EE16B
Designing Information Devices and Systems II

Lecture 14A
The Discrete Fourier Transform

Intro

- Last time:
  - Finite sequences as vectors
  - Linear convolutions as matrices
  - Change of basis
  - Begin discrete Fourier Transform

- Today
  - Discrete Fourier Transform

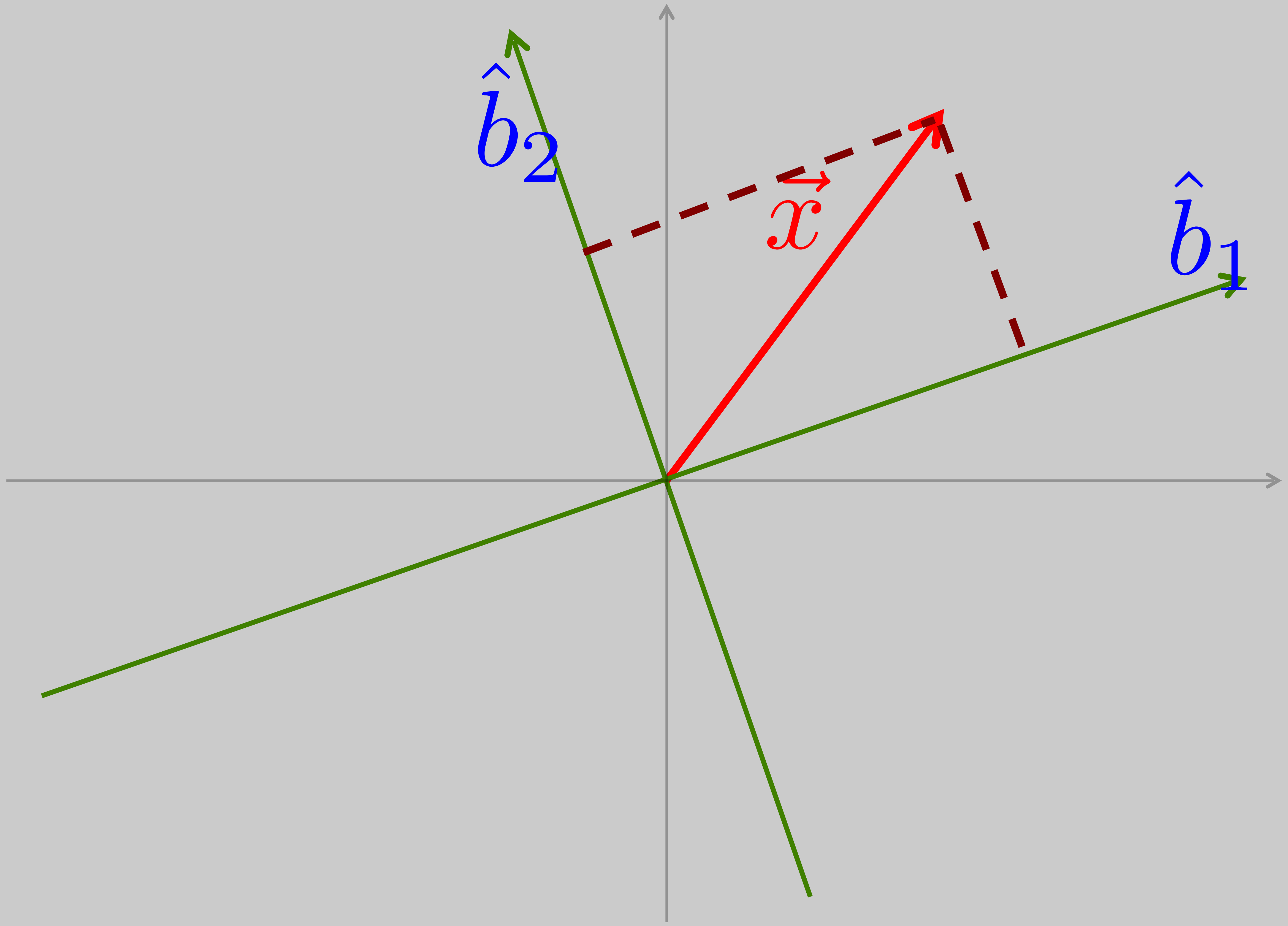
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

