EE16B
Designing Information Devices and Systems II

Lecture 13A
Finite Sequences
The Discrete Fourier Transform

Intro

- Last time:
  - LTI Systems
  - Convolution sum

- Today
  - Finite sequences as vectors
  - Linear convolutions as matrices
  - Change of basis
  - Begin discrete Fourier Transform

–Announcements:
  - Course evaluations please fill!

Bad news? – Big Game

98	1995	Stanford	29–24
99	1996	Stanford	42–21
100	1997	Stanford	21–20
101	1998	Stanford	10–3
102	1999	Stanford	31–13
103	2000	Stanford	36–30
104	2001	Stanford	35–28
105	2002	California	30–7
106	2003	California	28–16
107	2004	California	41–6
108	2005	California	27–3
109	2006	California	26–17
110	2007	Stanford	20–13
111	2008	California	37–16
112	2009	California	34–28
113	2010	Stanford	48–14
114	2011	Stanford	31–28
115	2012	Stanford	21–3
116	2013	Stanford	63–13
117	2014	Stanford	38–17
118	2015	Stanford	35–22
119	2016	Stanford	45–31
120	2017	Stanford	17–14
121	2018	Stanford	23–13
122	2019	California	24–20
122	2020	?	

Linear Time Invariant Systems

- Decompose x[n]:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

$$= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

- Compute output:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

Convolution sum

Sum of weighted, delayed impulse responses!

BIBO Stability of LTI systems

- LTI system is BIBO stable if, and only if h[n] is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Proof (if):

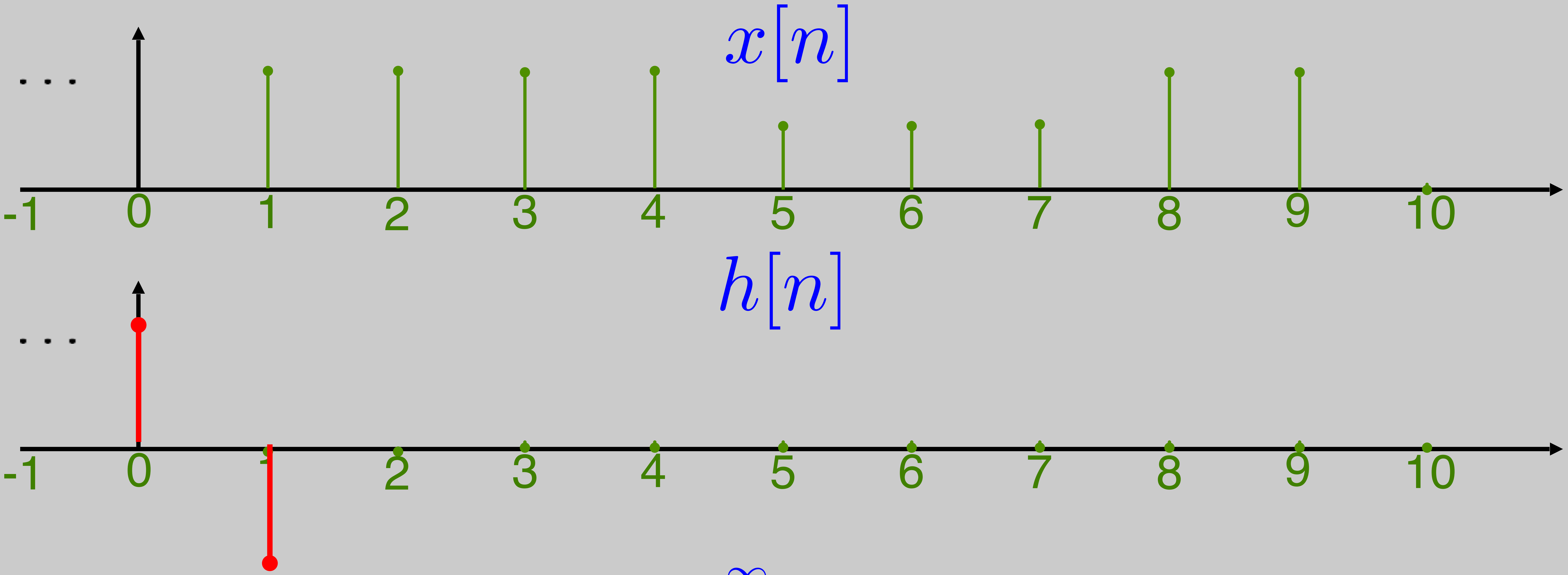
$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} x[m]h[n-m] \right| \leq \sum_{m=-\infty}^{\infty} |x[m]| \cdot |h[n-m]|$$

$$|x[m]| < M$$

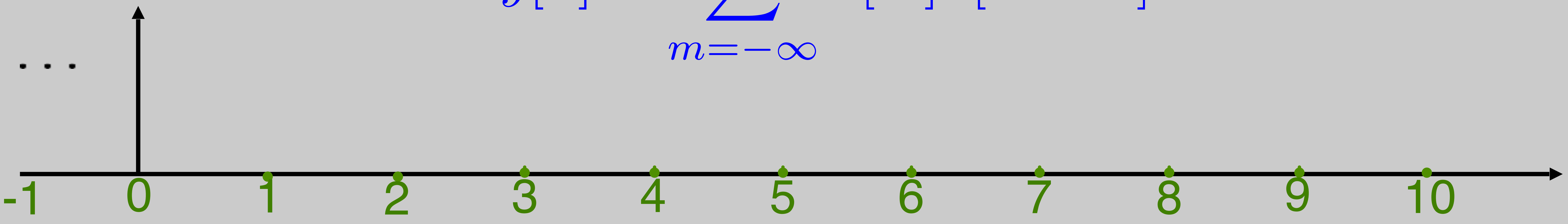
$$\leq M \sum_{m=-\infty}^{\infty} |h[n-m]| = M \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Only if in EE123

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



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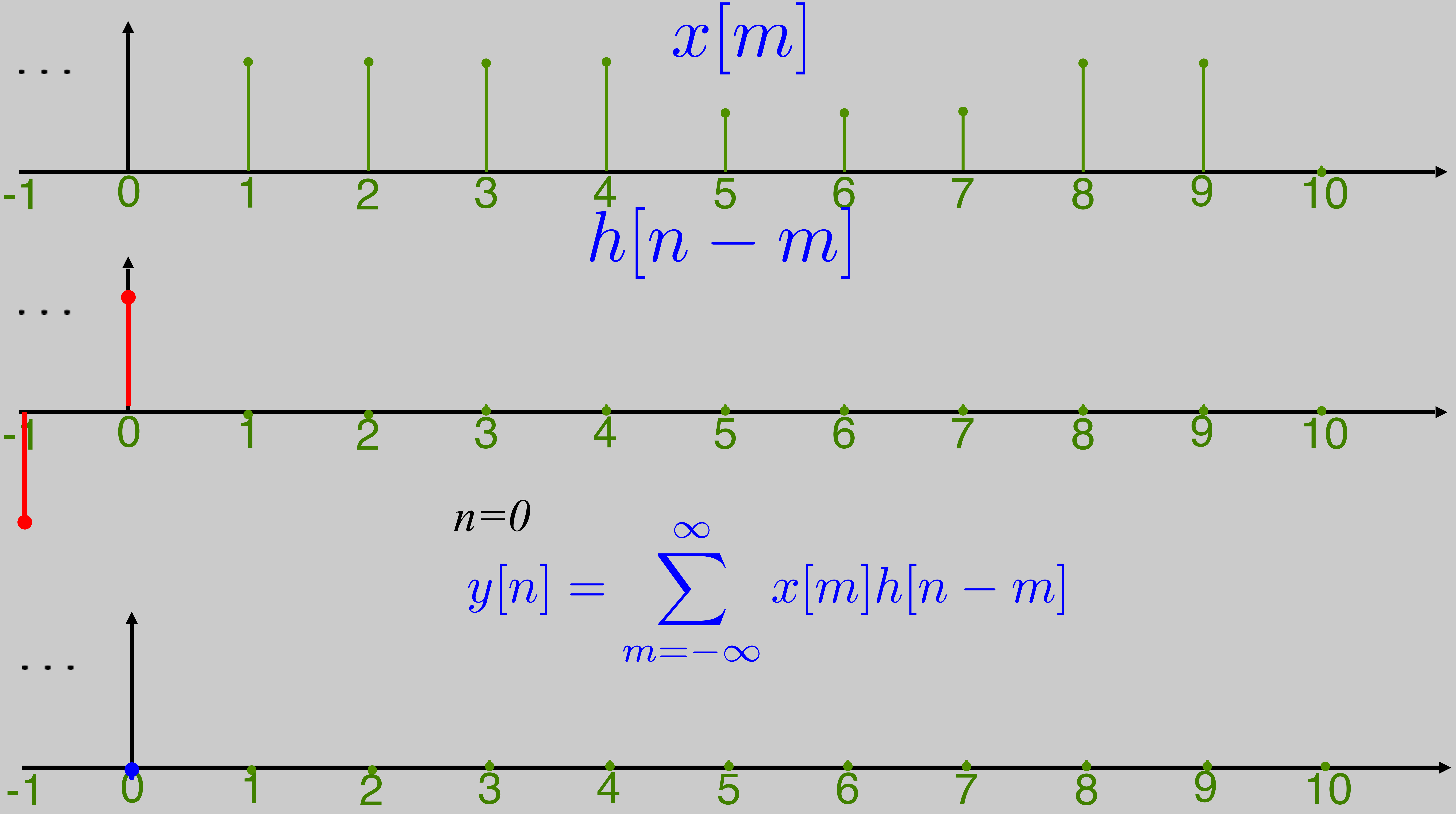

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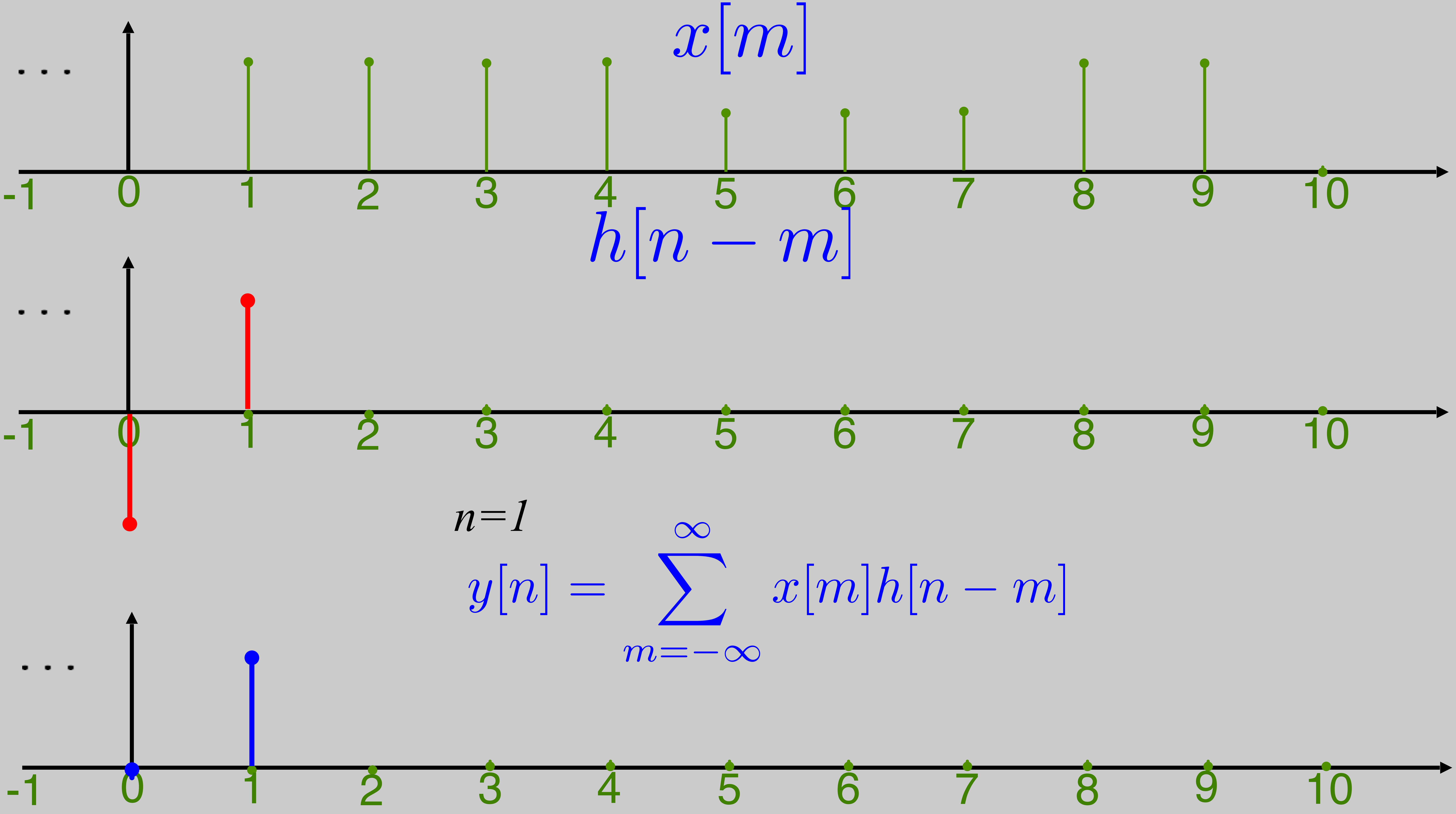

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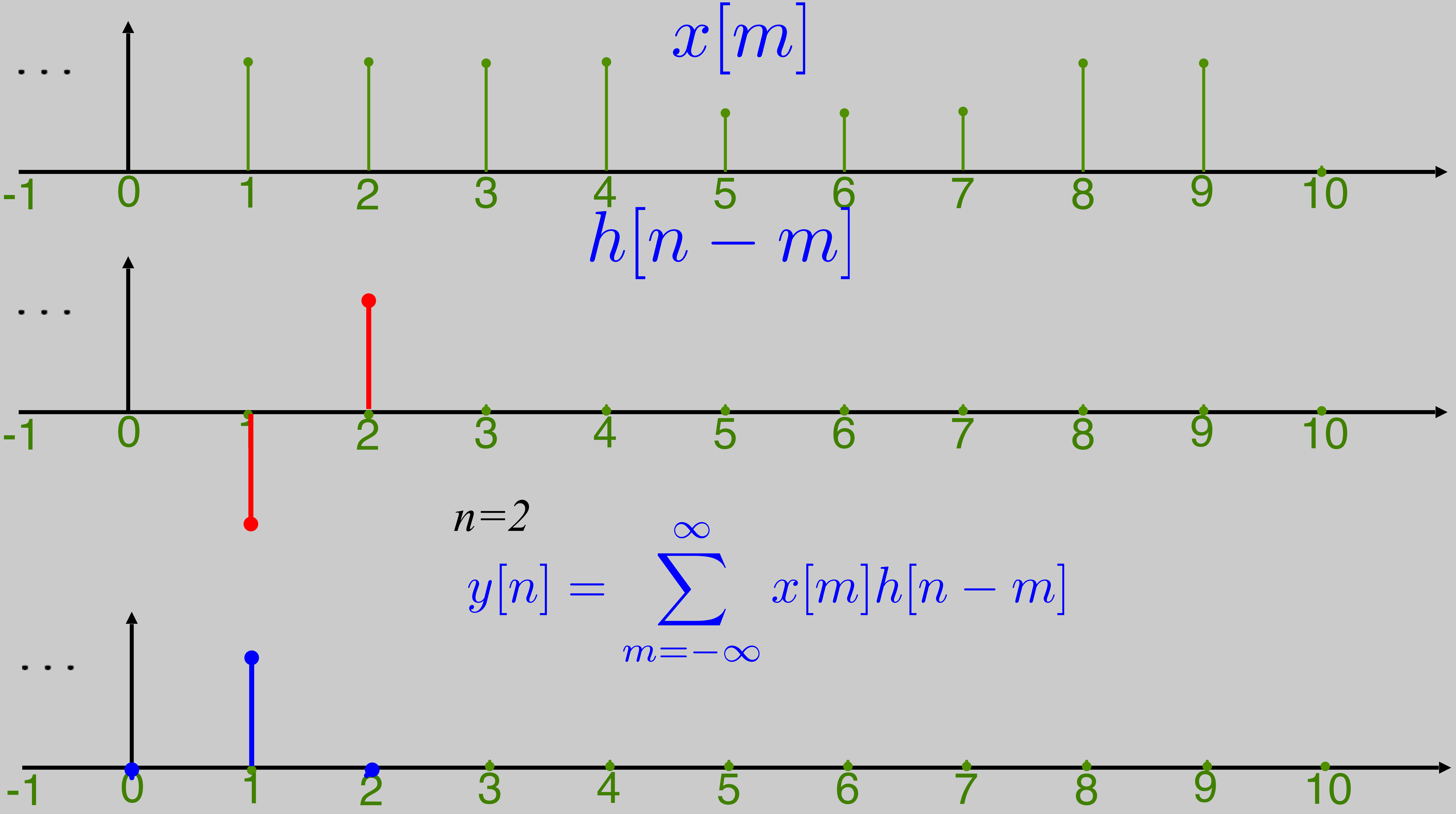

