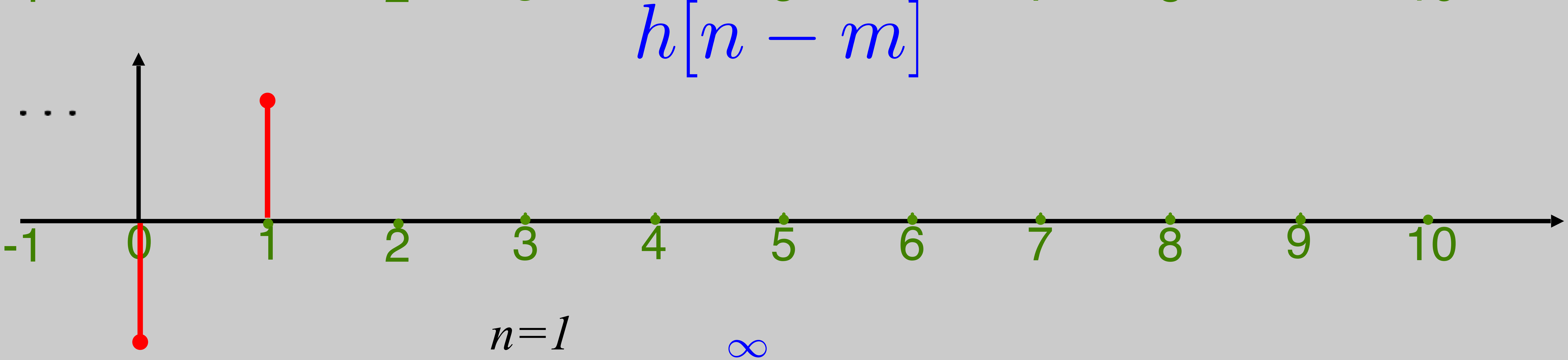


n=0

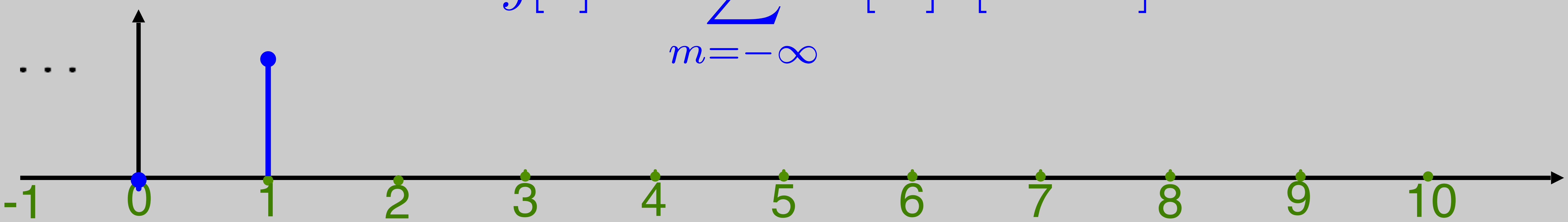
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution