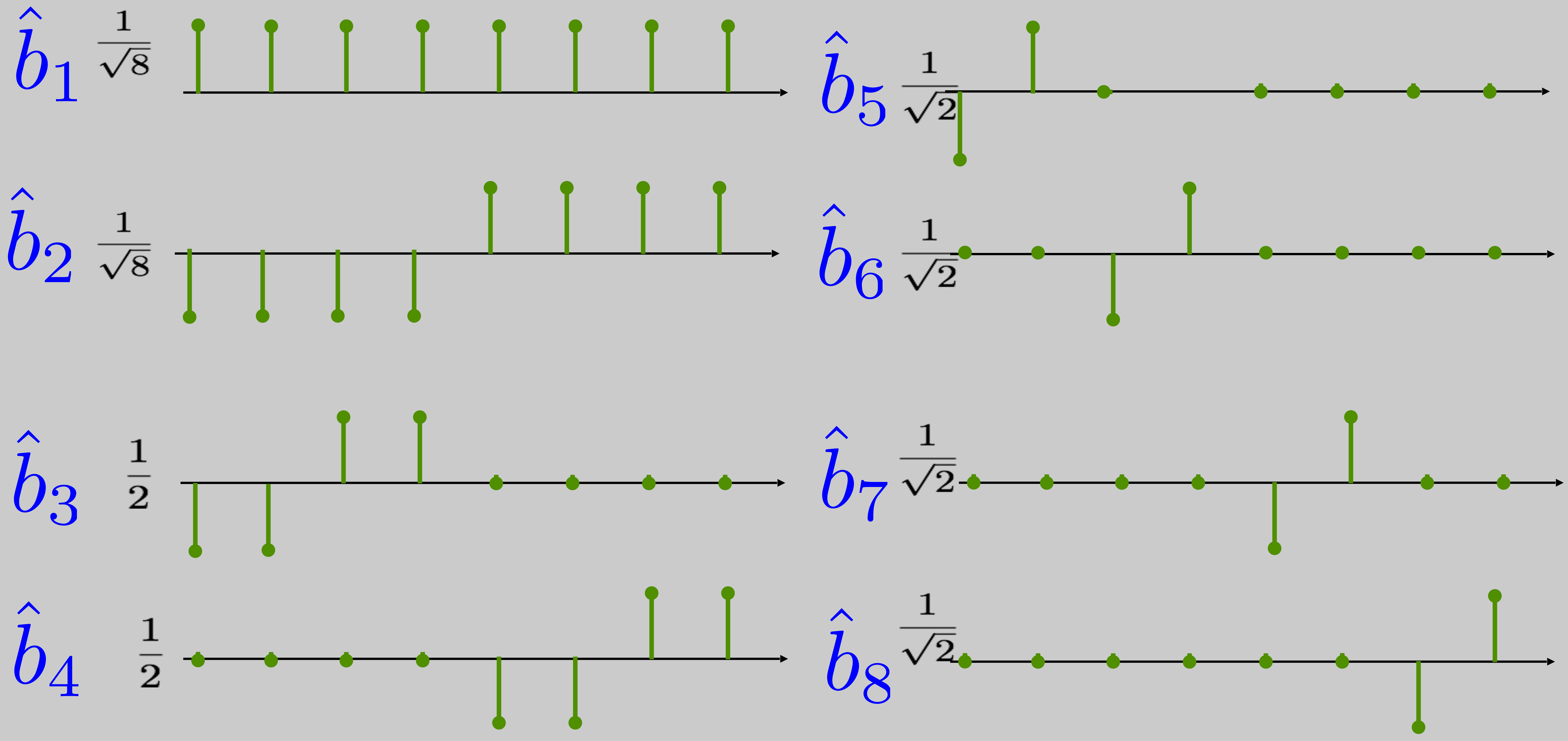
New coordinates:

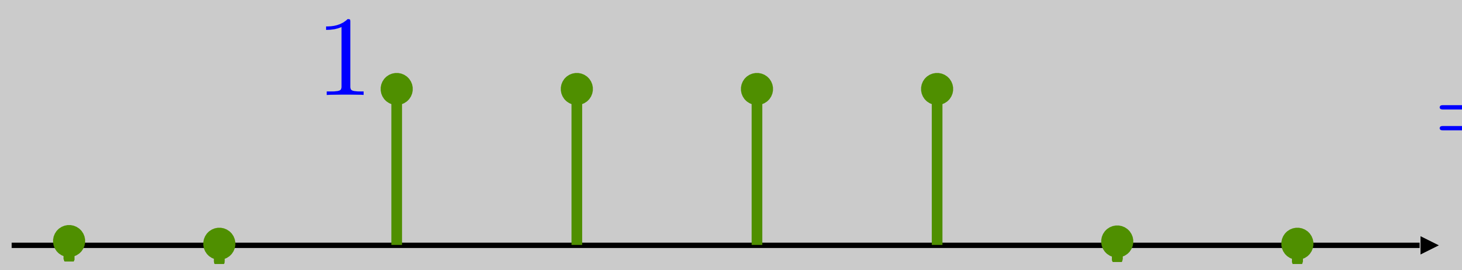
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



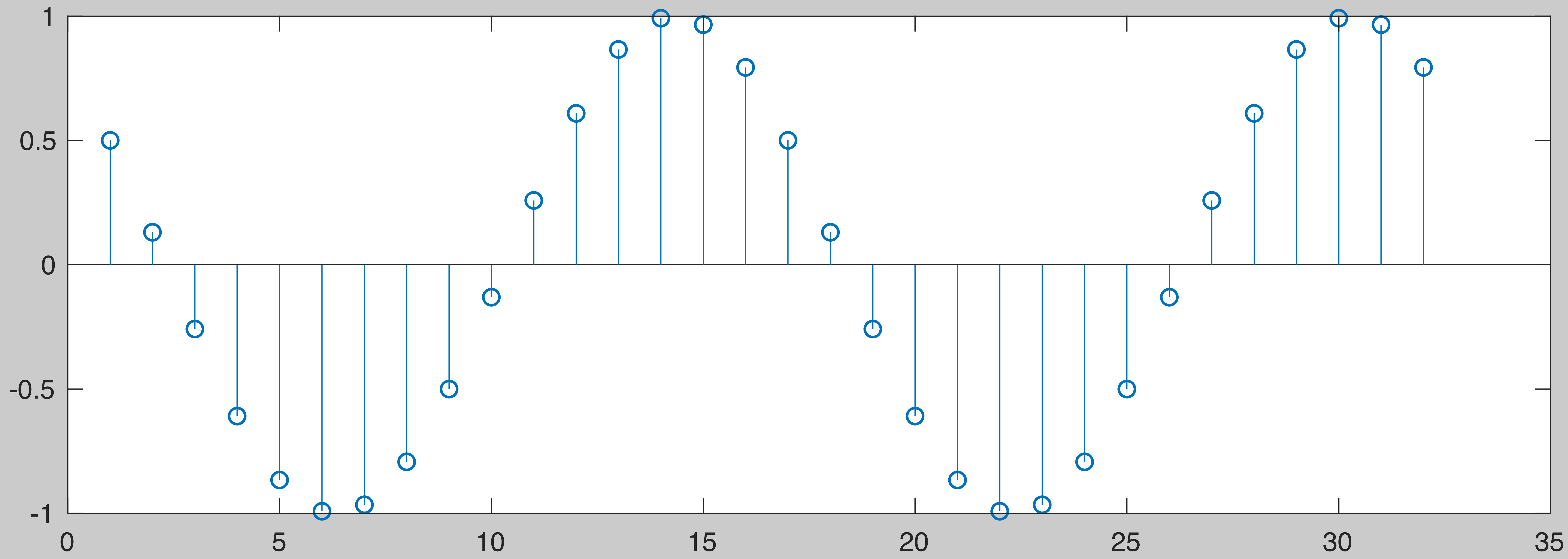
Change of basis





$$= \frac{4}{\sqrt{8}} \cdot \hat{b}_1 + 0 \cdot \hat{b}_2 + 1 \cdot \hat{b}_3 + (-1) \cdot \hat{b}_4 + 0 \cdot \hat{b}_5 + 0 \cdot \hat{b}_6 + 0 \cdot \hat{b}_7 + 0 \cdot \hat{b}_8$$

- How can we find the frequency of this N=32 length signal?