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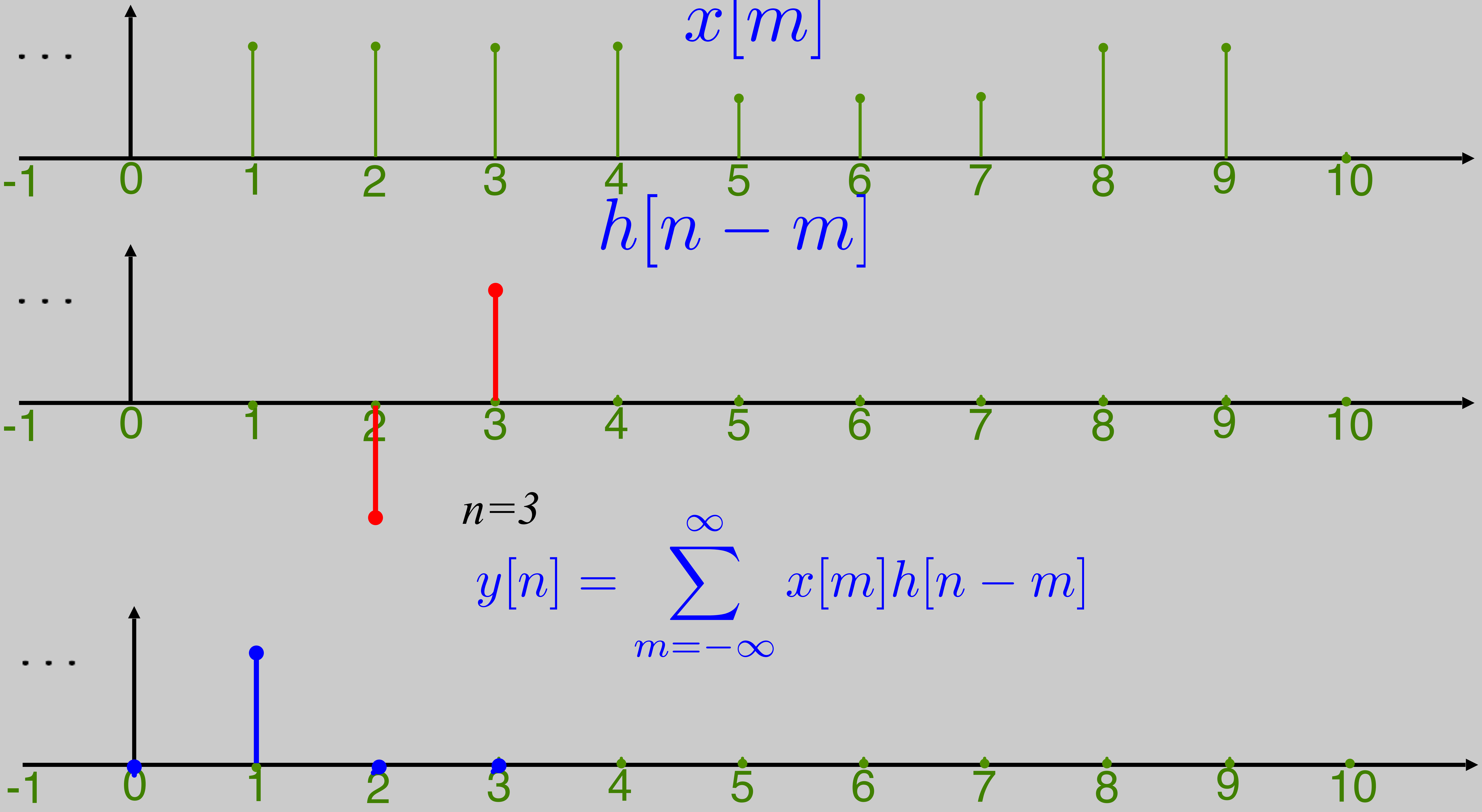

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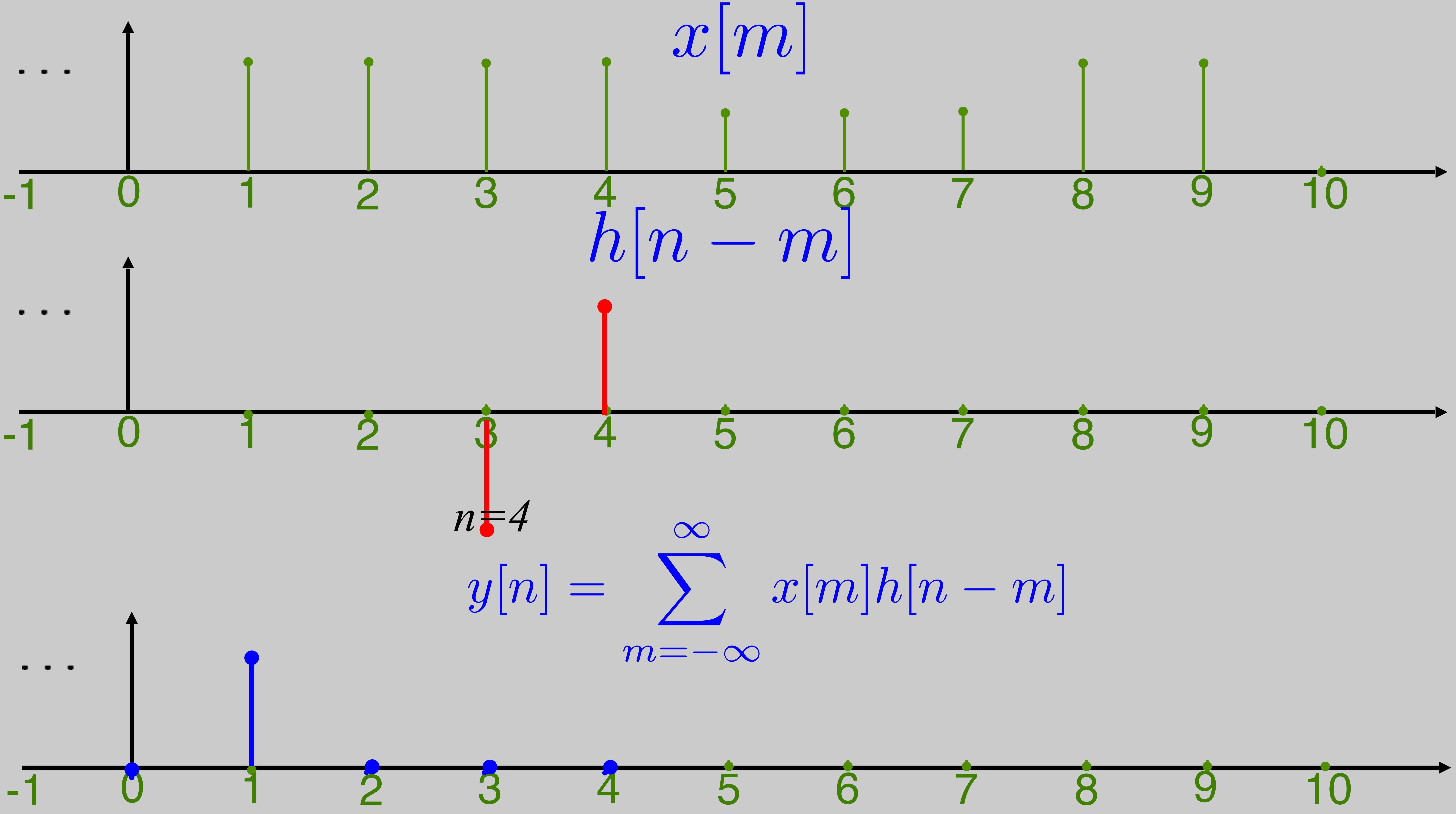

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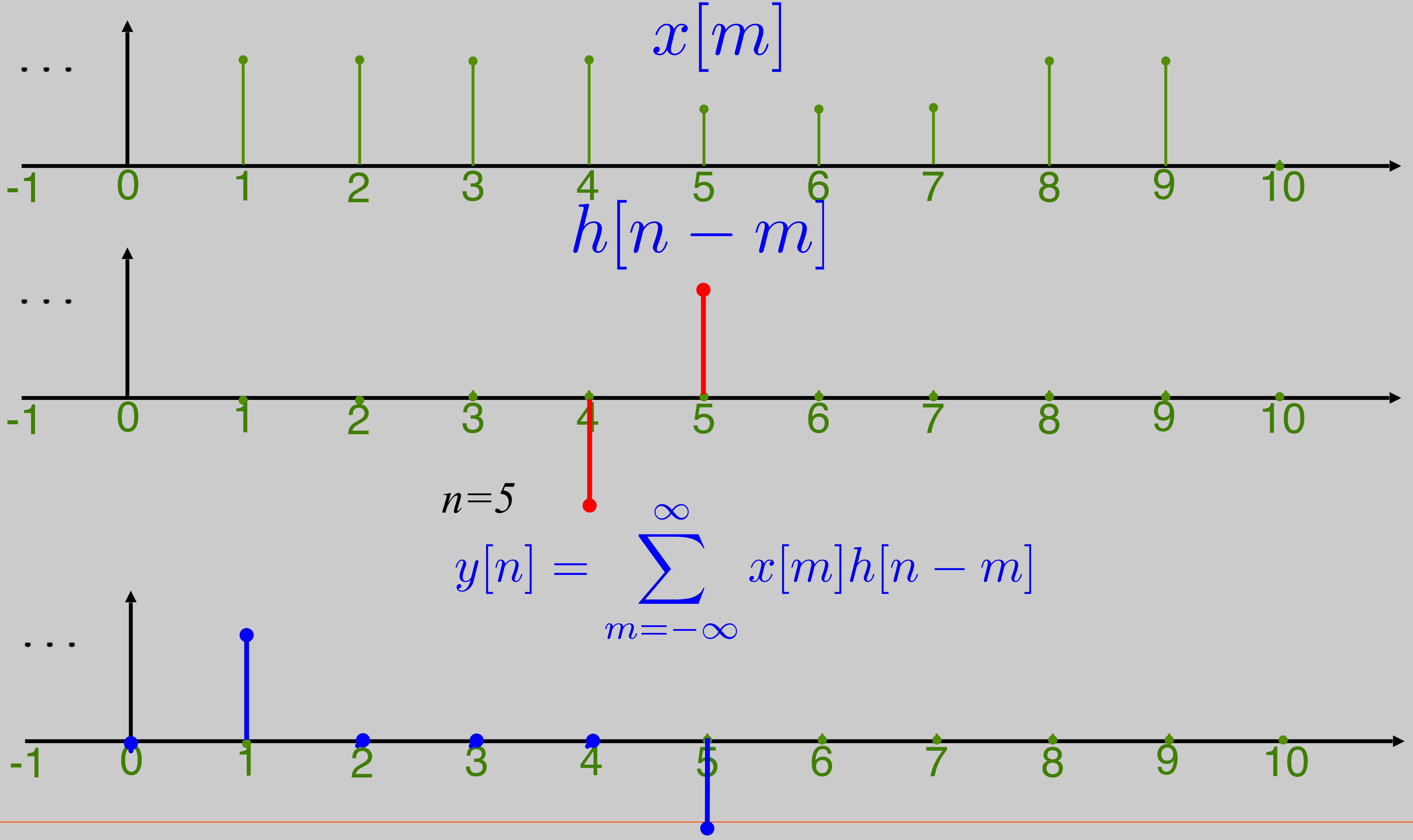
Graphical Example of Convolution



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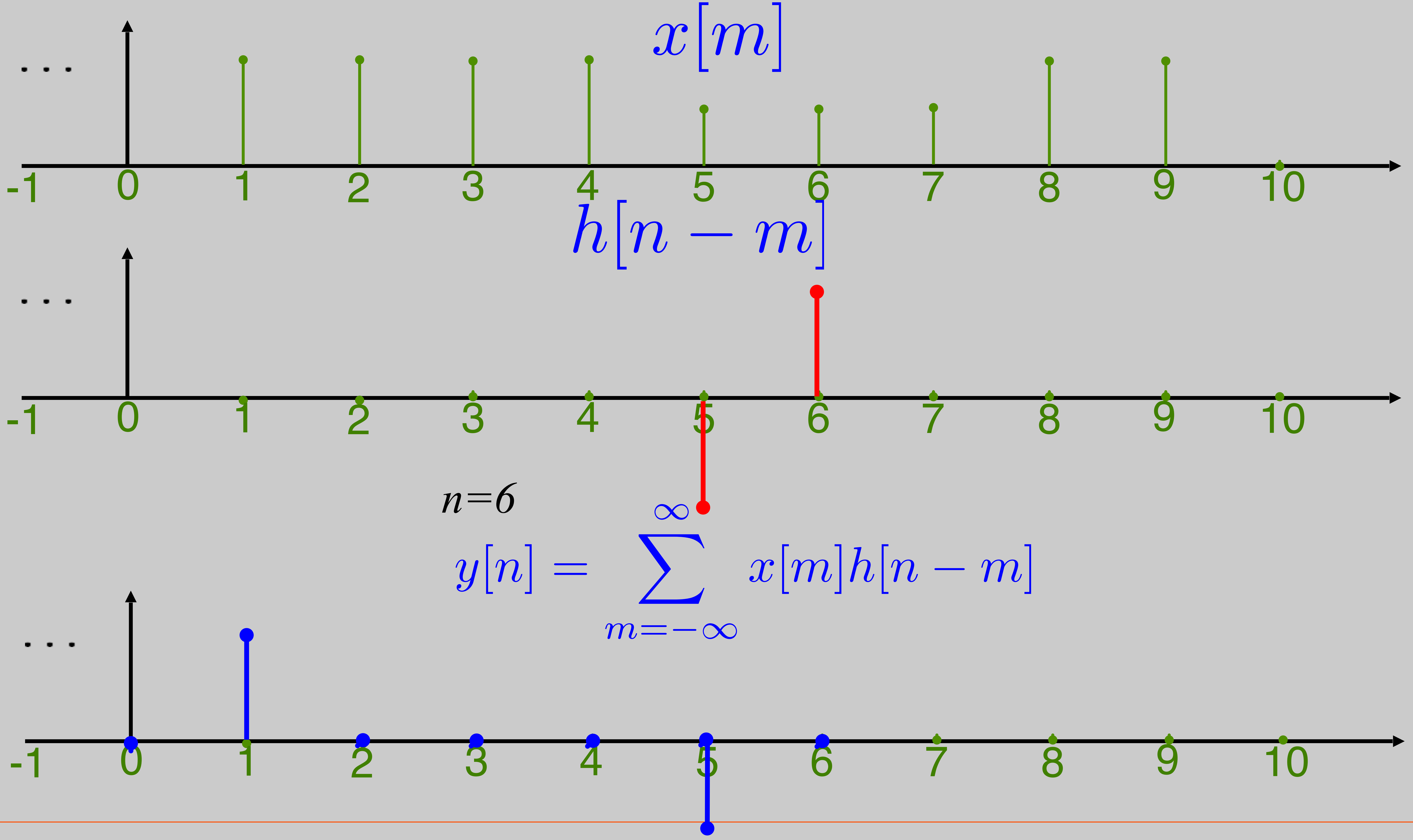
Graphical Example of Convolution