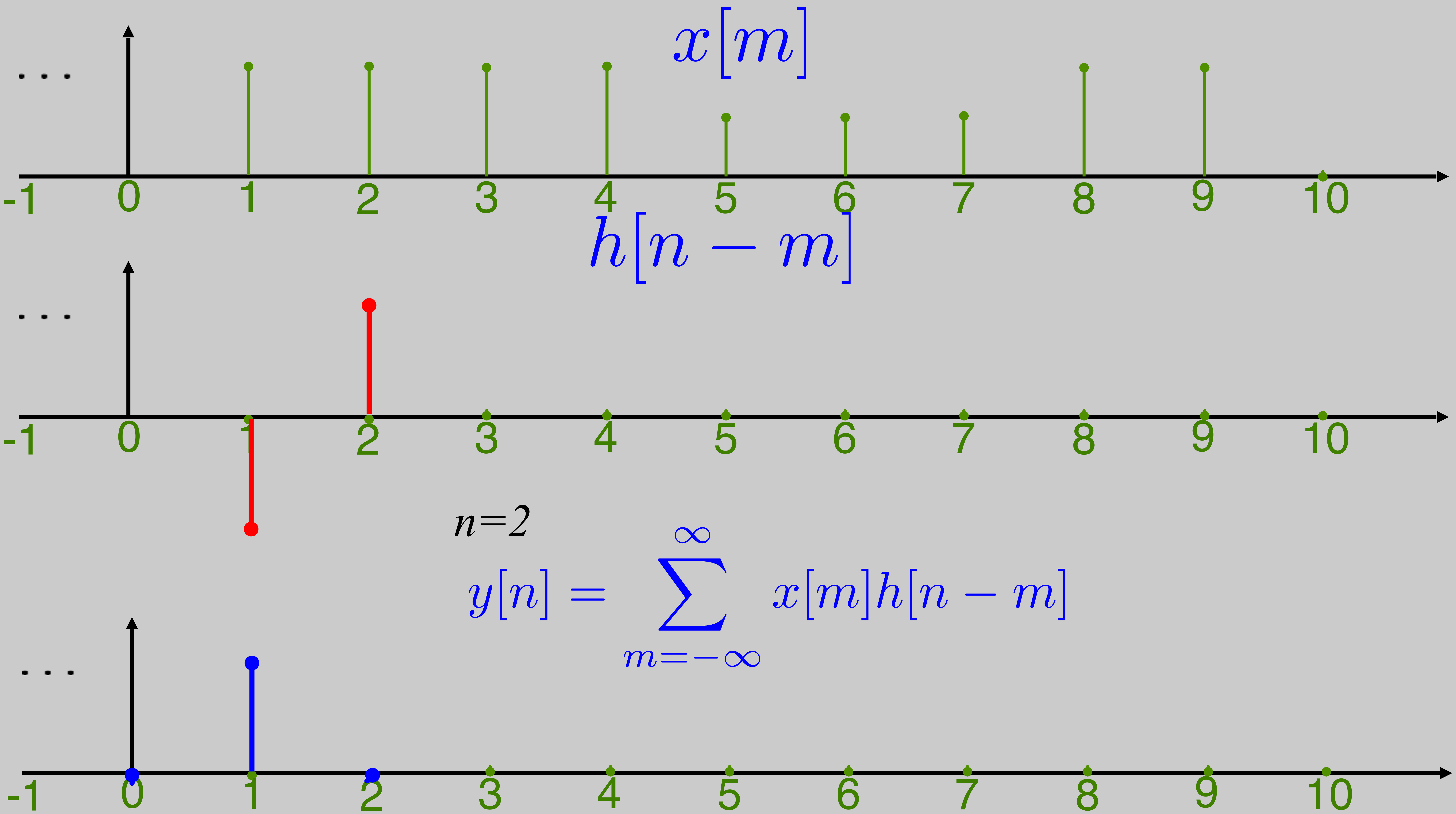




$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



Graphical Example of Convolution



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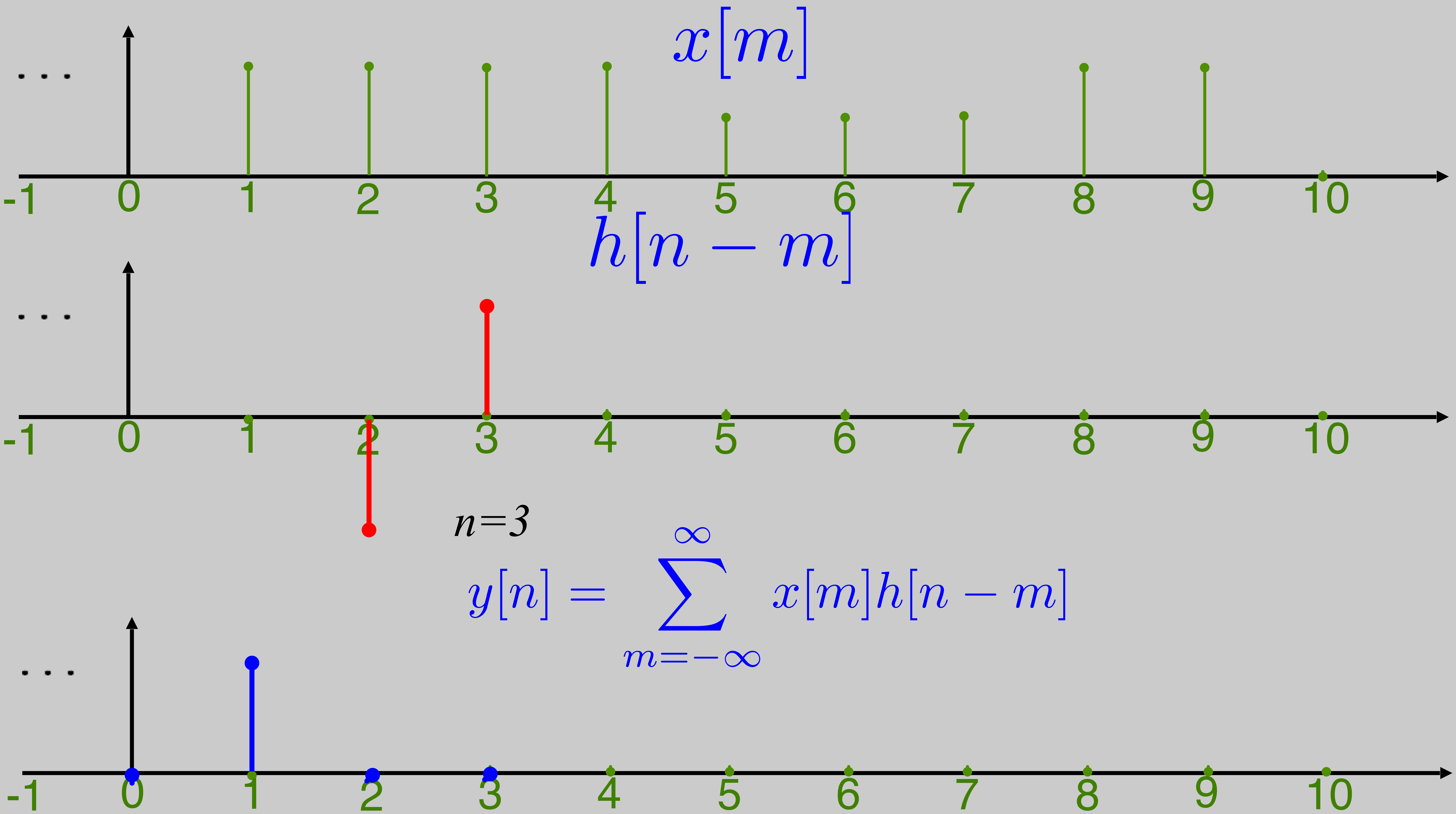
Graphical Example of Convolution