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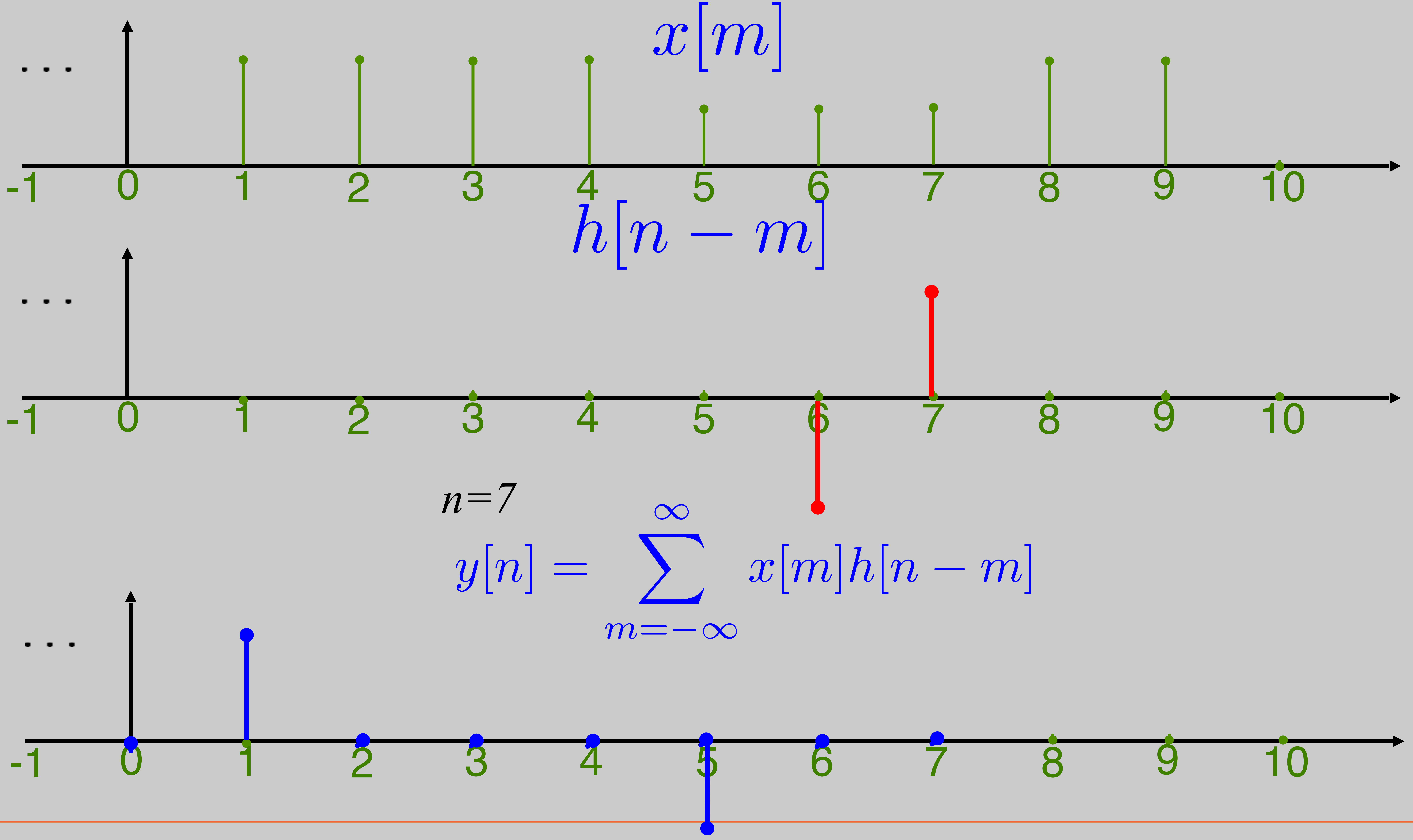
Graphical Example of Convolution



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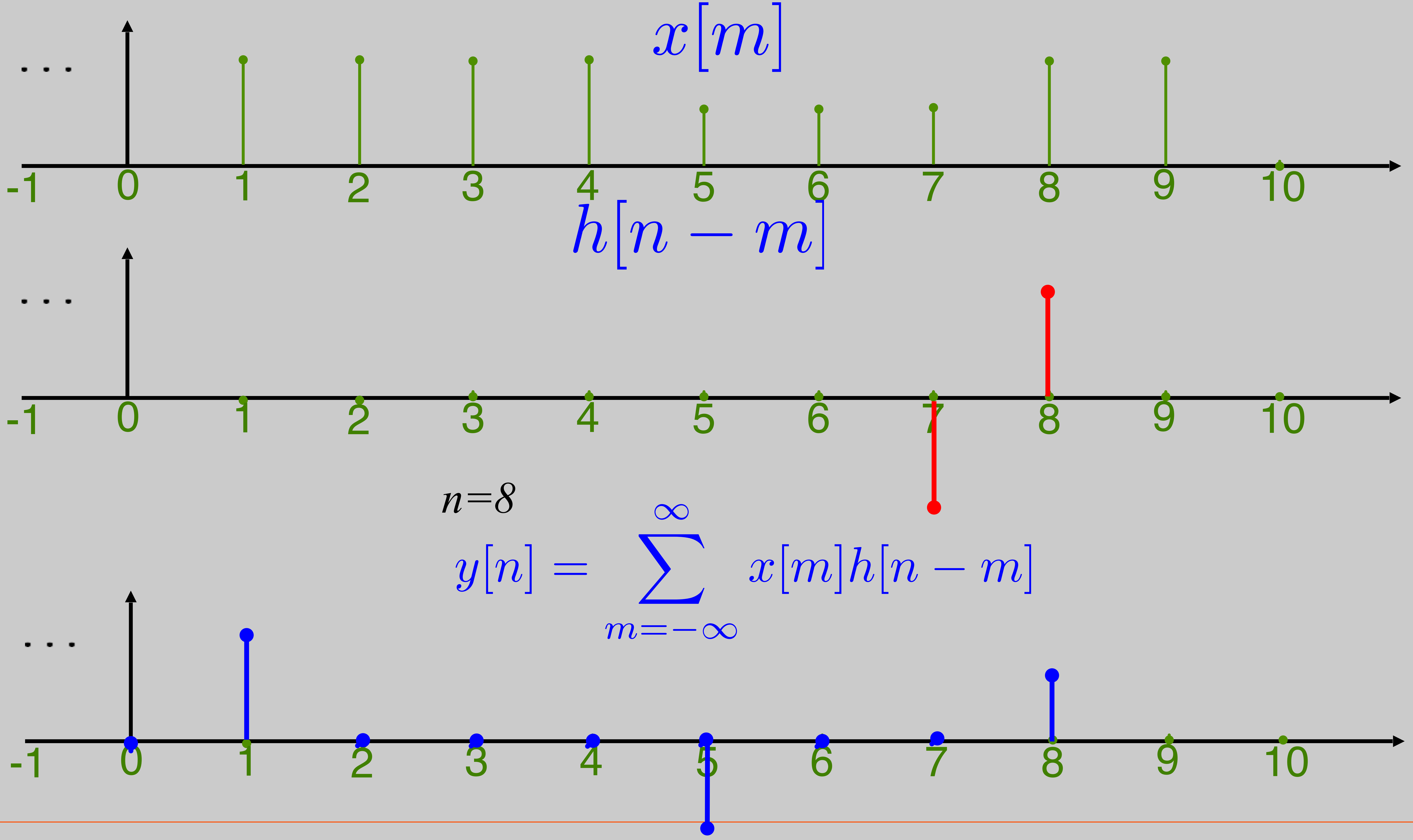
Graphical Example of Convolution



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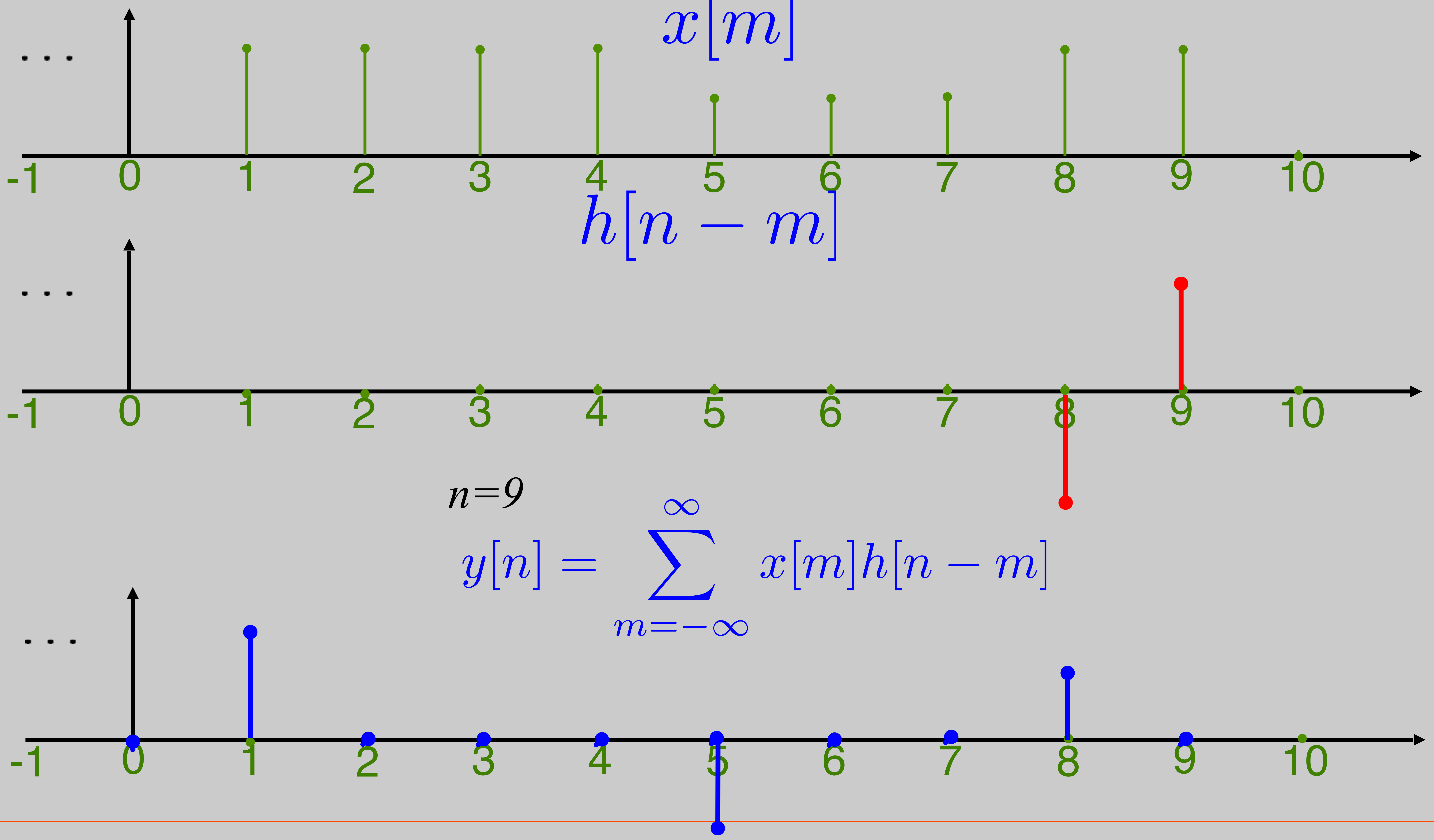
Graphical Example of Convolution



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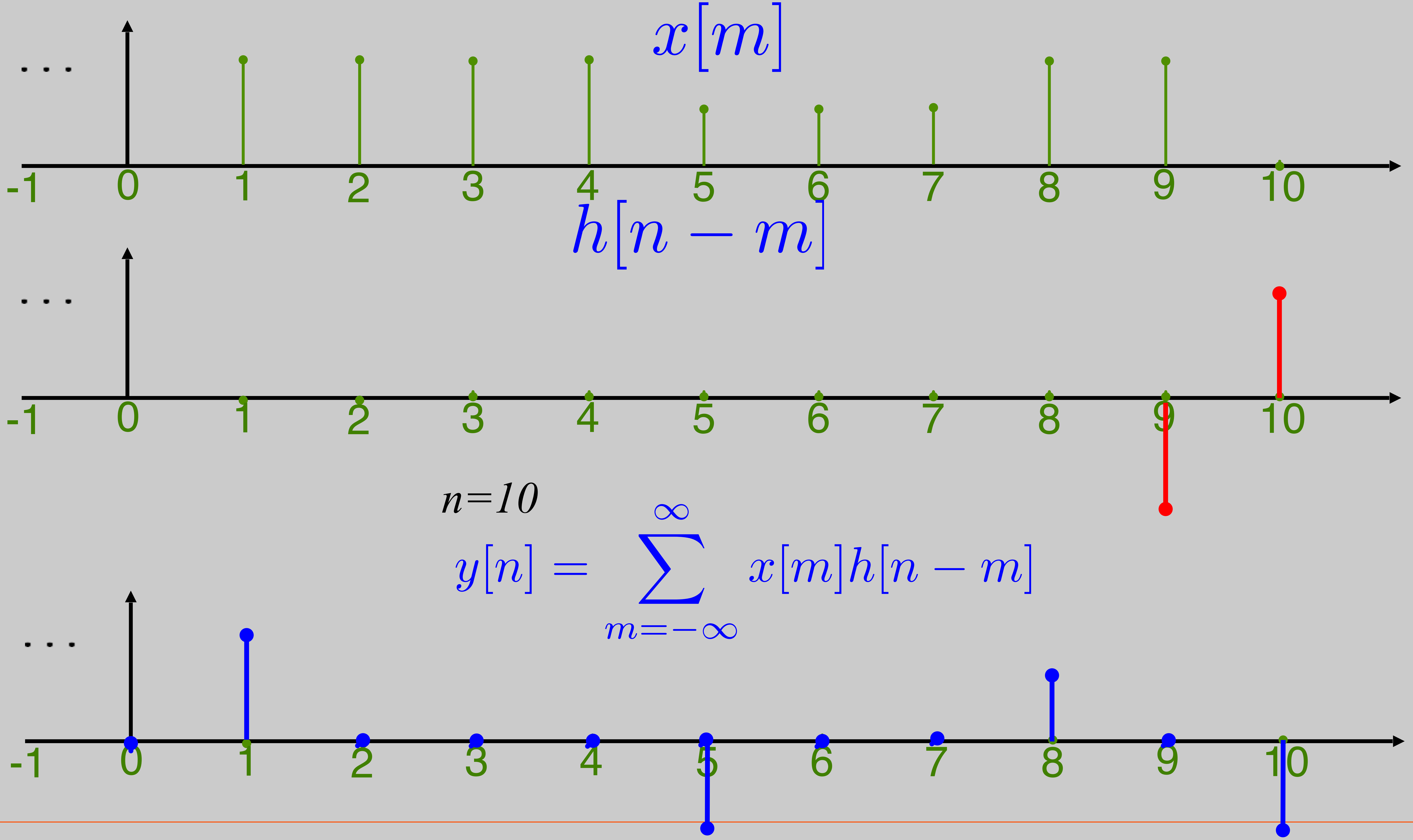
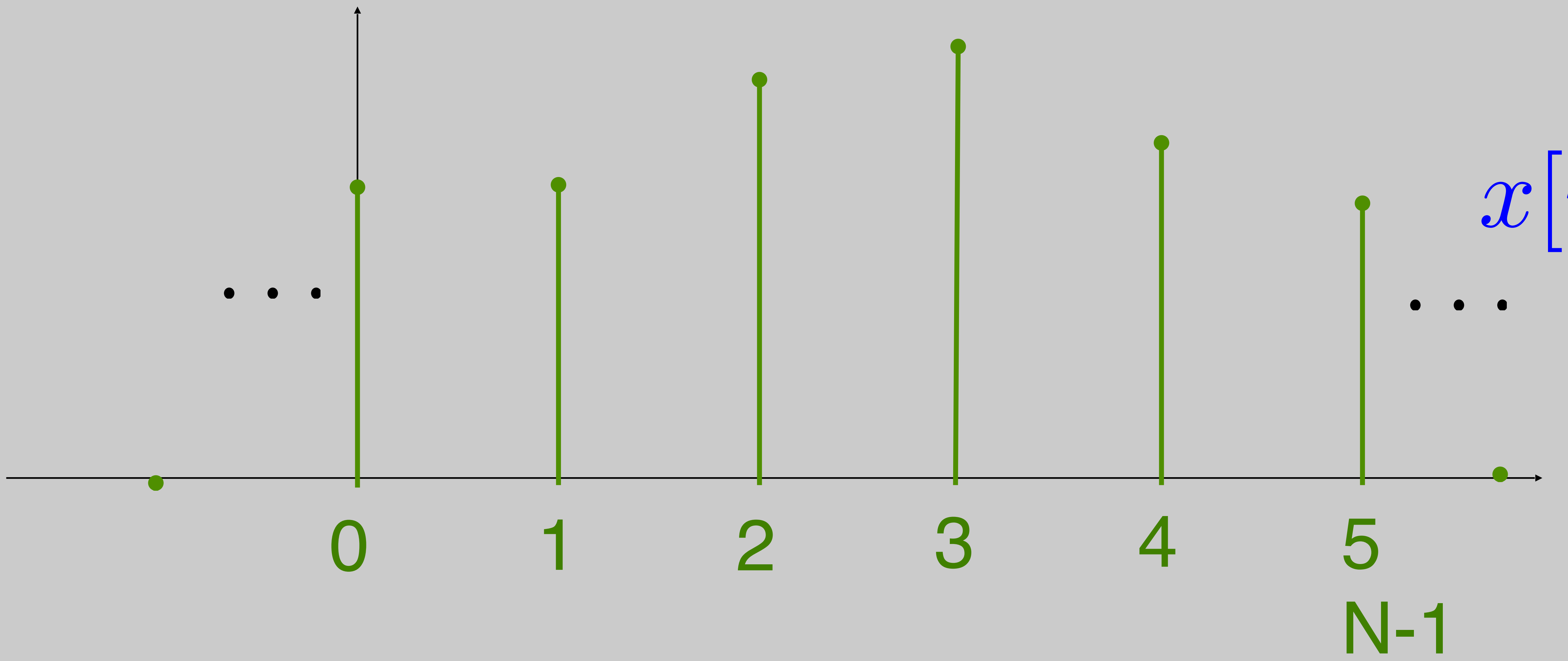
Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Finite Sequences