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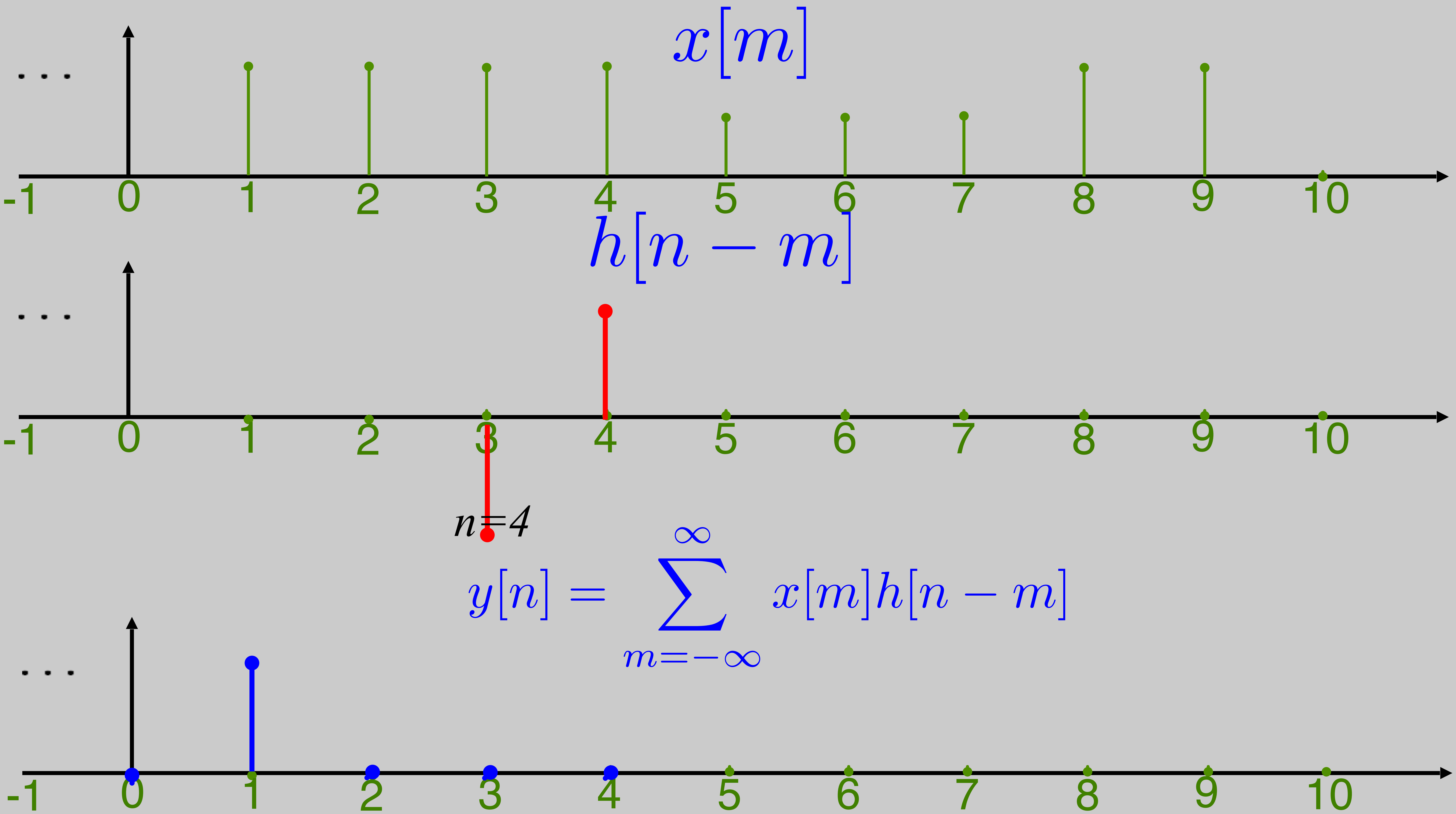
Graphical Example of Convolution



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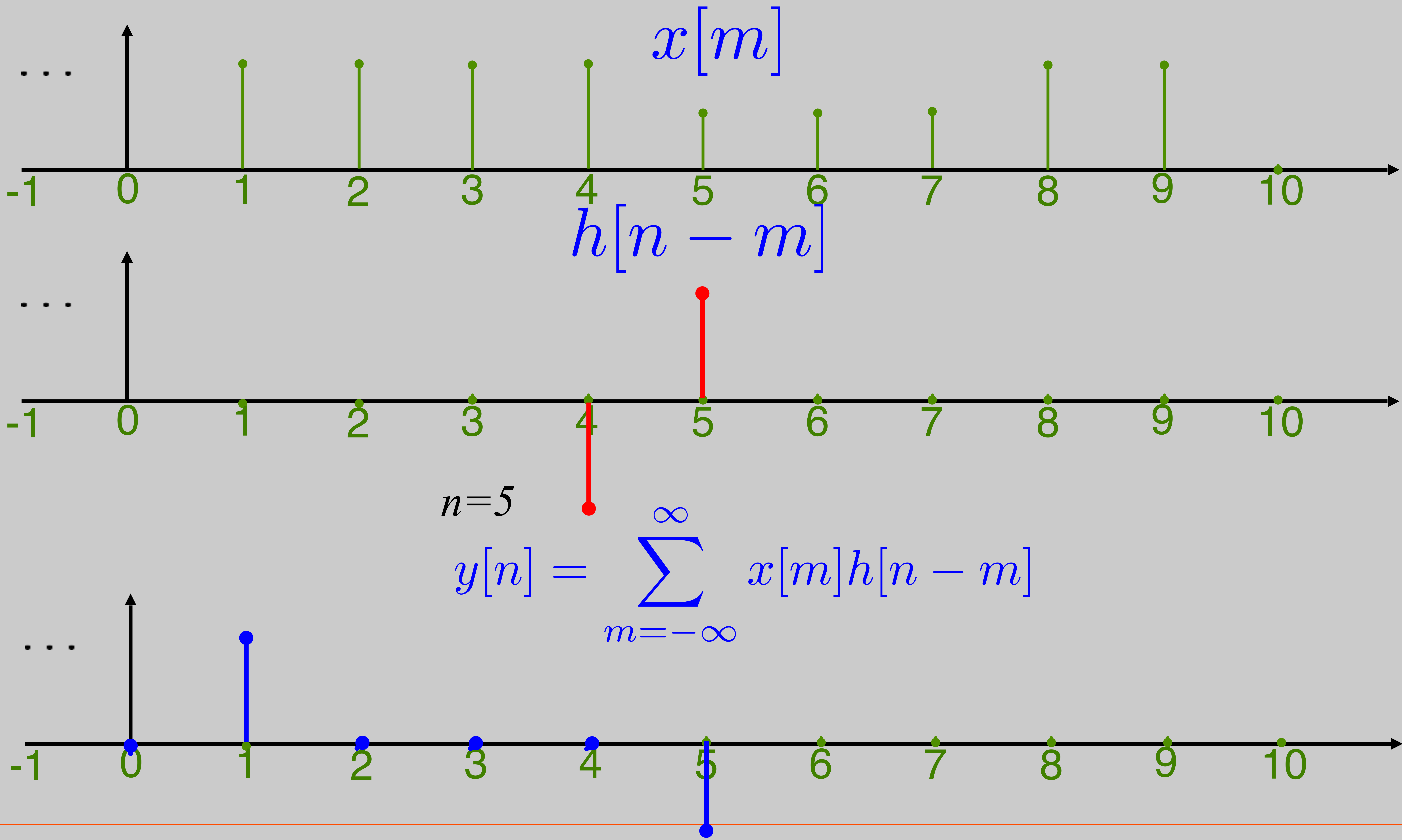
Graphical Example of Convolution