- Consider a finite sequence of length N

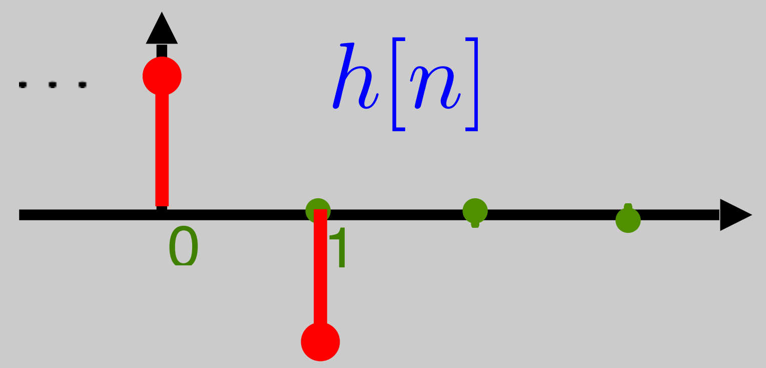


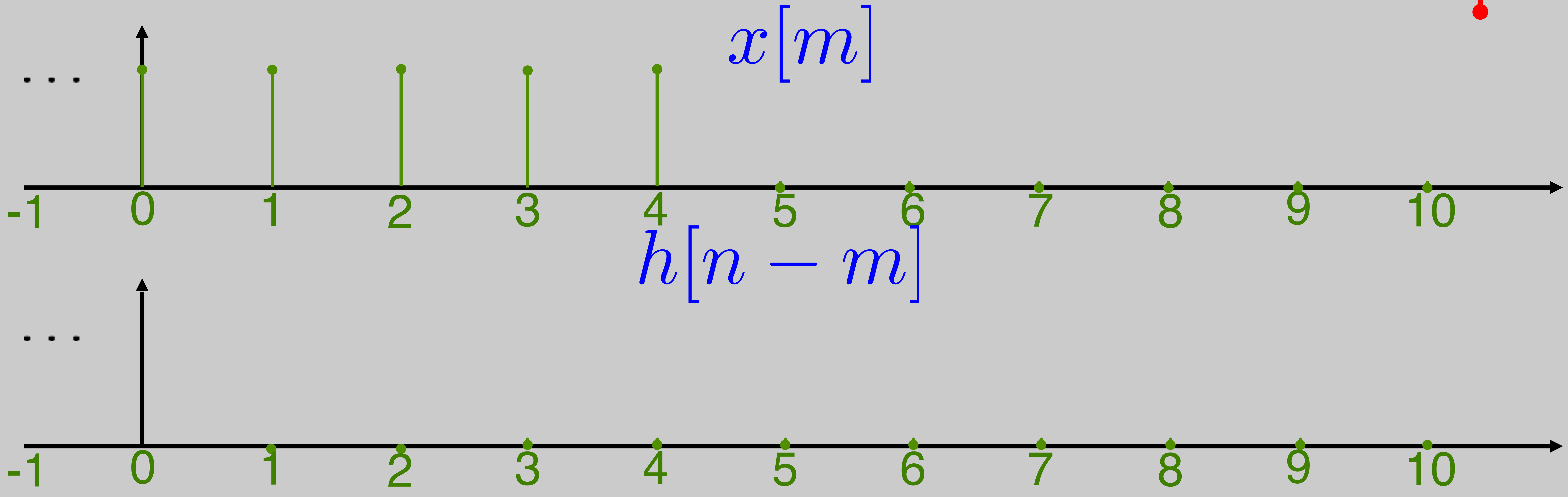
$$x[n] = \begin{cases} \text{something} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

- Can also be written as a vector

$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Convolution as a Matrix-Vector Operation





$$\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

Convolution of Finite Sequences

- x[n] is N-length sequence, h[n] is M-length
- Length of $x[n] * h[n]$ is N + M - 1
- Convolution matrix have Toepelitz structure

$$\begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \\ j & i & h & g & f \end{bmatrix}$$

Convolution with Circulant Matrices

- Linear convolution is non-square
  - Zero pad x, and cycle h

Circulant

$$\begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix}$$

Toeplitz

$$\begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ \; \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix}$$

Convolution with Circulant Matrices

- Linear convolution is non-square
- Zero pad x, and cycle h

Circulant

$$\begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix}$$

Toeplitz

$$\begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ 0 \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix}$$

Q: Why bother?

A: Circulant matrices have the coolest eigenvectors! (DFT basis)

Finite Sequences as Vectors

- Define an inner-product (for R^N):

$$< \vec{x}, \vec{y} >= \vec{x} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]y[n] =$$

$$= \vec{x}^T \vec{y}$$

So,

$$< \vec{x}, \vec{x} >= \sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} x^2[n] = ||\vec{x}||^2$$

$$\Rightarrow \vec{x}^T \vec{x} = ||\vec{x}||^2$$

Finite Sequences as Vectors

- What about complex?

$$x \cdot x = x^2 = (x_r + jx_i)(x_r + jx_i) = x_r^2 - x_i^2 + 2jx_rx_i \neq ||x||^2$$

but,

$$x^* \cdot x = (x_r - jx_i)(x_r + jx_i) = x_r^2 + x_i^2 = ||x||^2$$

- Transpose vs Transpost conjugate

$$\vec{x} = \begin{bmatrix} 1 \\ j \\ 1+j \end{bmatrix}$$

$$\vec{x}^T = \begin{bmatrix} 1 & j & 1+j \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} 1 & -j & 1-j \end{bmatrix}$$

Finite Sequences as Vectors

- Define Complex inner product

$$< \vec{x}, \vec{y} >= \overline{\vec{x}} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = \vec{x}^* \vec{y} = \vec{x}^H \vec{y}$$

$$\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow \vec{x}^* x = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = 2$$

Projections

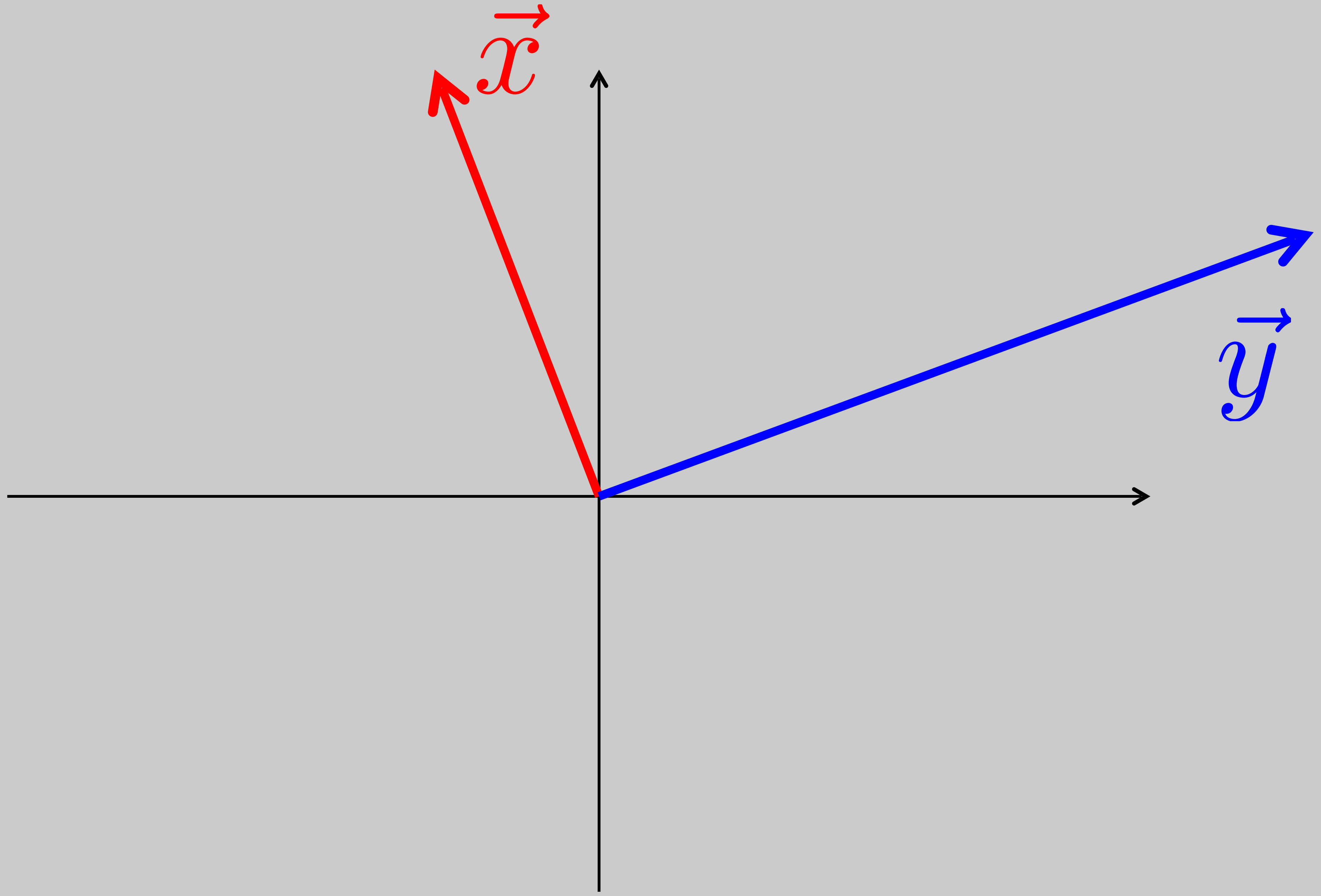
- Orthogonality:

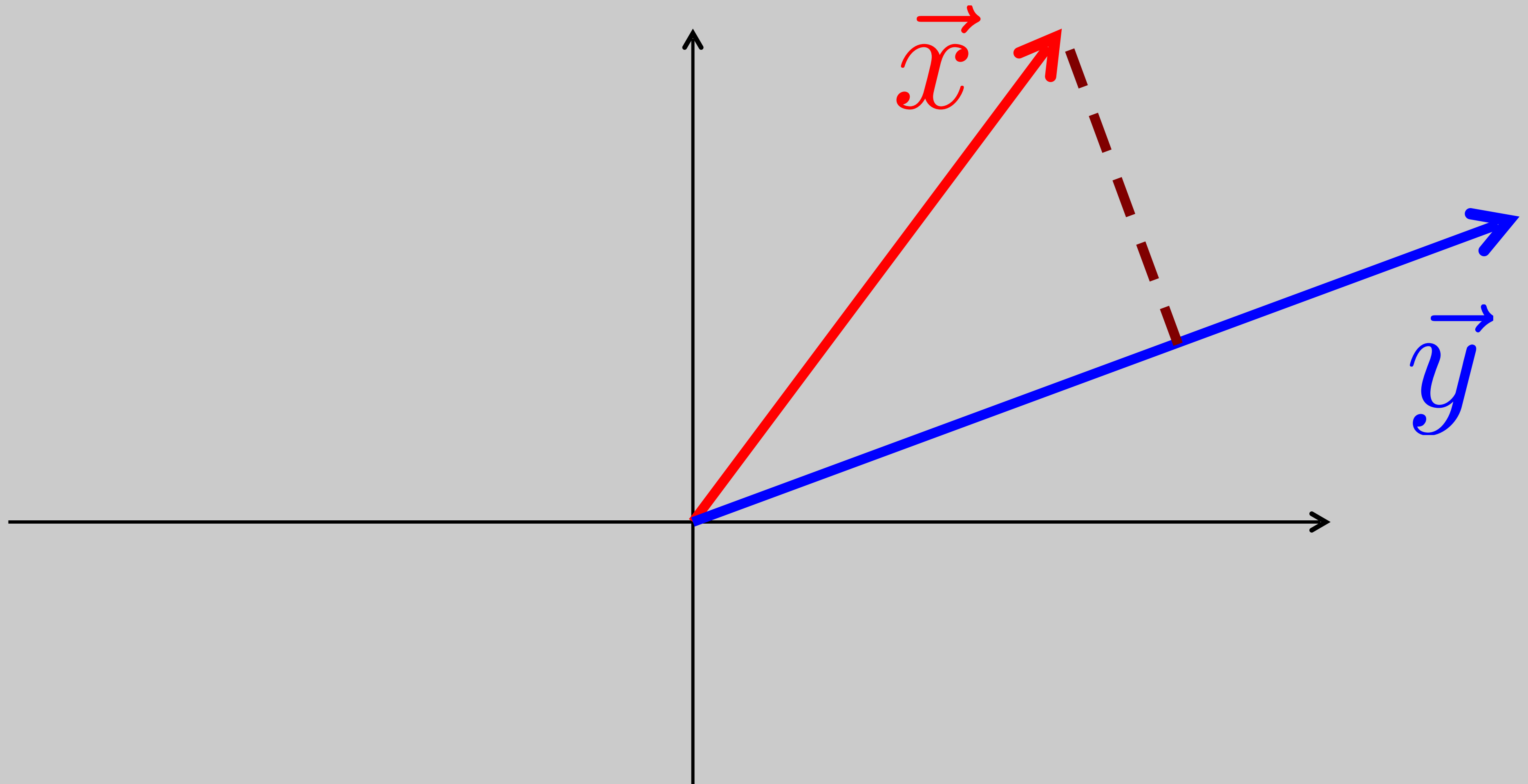
$$\vec{x}^*\vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = 0$$

- Unit vector: $||\hat{x}|| = 1$

$$\hat{x} = \frac{\vec{x}}{||\vec{x}||}$$

- Define projection as: $\dfrac{\vec{y}^* x}{||\vec{y}||}$



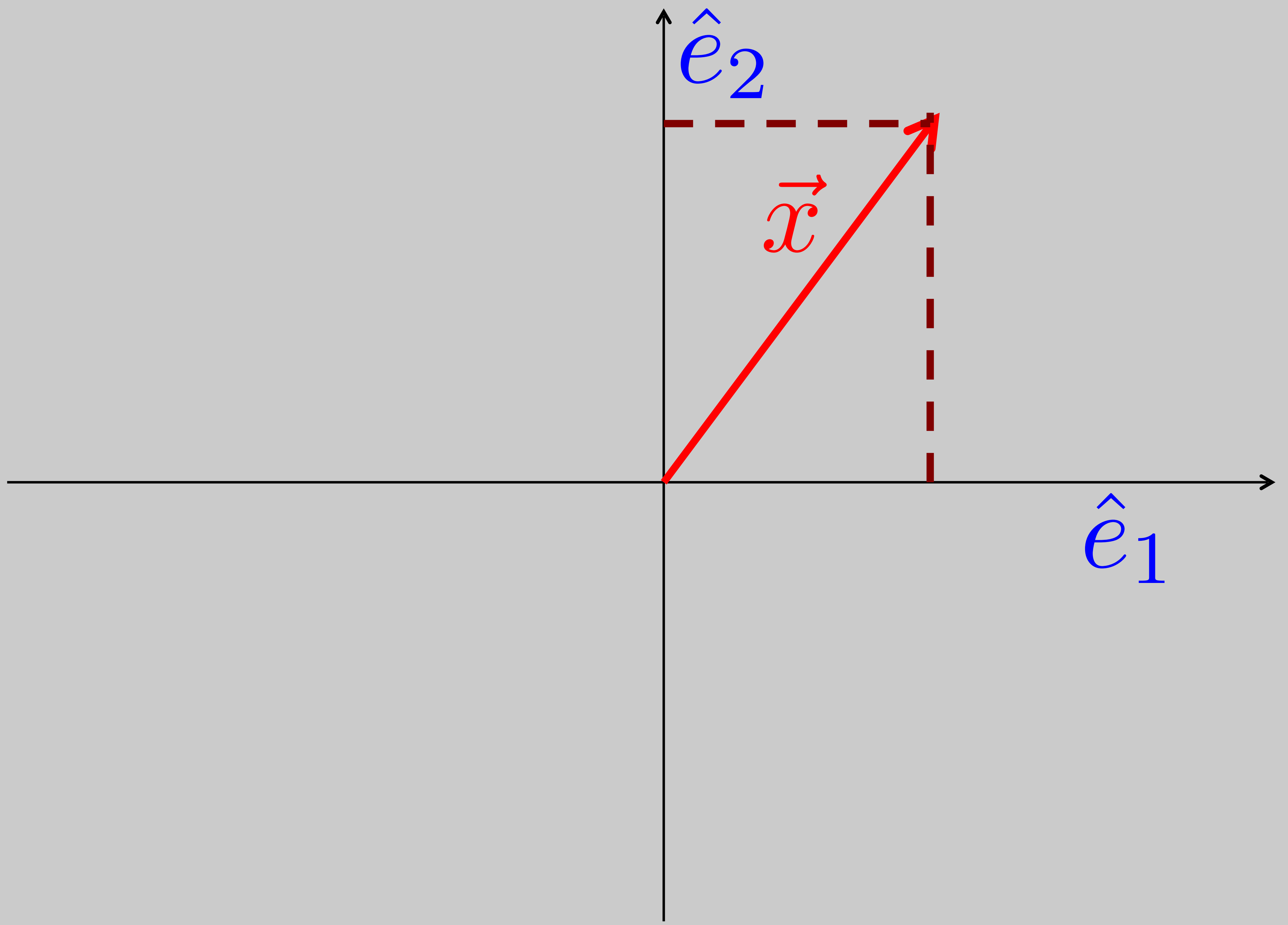


For which y the projection is maximized?

Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

$$\hat{e}_1^* \vec{x} = [1 \quad 0] \vec{x} = x_1$$
$$\hat{e}_2^* \vec{x} = [0 \quad 1] \vec{x} = x_2$$



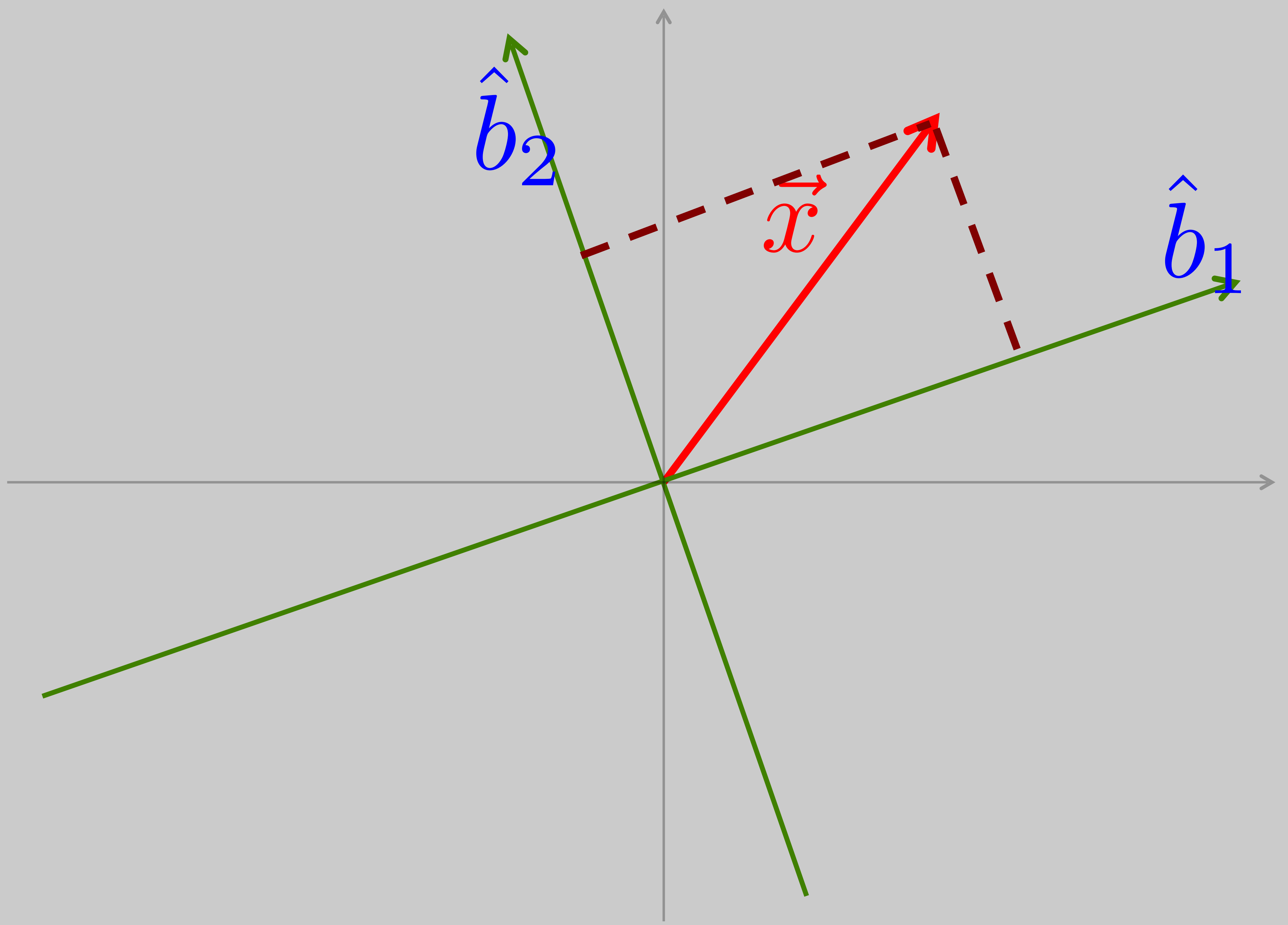
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

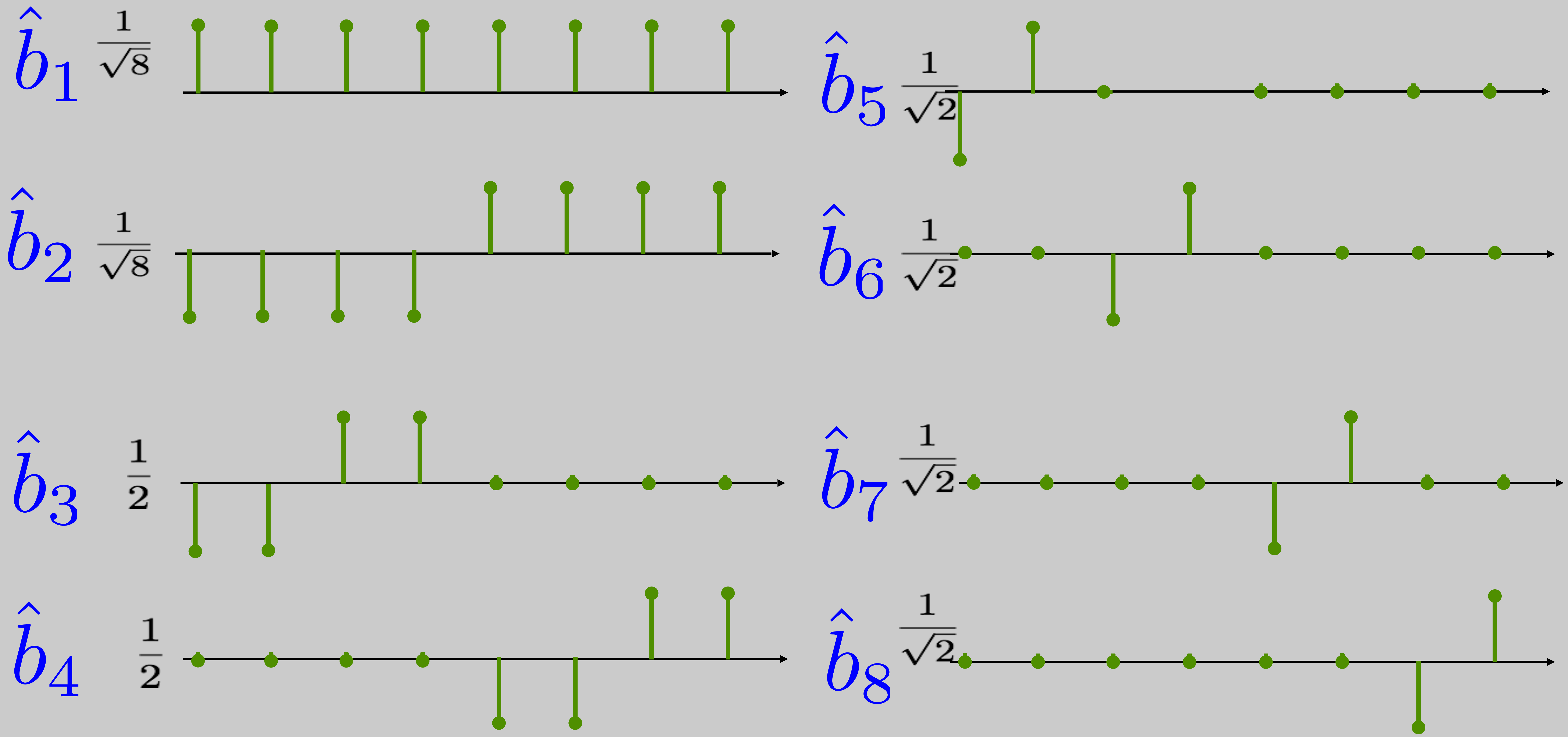
New coordinates:

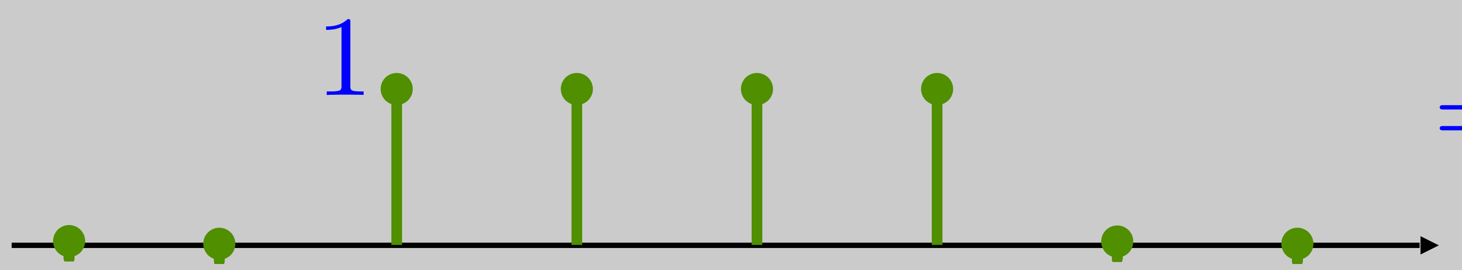
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



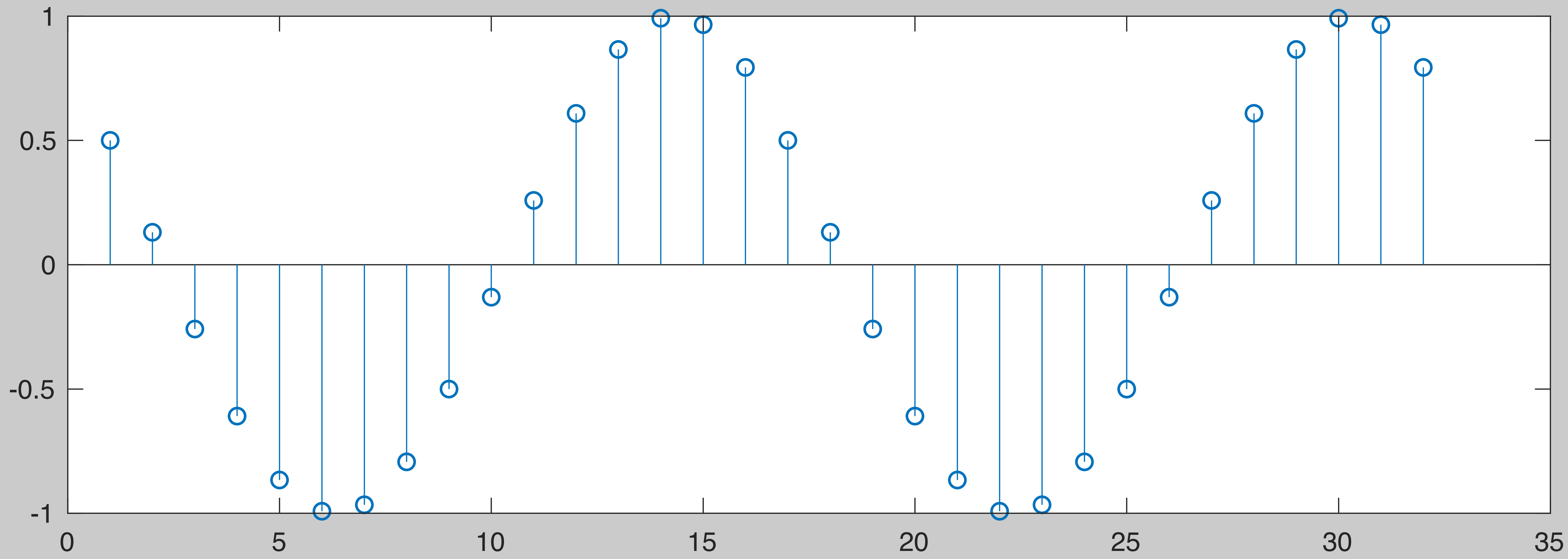
Change of basis





$$= \frac{4}{\sqrt{8}} \cdot \hat{b}_1 + 0 \cdot \hat{b}_2 + 1 \cdot \hat{b}_3 + (-1) \cdot \hat{b}_4 + 0 \cdot \hat{b}_5 + 0 \cdot \hat{b}_6 + 0 \cdot \hat{b}_7 + 0 \cdot \hat{b}_8$$

- How can we find the frequency of this N=32 length signal?



Project on unit sinusoidal vectors?

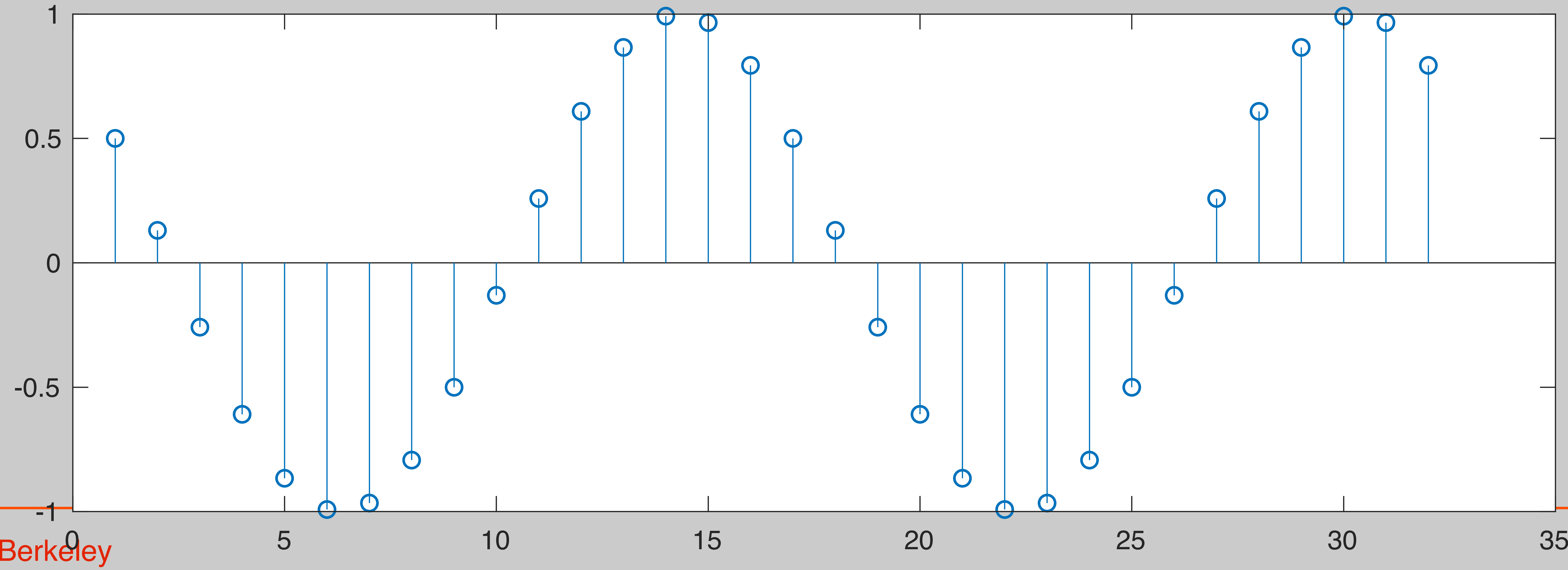
Complex Exponential Basis

- Phase is a problem! (inside a cosine)

$$\vec{x} \quad \Big| \quad x[n] = \cos(\omega_0 n + \phi_0)$$

- Solution: Phase is a coefficient for complex exponentials!

$$\vec{x} \quad \Big| \quad x[n] = \frac{1}{2}e^{j\omega n} \cdot e^{j\phi} \; + \frac{1}{2}e^{-j\omega n} \cdot e^{-j\phi}$$



Frequency Analysis Through Projections

- N-length normalized discrete frequency:

$$u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n} \qquad 0 \leq n < N \qquad 0 \leq \omega < 2\pi$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix} \Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$

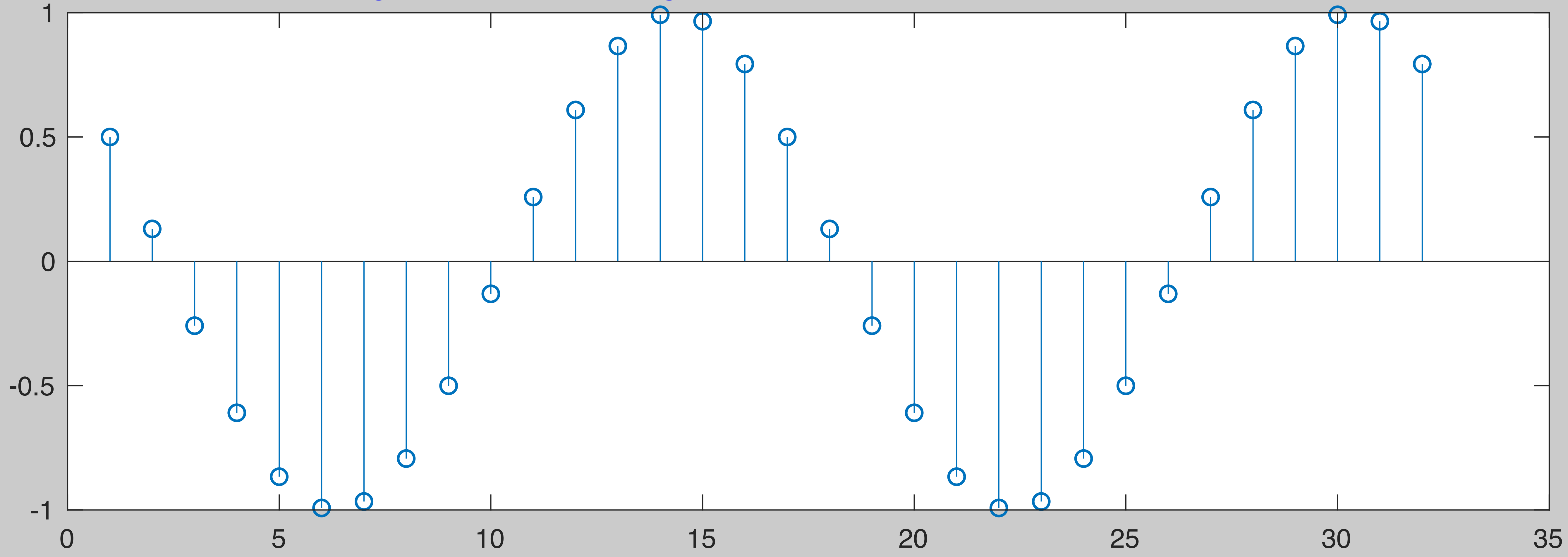
$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Also the DTFT of the finite sequence x

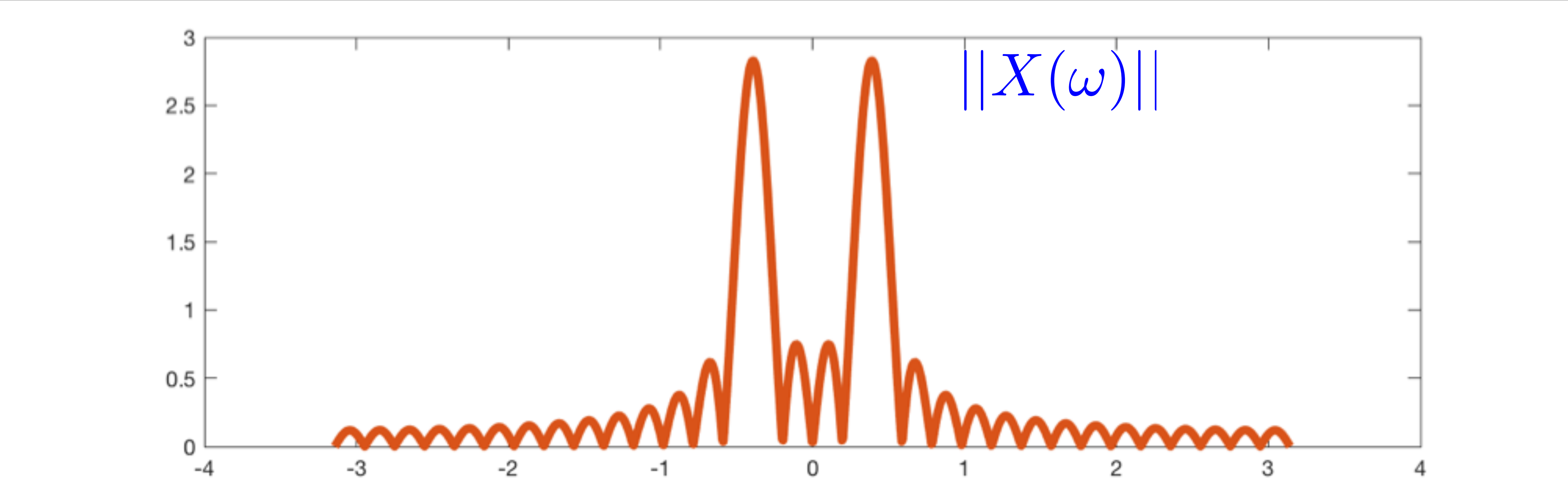
Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}n\right)$

N = 32



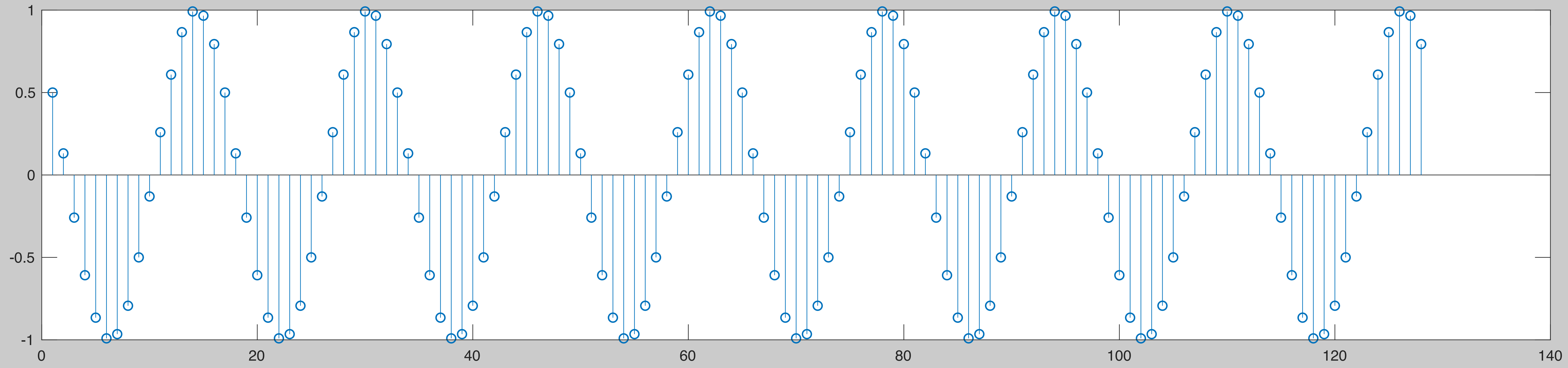
$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



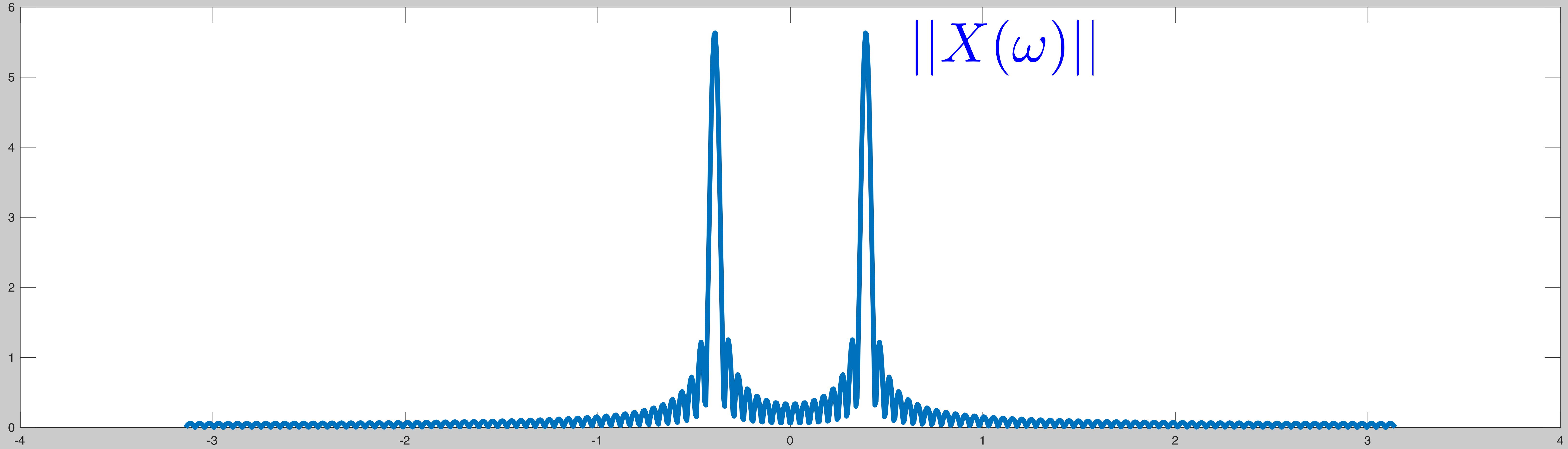
Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}\right)$

N = 128



$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$



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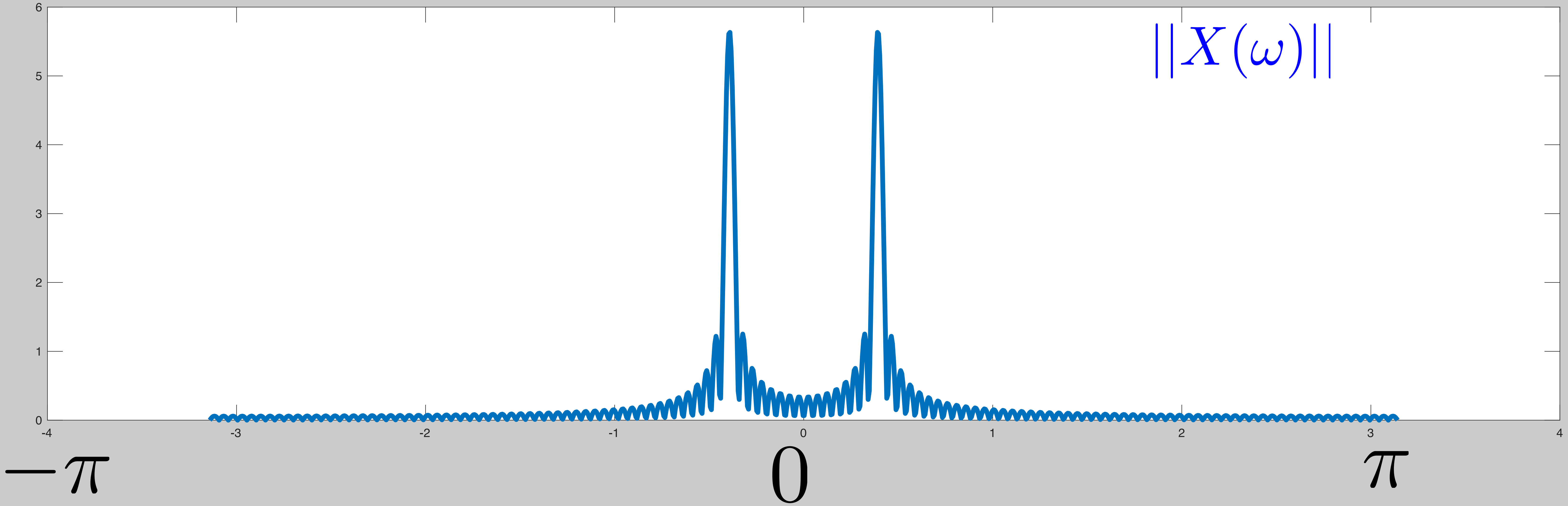
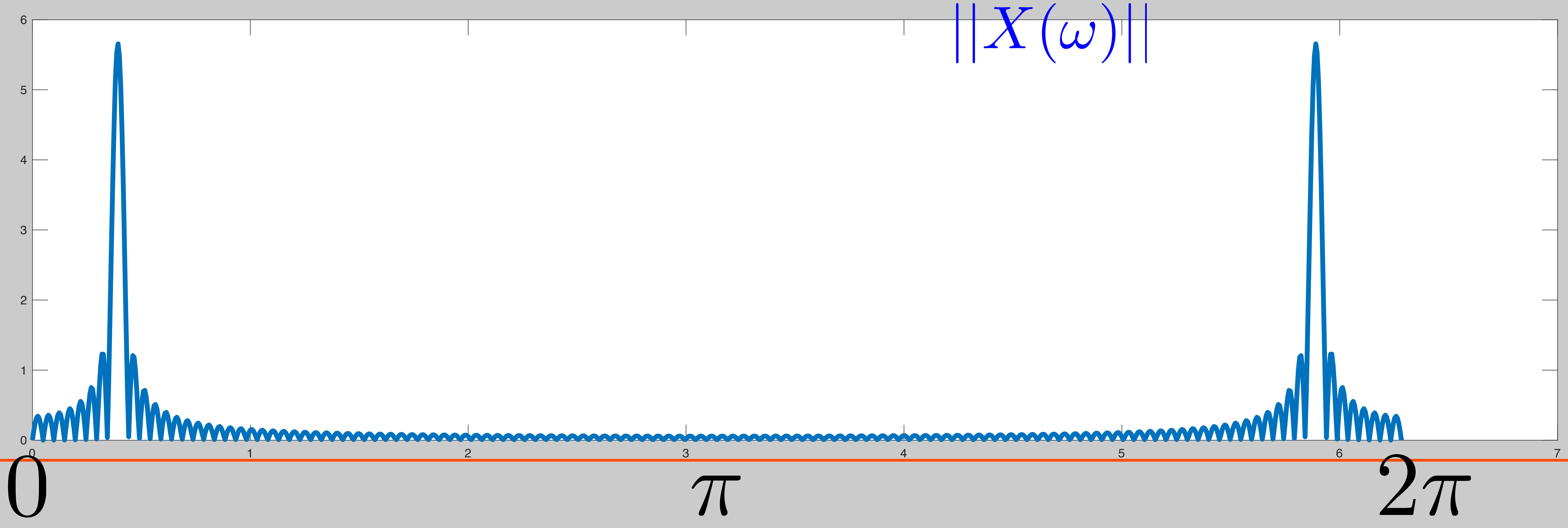
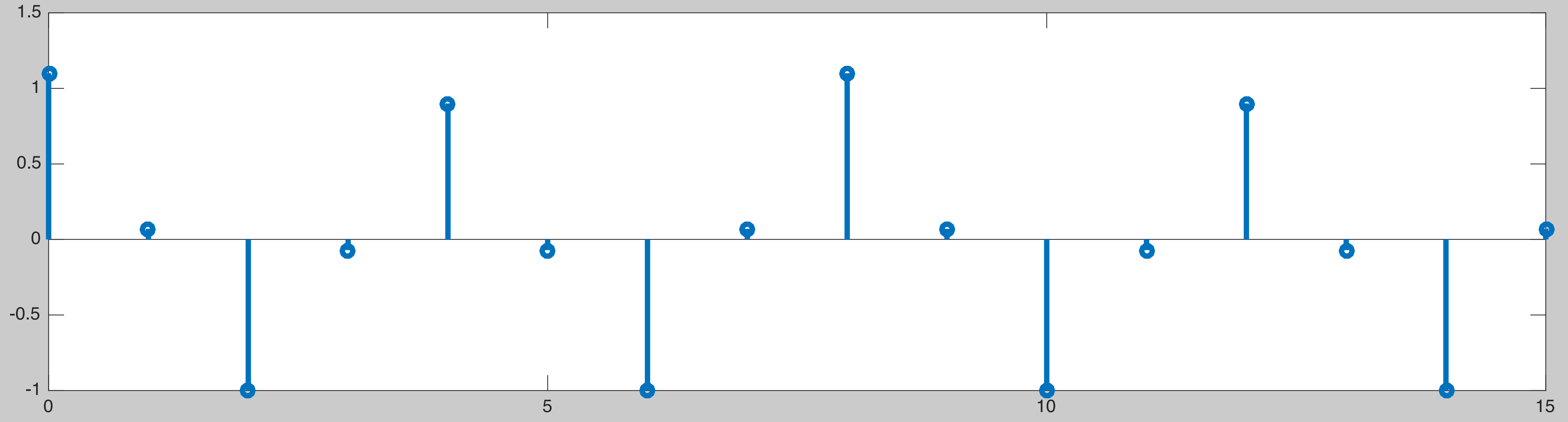