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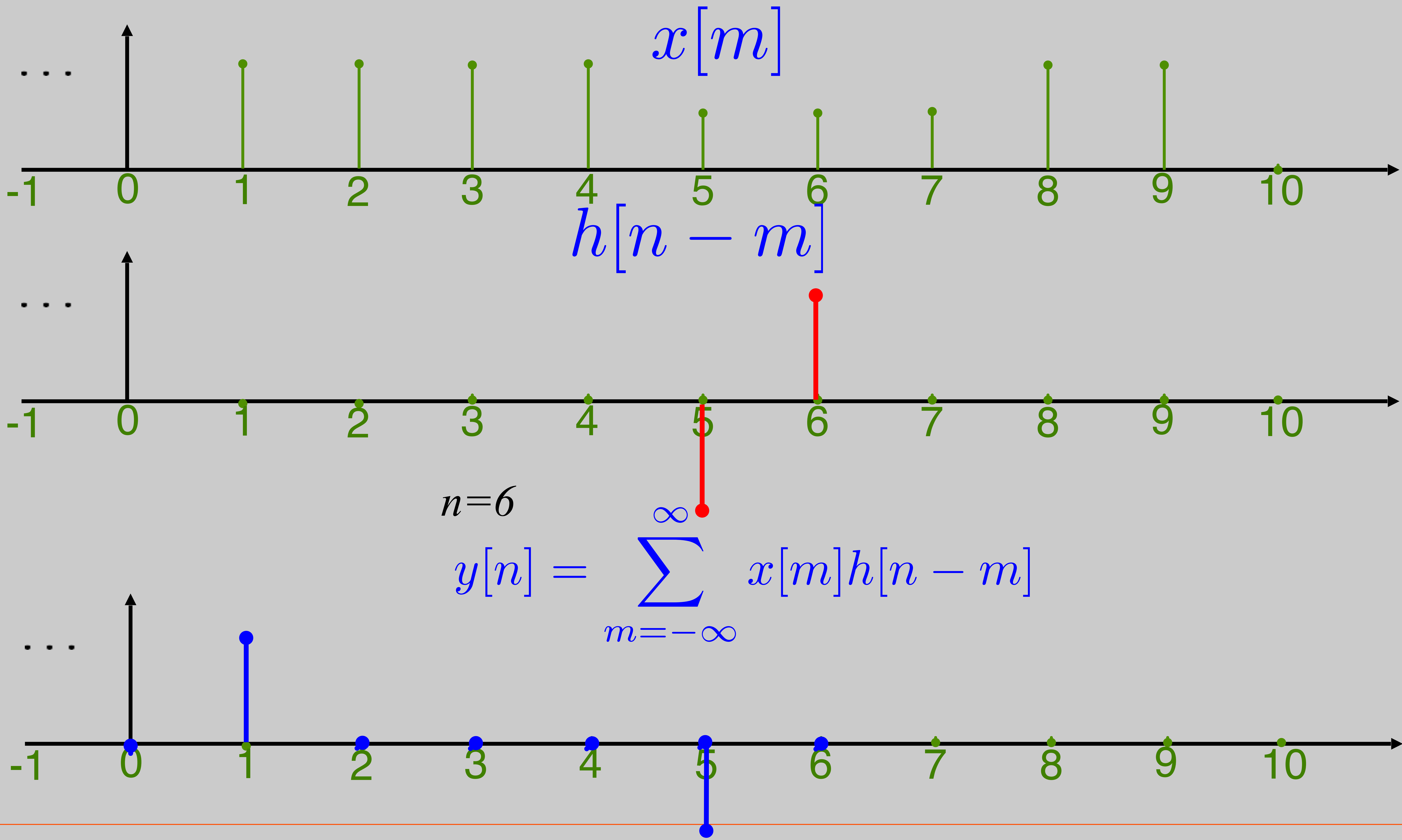
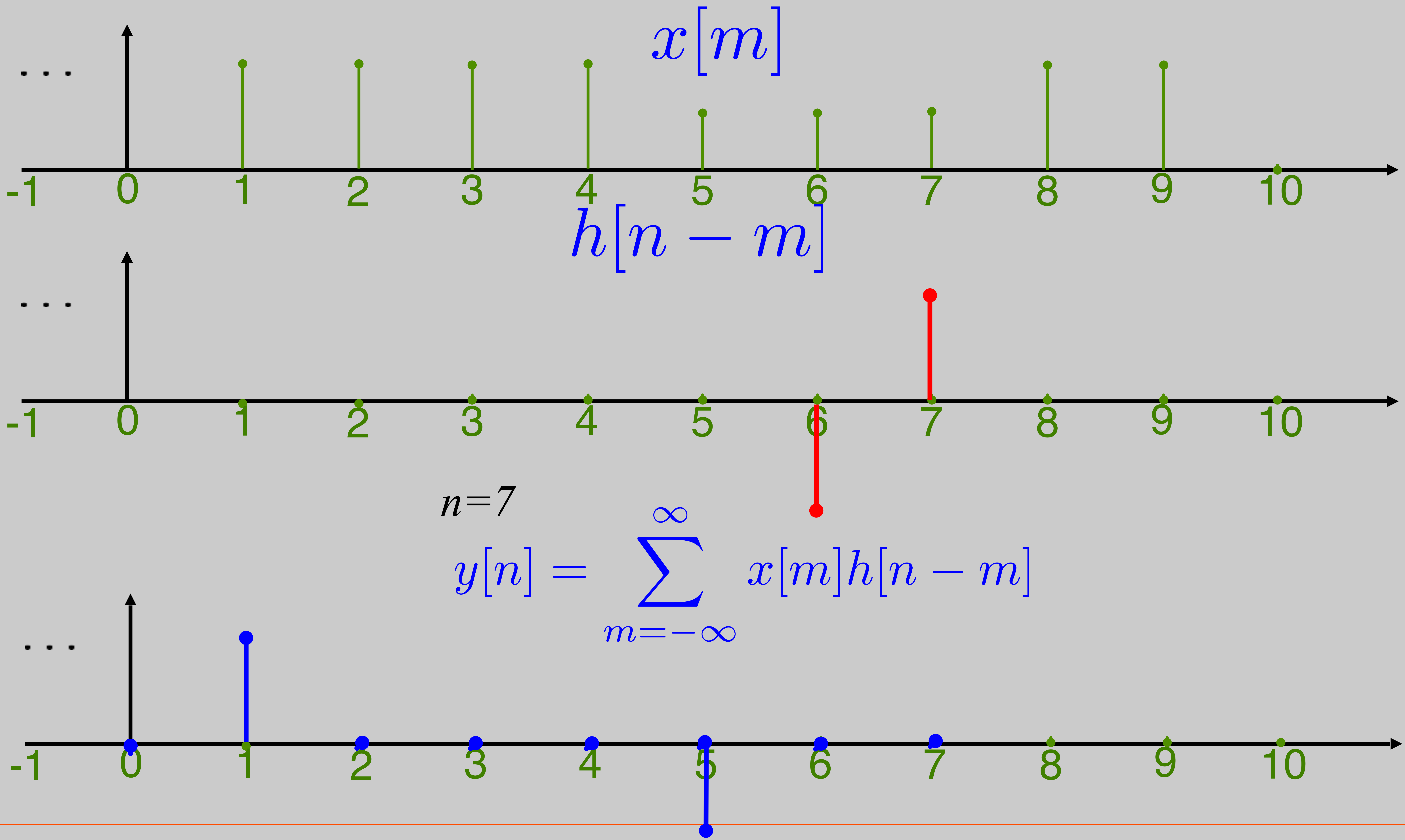
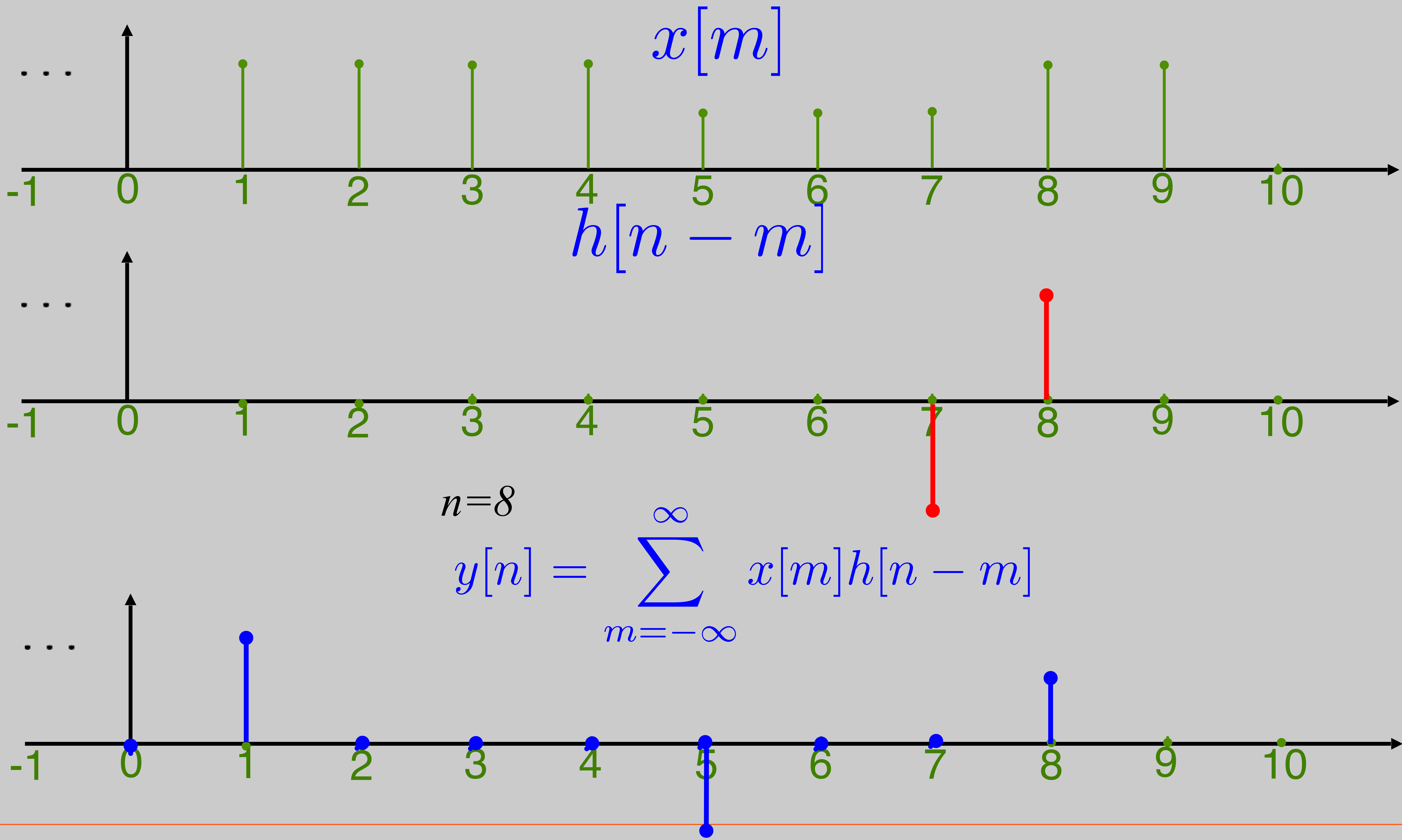
Graphical Example of Convolution