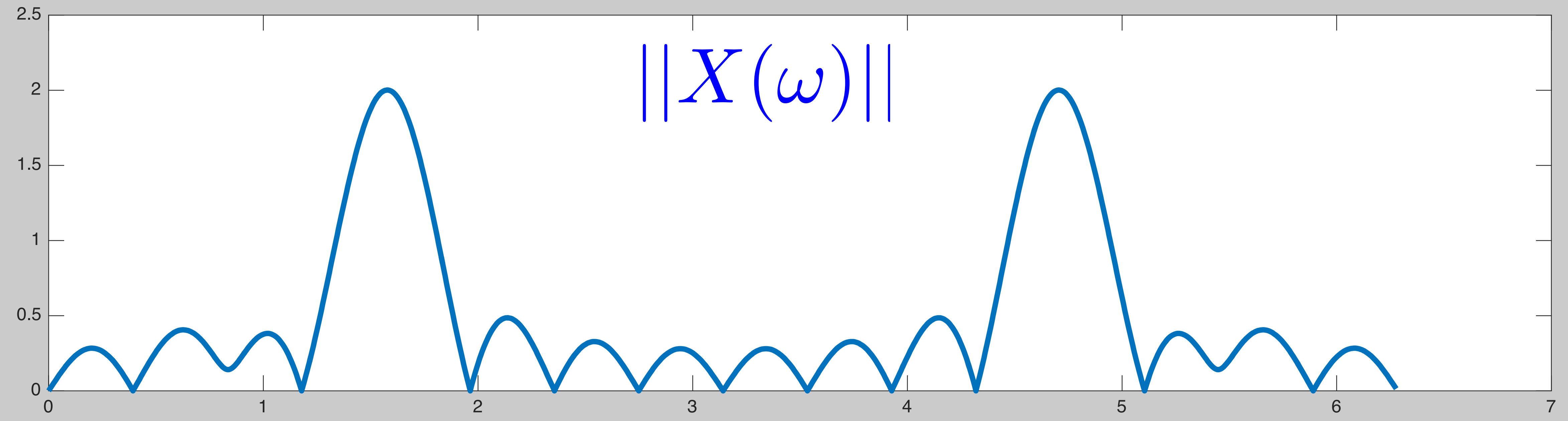
Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$

N = 16



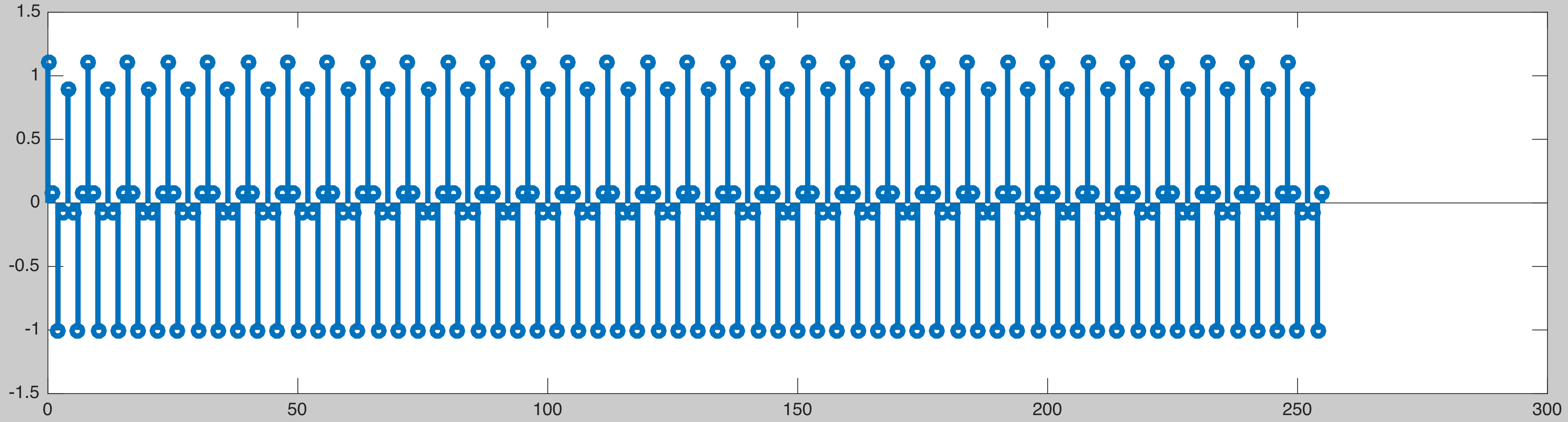
$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



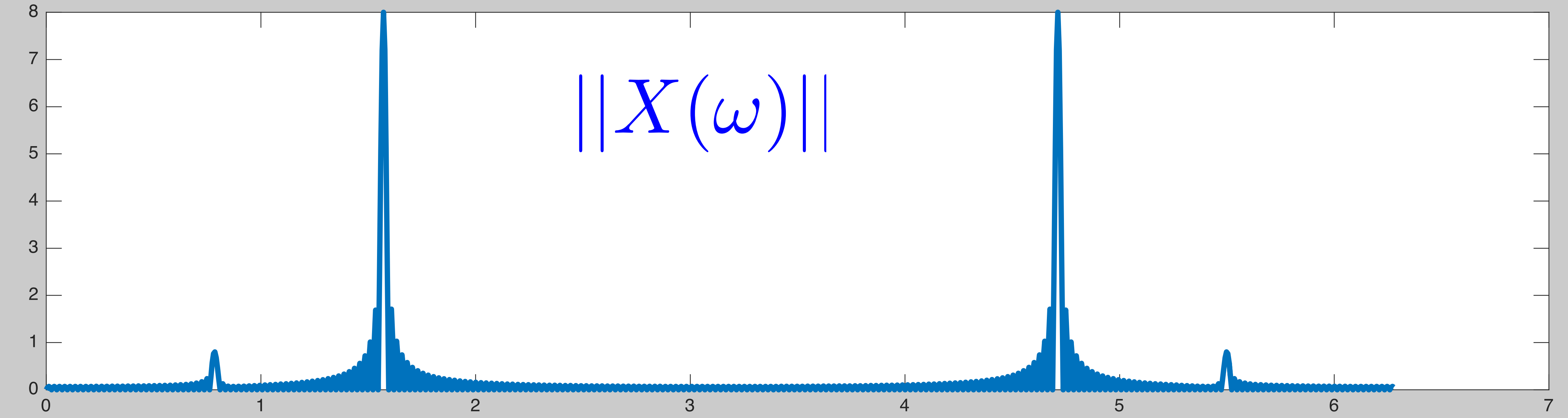
Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$

N = 256



$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



Discrete-Time-Fourier-Transform

- DTFT (not DFT)

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Discrete Fourier Transform (DFT)

- For $u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n}$, pick a set of N frequencies, which will result in an orthogonal basis

- Choose: $$\omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N} n}$$

$$k \in [0, N-1]$$

$$n \in [0, N-1]$$

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \; k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

DFT vs DTFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix}$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X[k] = \vec{u}_k^* \vec{x}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

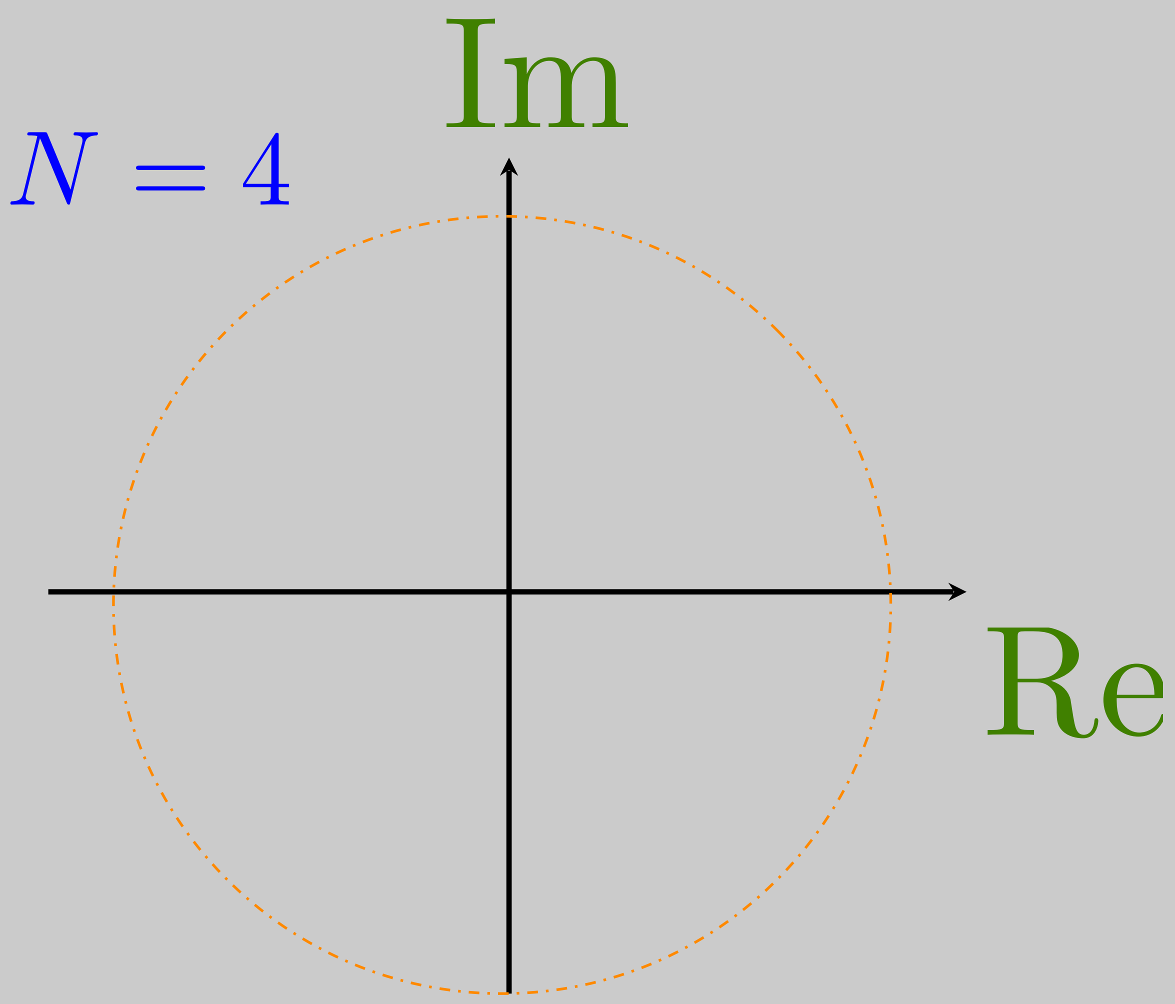
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$k = 1, N = 4$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$



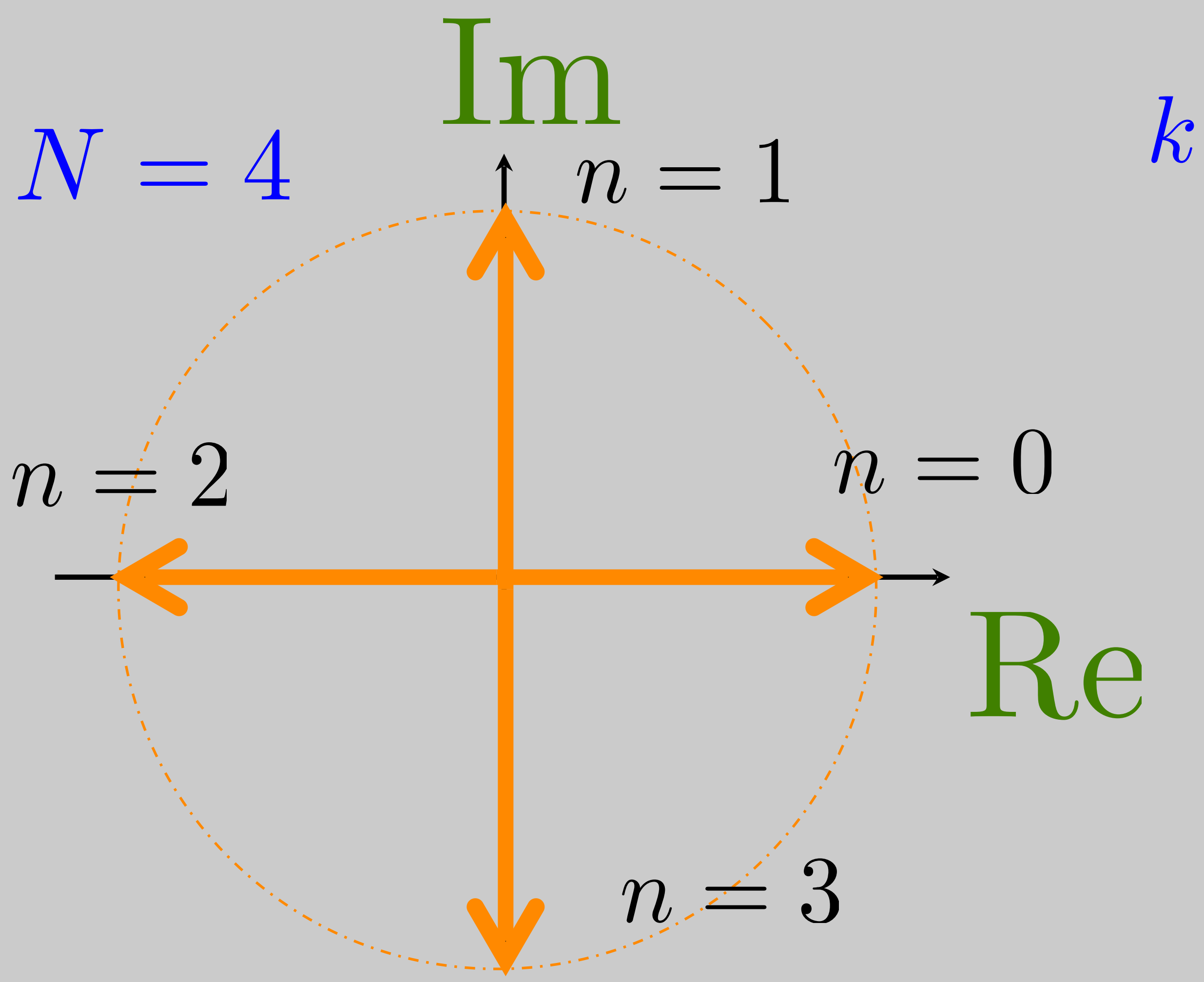
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