Graphical Example of Convolution

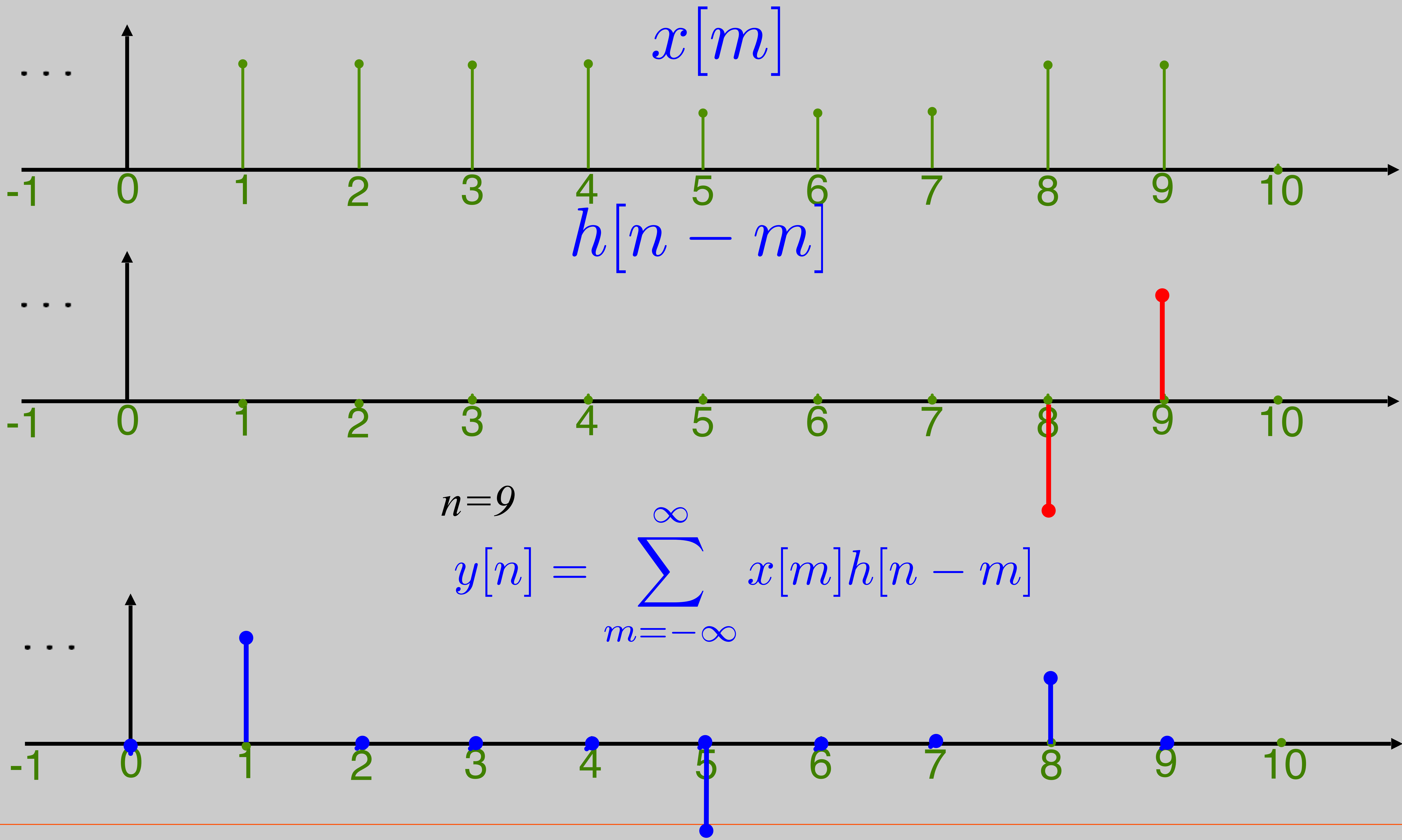


Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

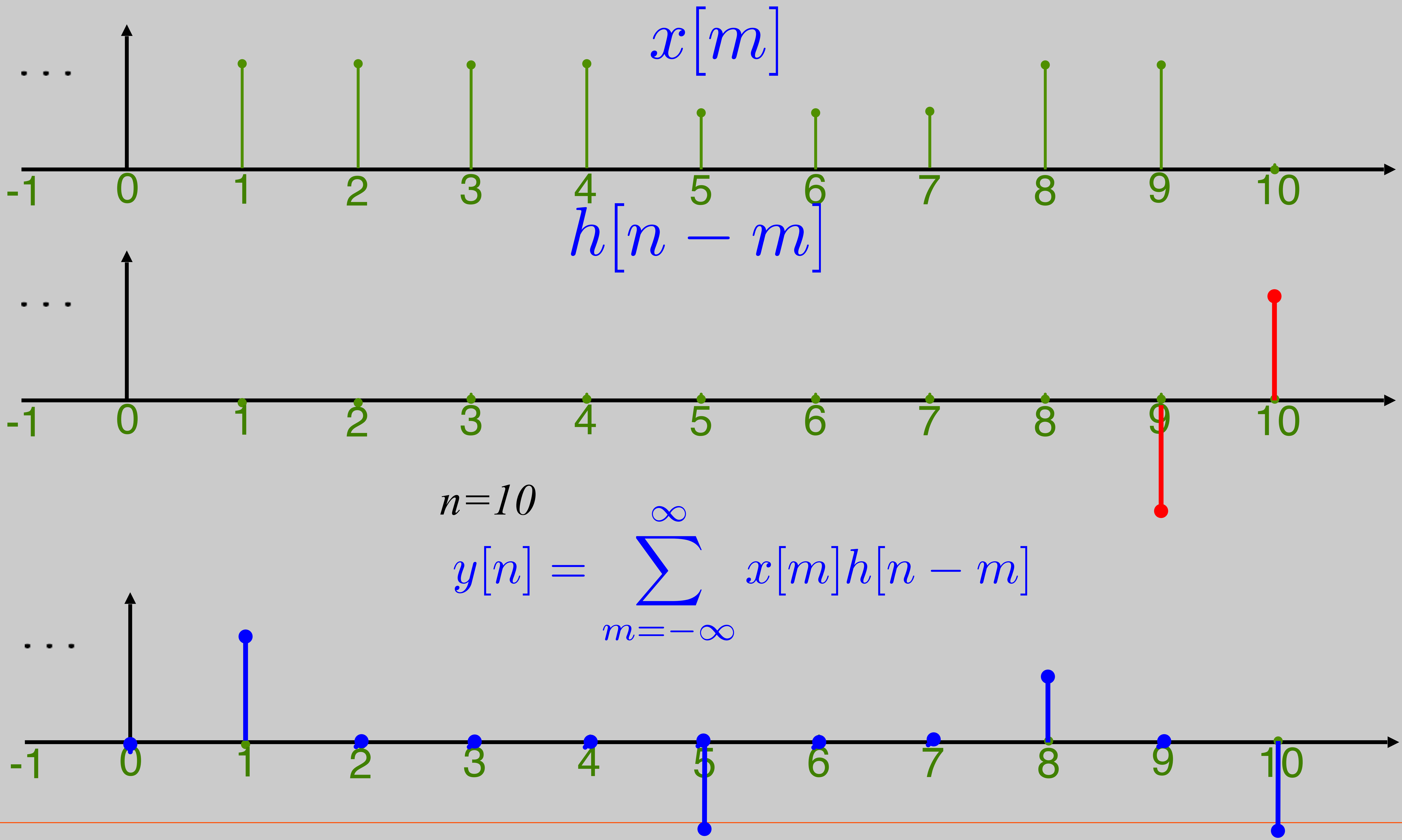
Graphical Example of Convolution



n=9

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Example

BIBO Stability of LTI systems

- LTI system is BIBO stable if, and only if h[n] is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Proof (if):

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} x[m]h[n-m] \right| \leq \sum_{m=-\infty}^{\infty} |x[m]| \cdot |h[n-m]|$$

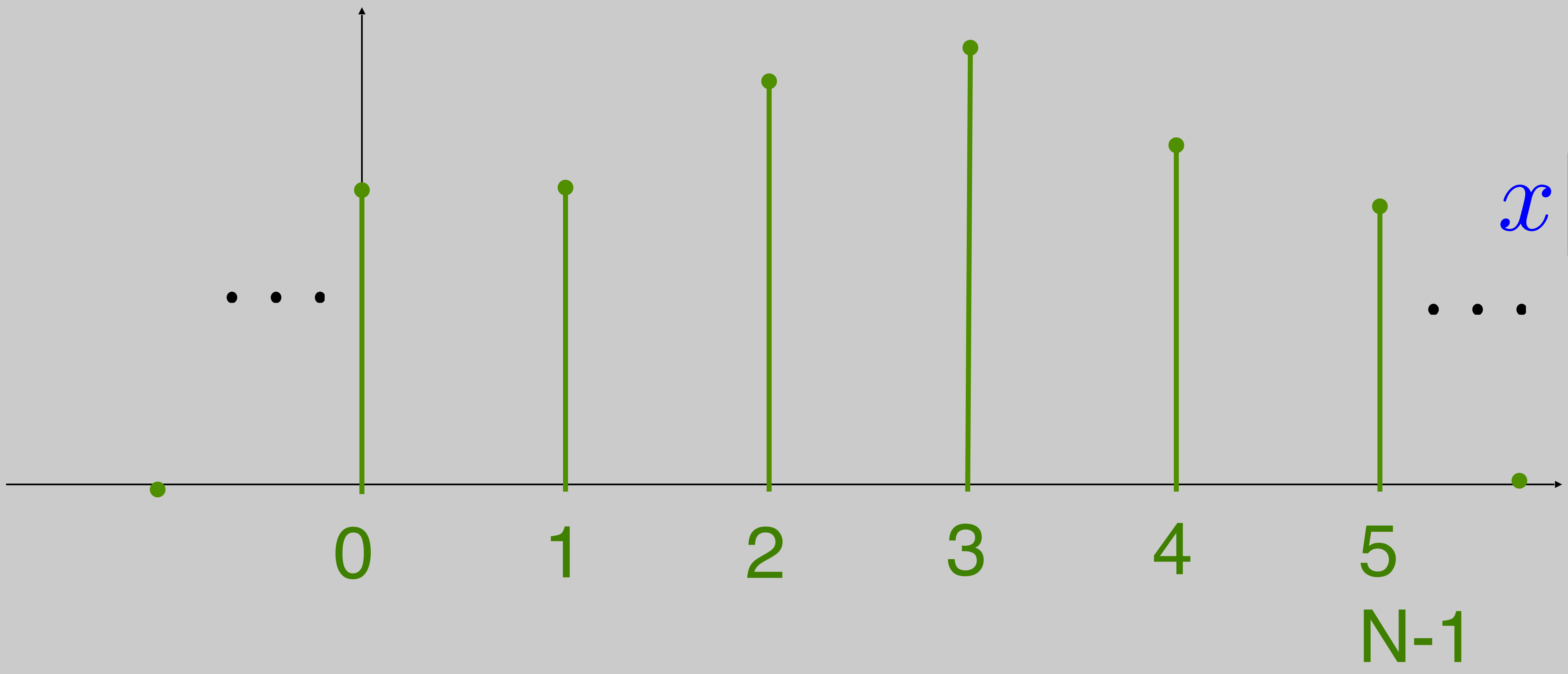
$$|x[m]| < M$$

$$\leq M \sum_{m=-\infty}^{\infty} |h[n-m]| = M \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Only if in EE123

Finite Sequences

- Consider a finite sequence of length N



$$x[n] = \begin{cases} \text{something} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

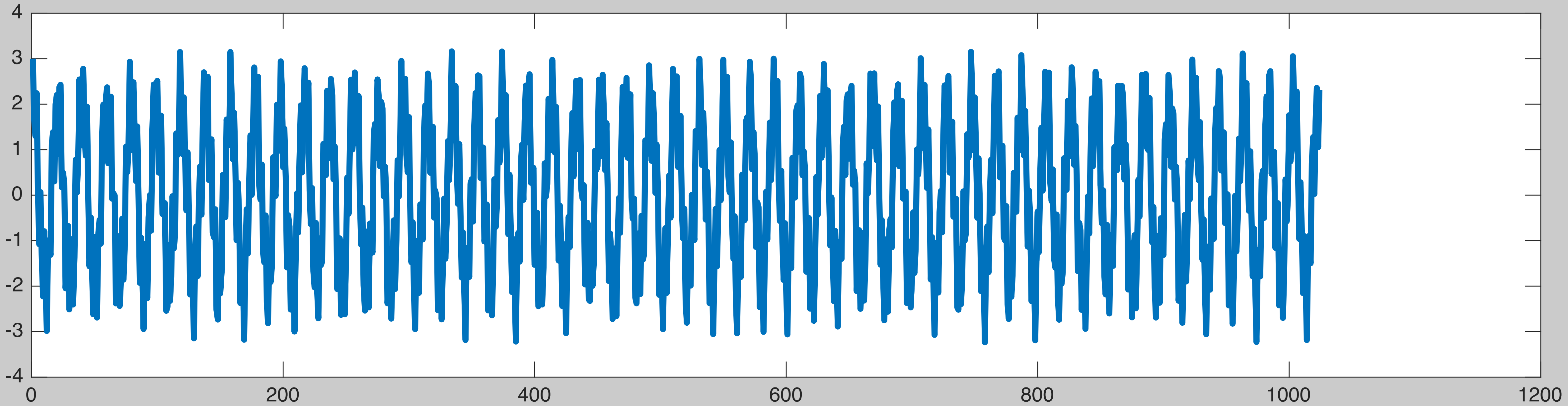
- Can also be written as a vector

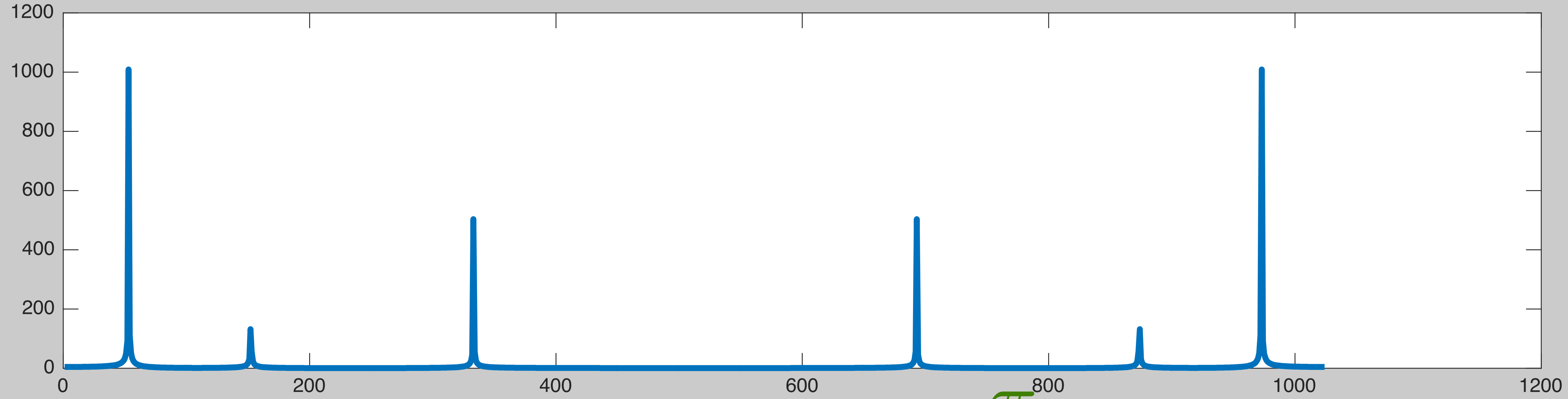
$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Why?

- To compute this:

$x[n]$ $\Rightarrow$ $X[k]$





$\frac{\pi}{2}$ π $-\frac{\pi}{2}$

Finite Sequences as Vectors

- Define an inner-product (for R^N):

$$< \vec{x}, \vec{y} > = \vec{x} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]y[n] =$$

$$= \vec{x}^T \vec{y}$$

So,

$$< \vec{x}, \vec{x} > = \sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} x^2[n] = ||\vec{x}||^2$$

$$\Rightarrow \vec{x}^T \vec{x} = ||\vec{x}||^2$$

Finite Sequences as Vectors

- What about complex?

$$x \cdot x = x^2 = (x_r + jx_i)(x_r + jx_i) = x_r^2 - x_i^2 + 2jx_r x_i \neq ||x||^2$$

but,

$$x^* \cdot x = (x_r - jx_i)(x_r + jx_i) = x_r^2 + x_i^2 = ||x||^2$$

- Transpose vs Transpost conjugate

$$\vec{x} = \begin{bmatrix} 1 \\ j \\ 1+j \end{bmatrix}$$

$$\vec{x}^T = \begin{bmatrix} 1 & j & 1+j \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} 1 & -j & 1-j \end{bmatrix}$$

Finite Sequences as Vectors

- Define Complex inner product

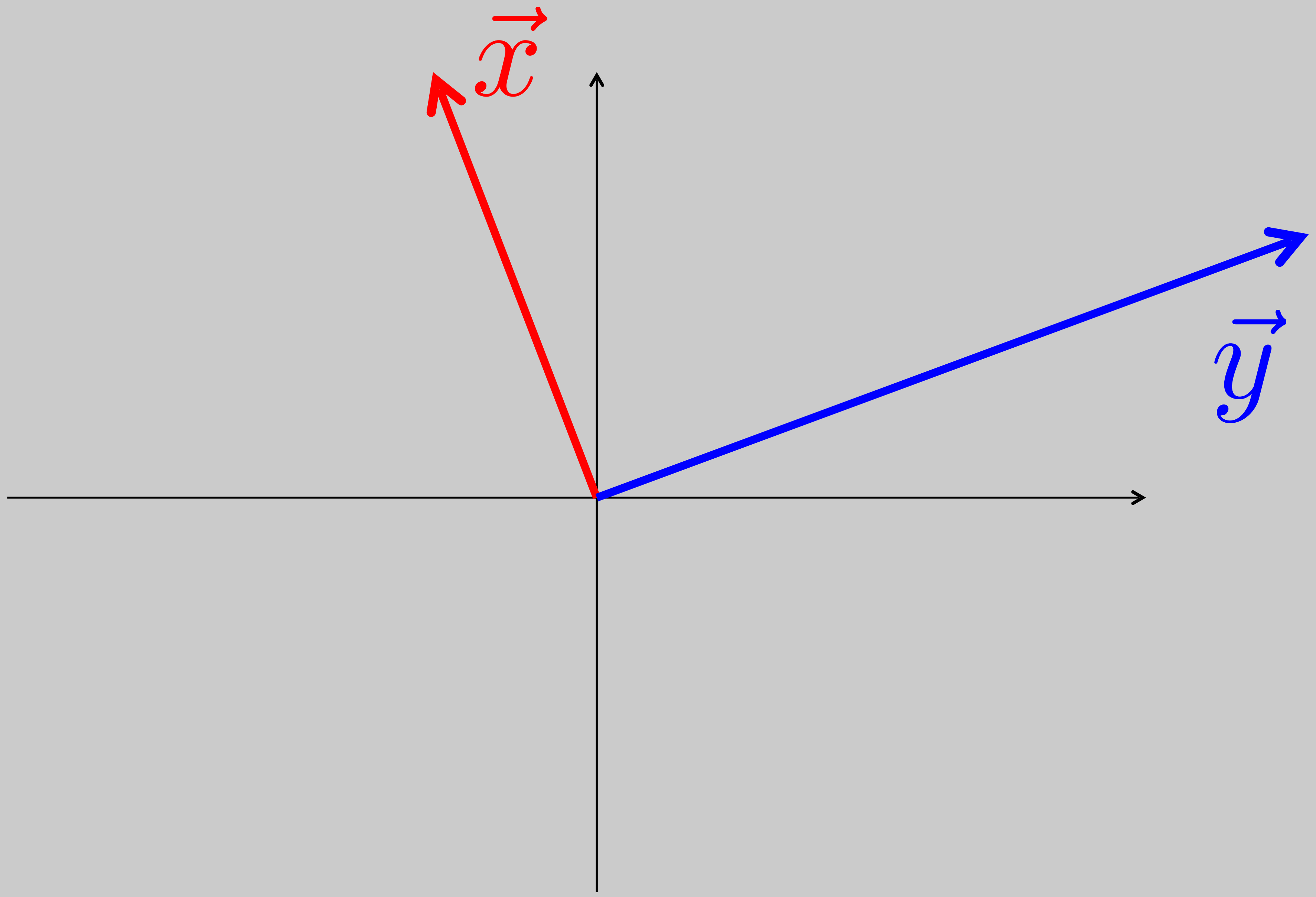
$$< \vec{x}, \vec{y} >= \overline{\vec{x}} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = \vec{x}^* \vec{y} = \vec{x}^H \vec{y}$$

$$\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow \vec{x}^* x = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = 2$$