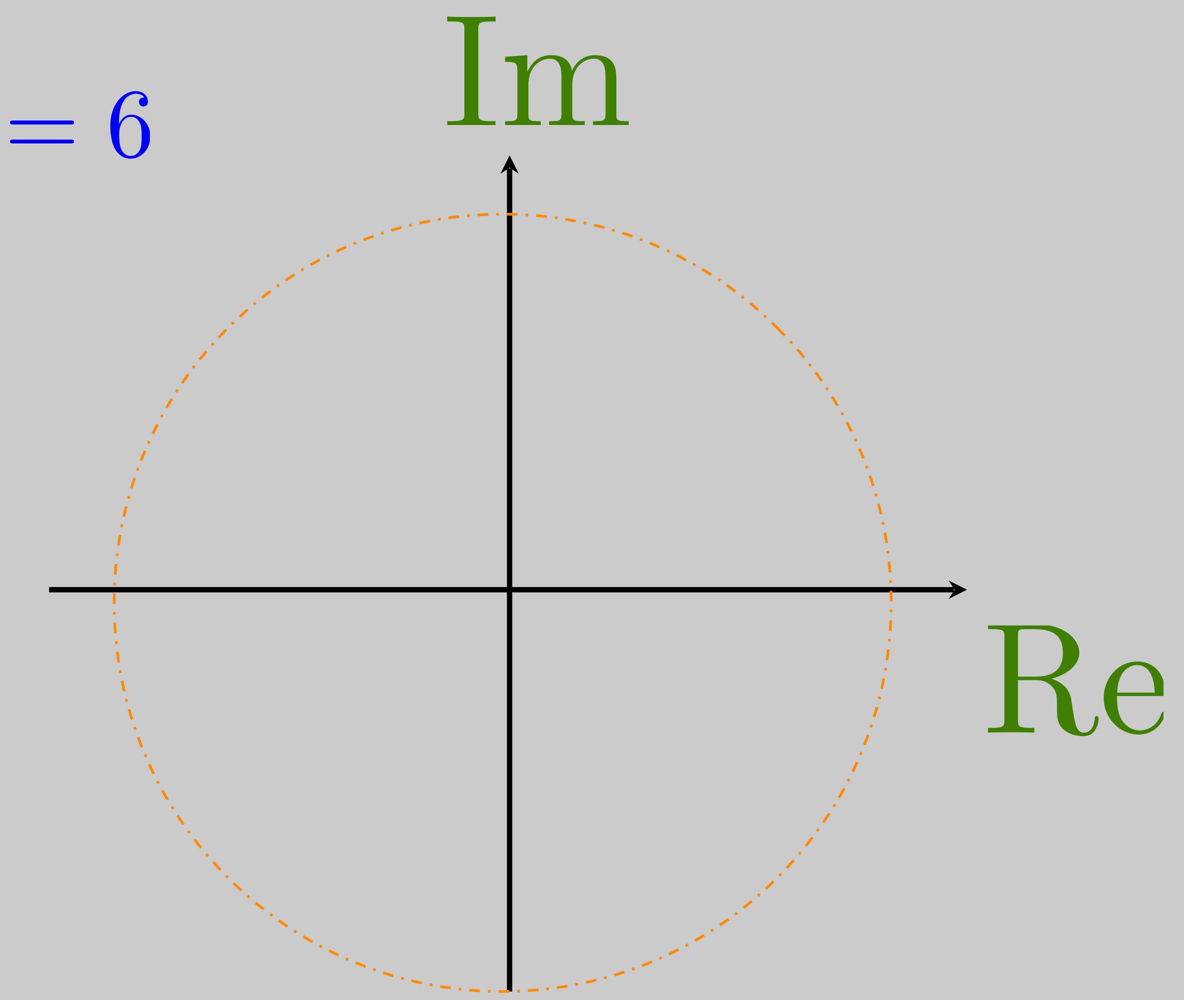
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$



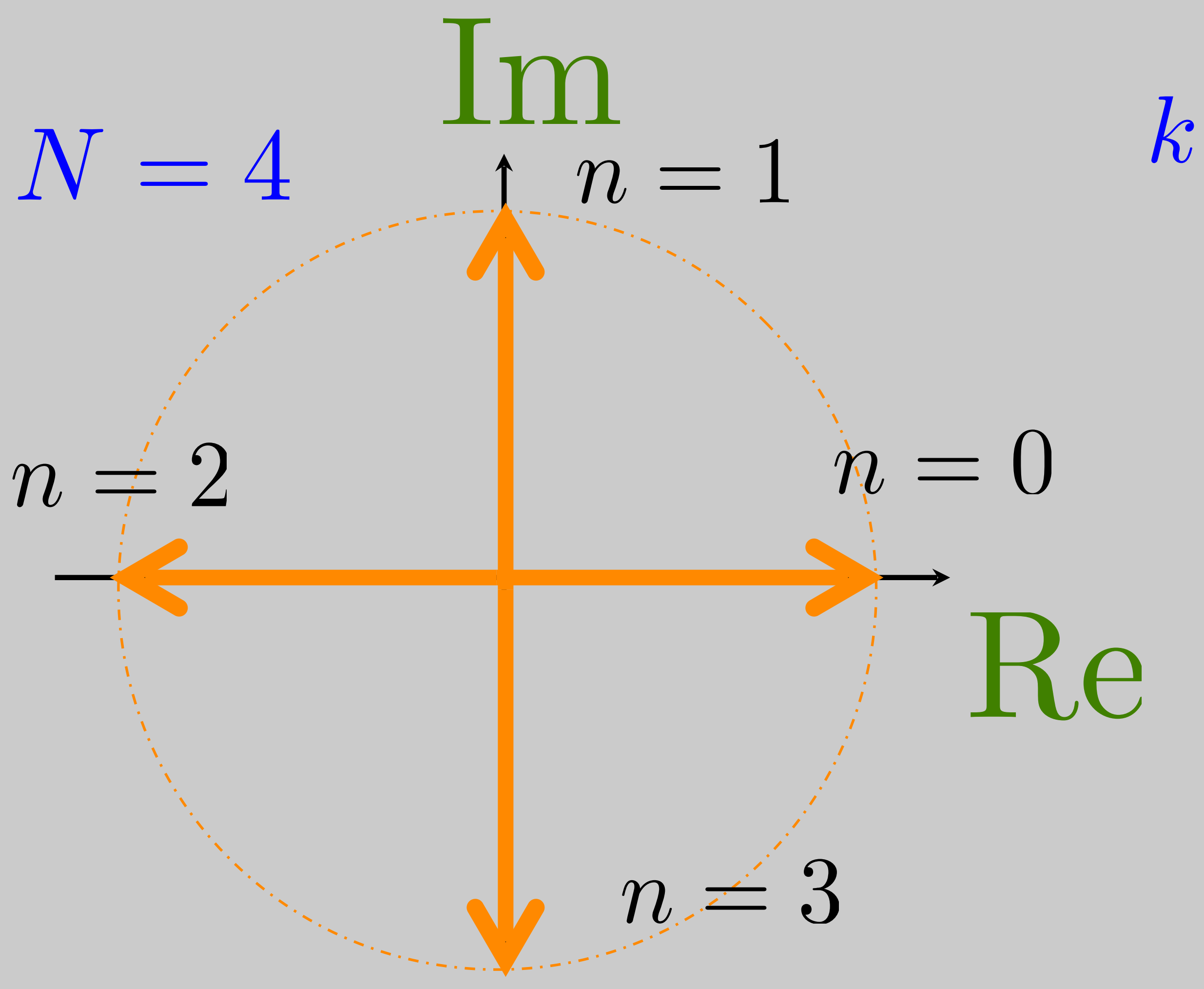
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

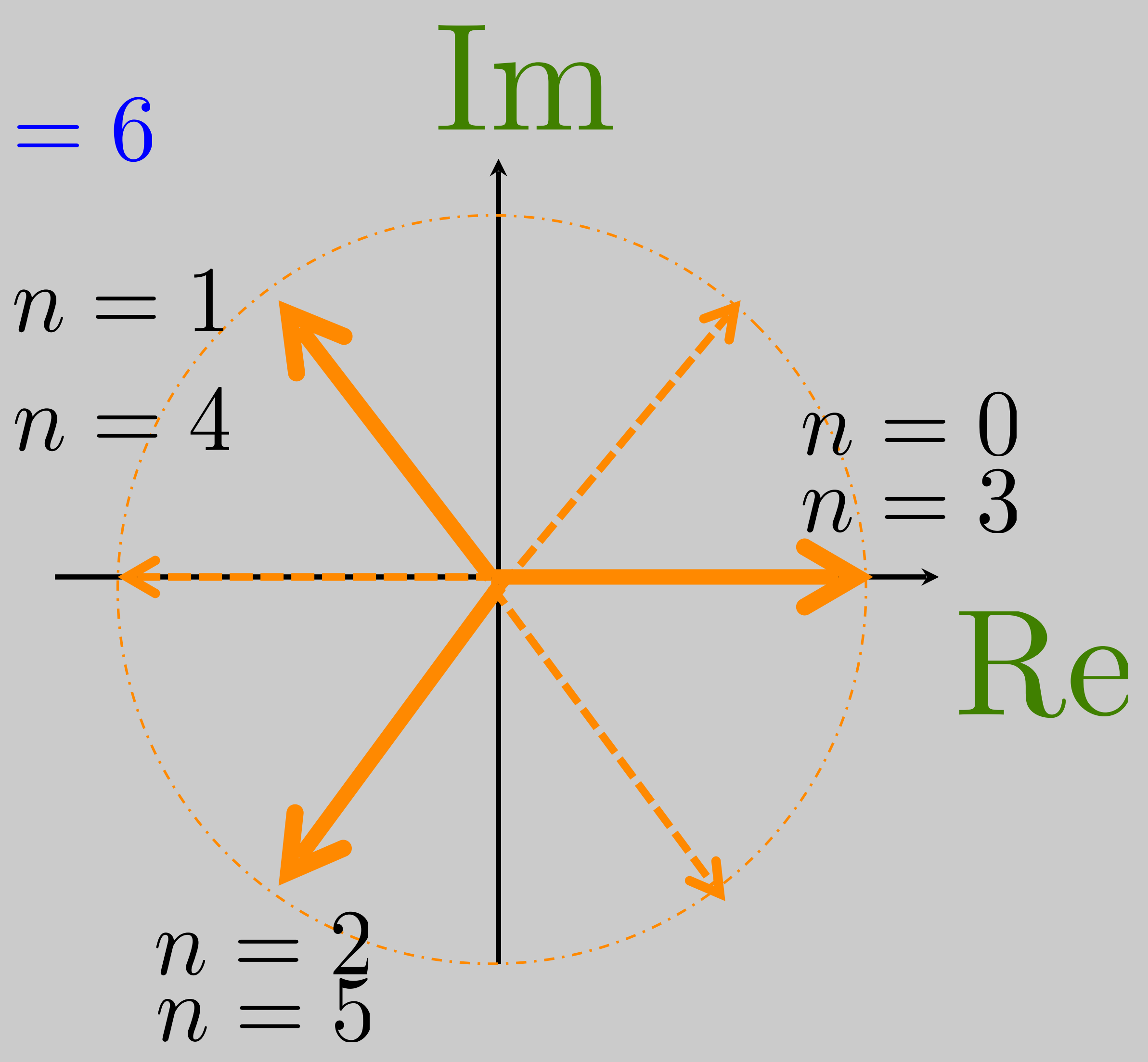
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$



$$\sum_{n=0}^{N-1} W_N^{nk} =?= \begin{cases} N & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Orthonormality of DFT Basis

- DFT basis vectors are orthonormal. Proof:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$\vec{u}_k^* \vec{u}_m = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{-nk} W_N^{nm} = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{n(m-k)} = \begin{cases} 1 & k=m \\ 0 & k \neq m \end{cases}$$

Example

$$N = 16$$

$$\vec{u}_k = \frac{1}{\sqrt{16}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{16}} \\ e^{j\frac{2\pi k \cdot 1}{16}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (15)}{16}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_{16}^{k\cdot 0} \\ W_{16}^{k\cdot 1} \\ \vdots \\ W_{16}^{k\cdot 15} \end{bmatrix}$$

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n) \quad = 0.5(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}} + 0.1e^{j\frac{\pi n}{4}} + 0.1e^{-j\frac{\pi n}{2}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{-j\frac{2\pi 4n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{-j\frac{2\pi 2n}{16}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{j\frac{2\pi 12n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{j\frac{2\pi 14n}{16}})$$

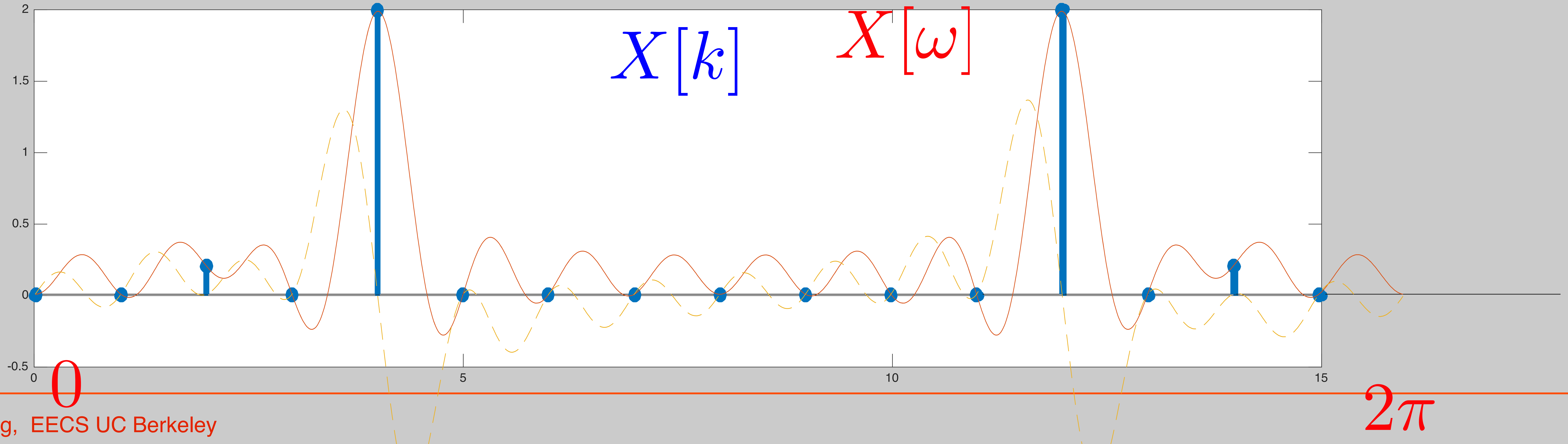
$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

Example

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$$

$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

$$= 0.2\vec{u}_2 + 2\vec{u}_4 + 2\vec{u}_{12} + 0.2\vec{u}_{14}$$

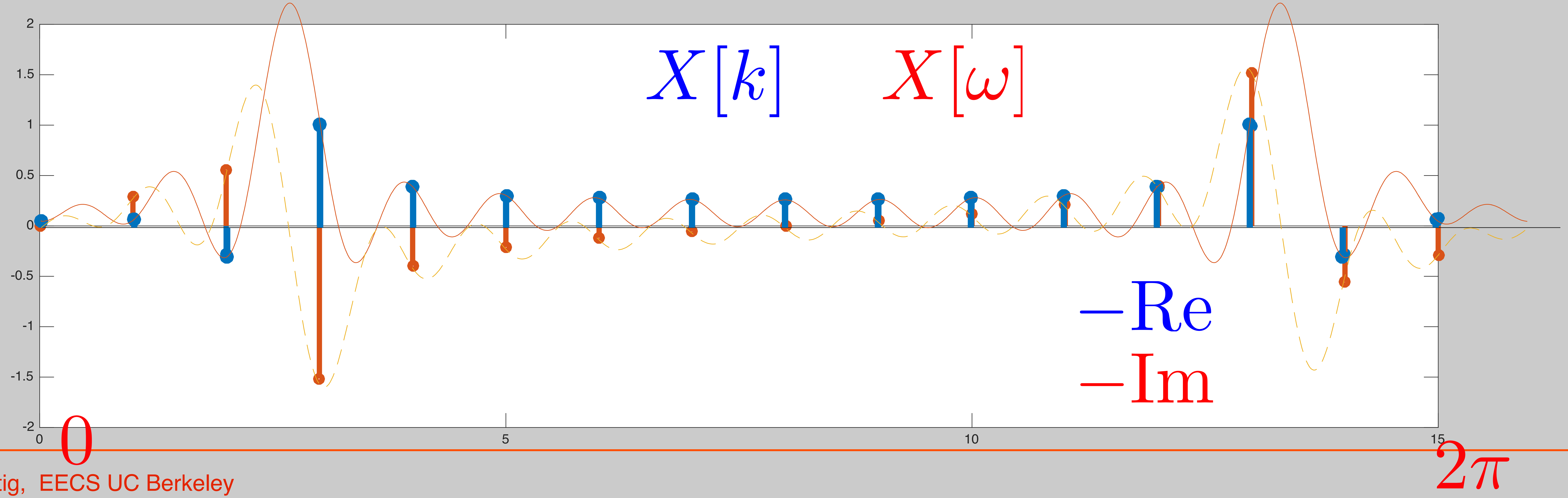


Example 2

What if there is no integer k to fit the frequency

$$\omega_k = \frac{2\pi k}{N}$$

$$x[n] = \cos(\frac{\pi}{3}n) + 0.1\cos(\frac{\pi}{6}n)$$



DFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \qquad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\vec{X} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^*}_{\triangleq F^*} \vec{x}$$

DFT

- DFT Analysis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} - & \vec{u}_0^* & - \\ - & \vec{u}_1^* & - \\ & \vdots & \\ - & \vec{u}_{N-1}^* & - \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\vec{X} = F^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

DFT

- DFT Synthesis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = F\vec{X} \qquad = F(F^*\vec{x})$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{+nk}$$

Quiz

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

Compute a 2 point DFT of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{u}_1 =$

$\vec{u}_2 =$

$\vec{u}_1^* \vec{x} =$

$\vec{u}_2^* \vec{x} =$

$\vec{X} =$

Example cont

- DFT_2 matrix:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{X} = F^* \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$