Projections

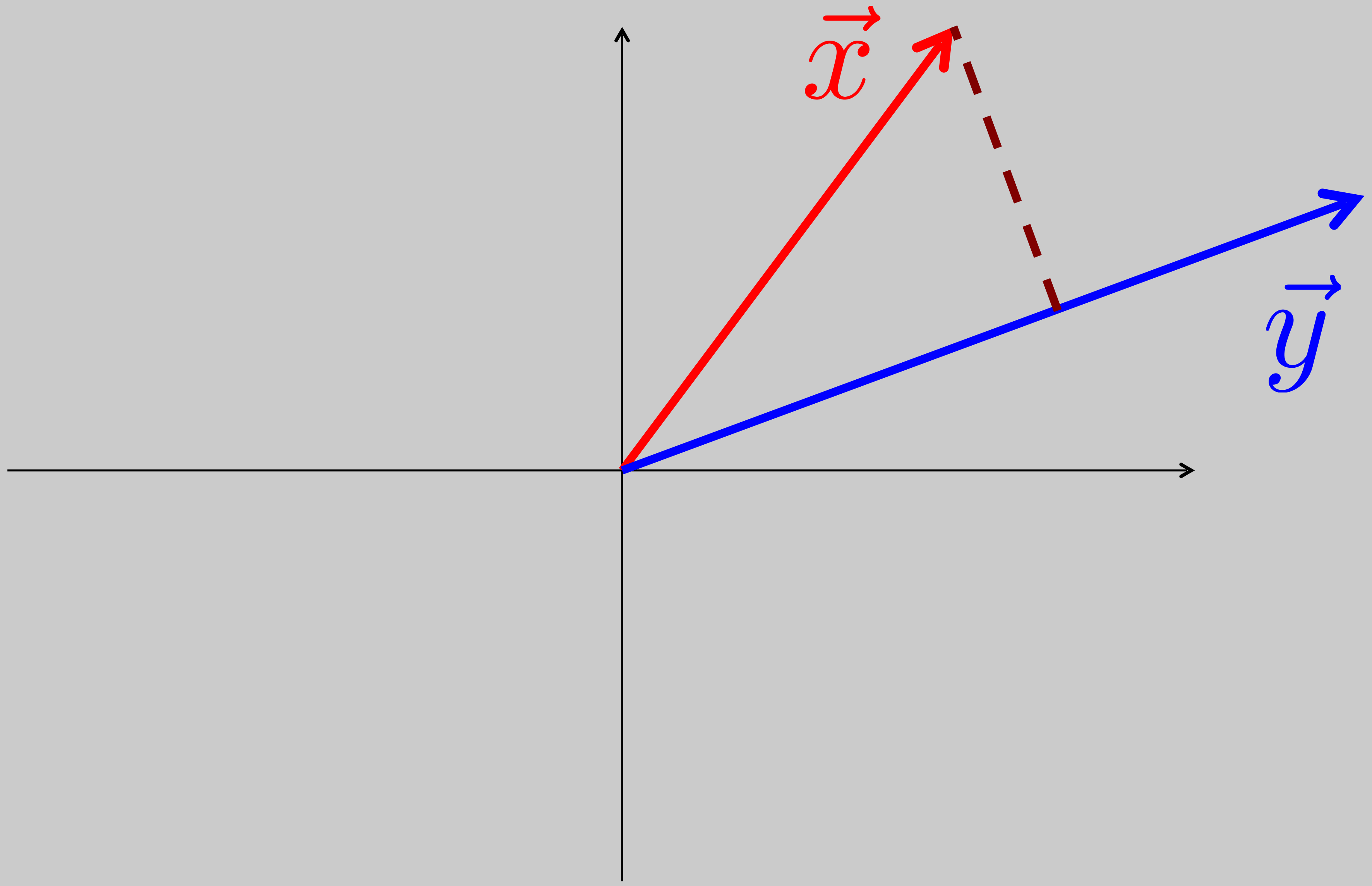
- Orthogonality:

$$\vec{x}^*\vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = 0$$



- Unit vector: $||\hat{x}|| = 1$

$$\hat{x} = \frac{\vec{x}}{||\vec{x}||}$$



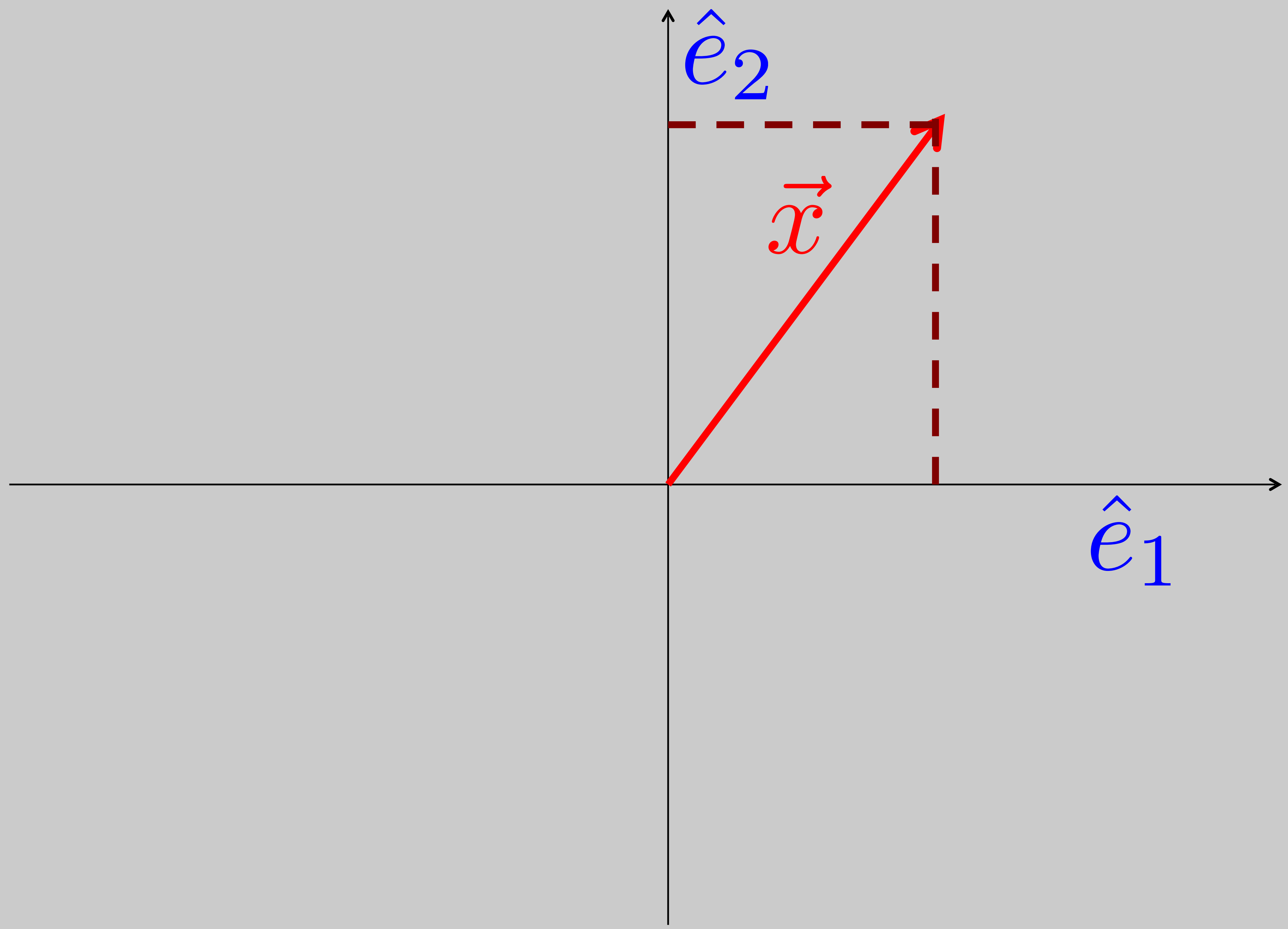
- Define projection as: $\frac{\vec{y}^* x}{||\vec{y}||}$

Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

$$\hat{e}_1^* \vec{x} = [1 \quad 0]\vec{x} = x_1$$

$$\hat{e}_2^* \vec{x} = [0 \quad 1]\vec{x} = x_2$$



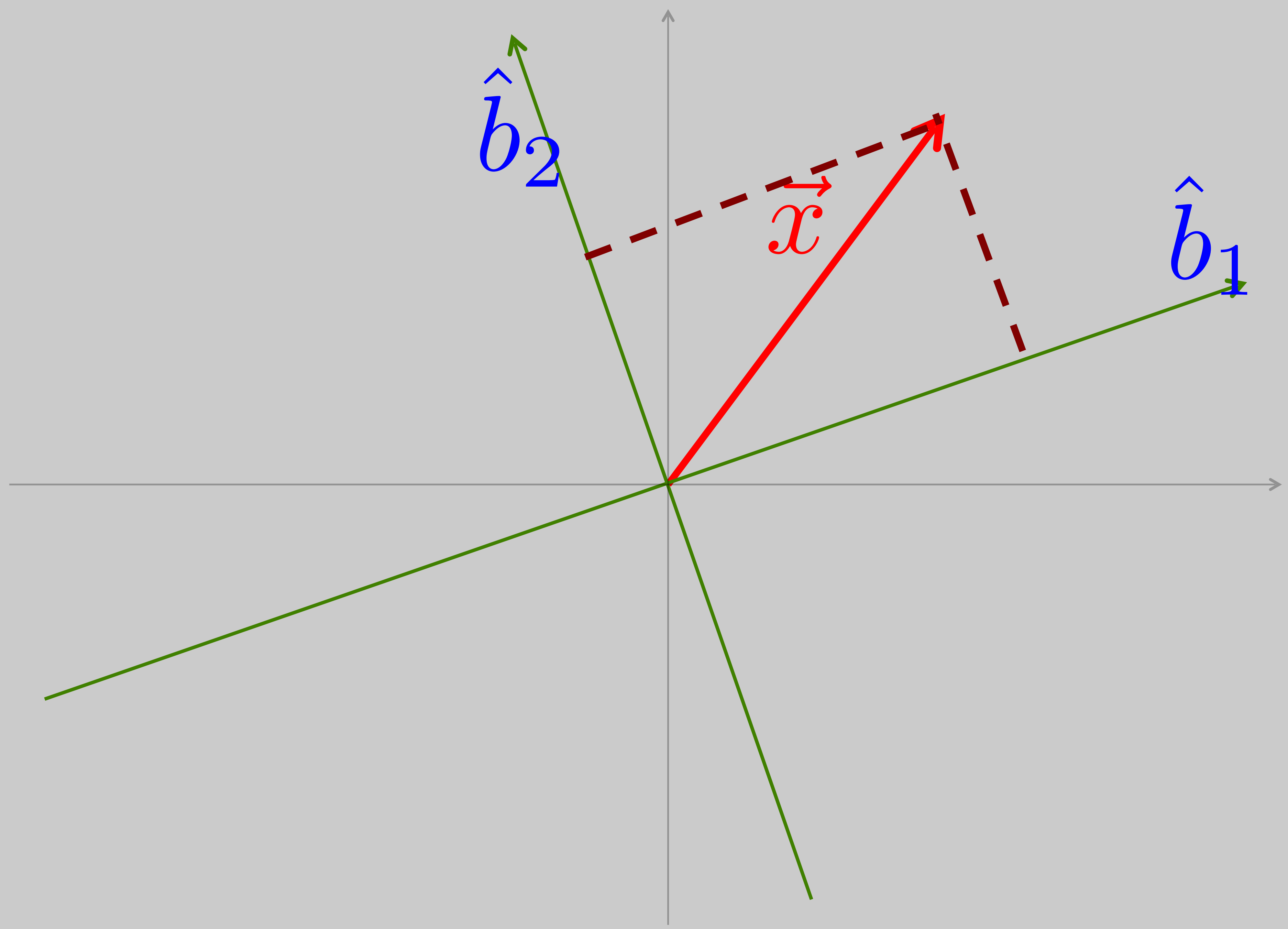
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

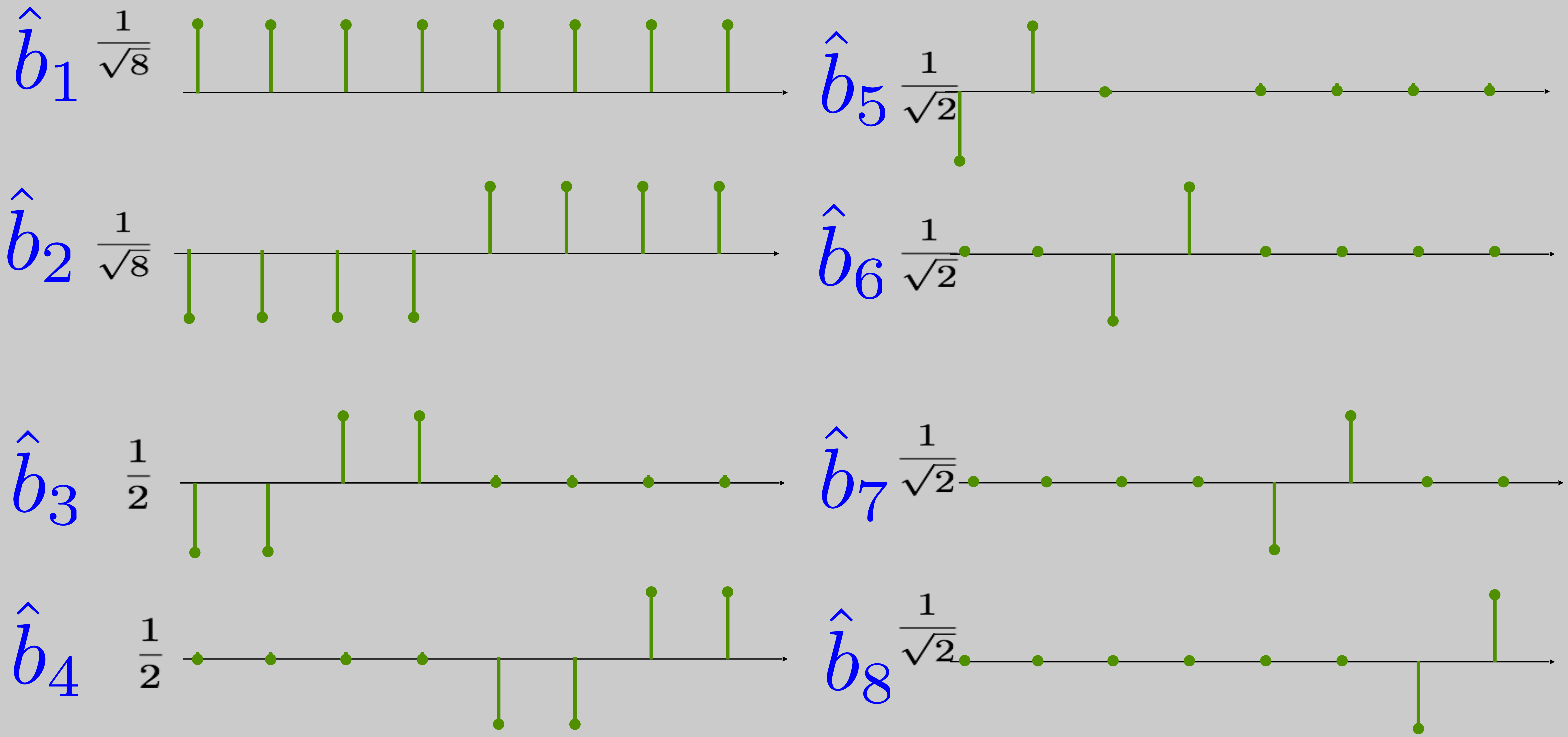
New coordinates:

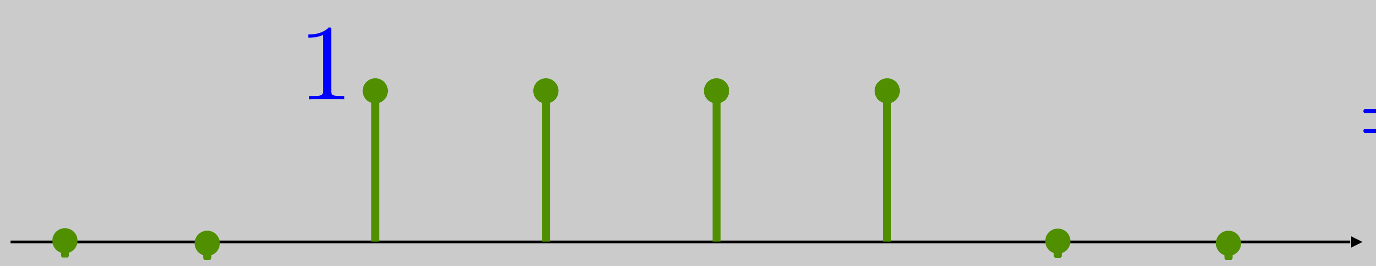
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



Change of basis





$$= \quad \hat{b}_1 + \quad \hat{b}_2 + \quad \hat{b}_3 + \quad \hat{b}_4 + \quad \hat{b}_5 + \quad \hat{b}_6 + \quad \hat{b}_7 + \quad \hat{b}_8$$