Project on unit sinusoidal vectors?

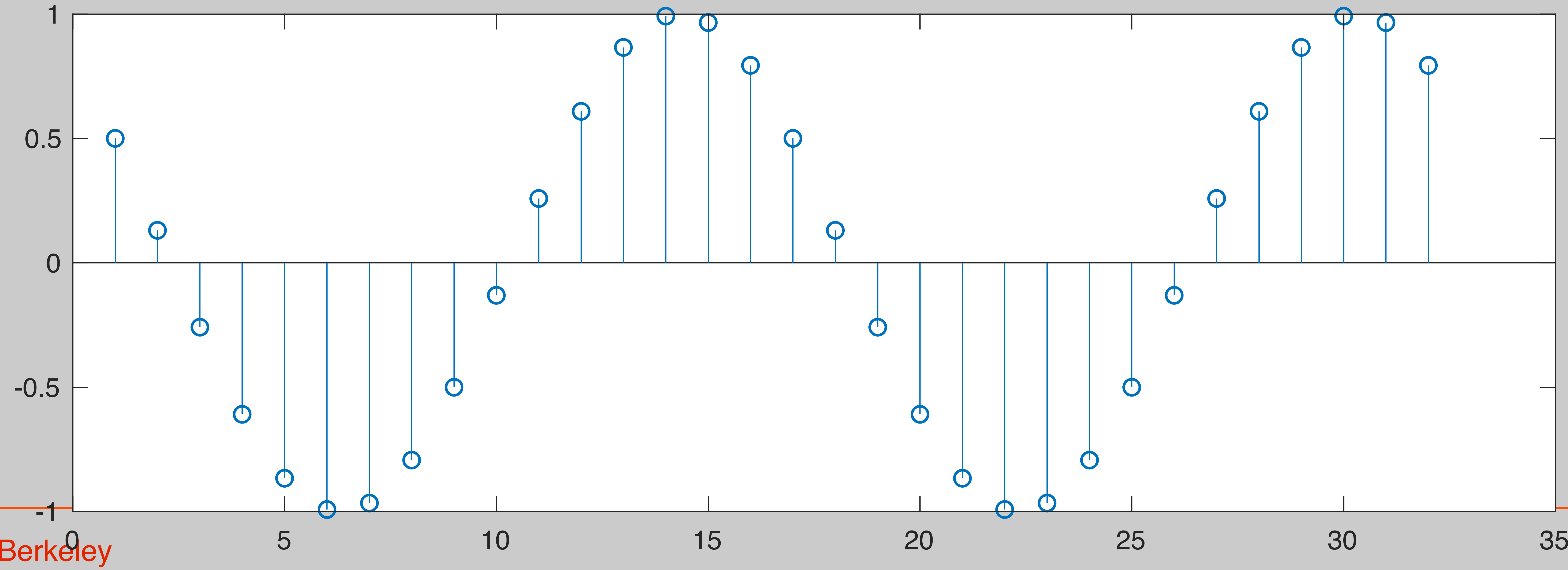
Complex Exponential Basis

- Phase is a problem! (inside a cosine)

$$\vec{x} \quad \Big| \quad x[n] = \cos(\omega_0 n + \phi_0)$$

- Solution: Phase is a coefficient for complex exponentials!

$$\vec{x} \quad \Big| \quad x[n] = \frac{1}{2}e^{j\omega n} \cdot e^{j\phi} \; + \frac{1}{2}e^{-j\omega n} \cdot e^{-j\phi}$$



Frequency Analysis Through Projections

- N-length normalized discrete frequency:

$$u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n} \qquad 0 \le n < N \qquad 0 \le \omega < 2\pi$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix} \Rightarrow X(\omega) = \vec{u}_w^{*} \vec{x}$$

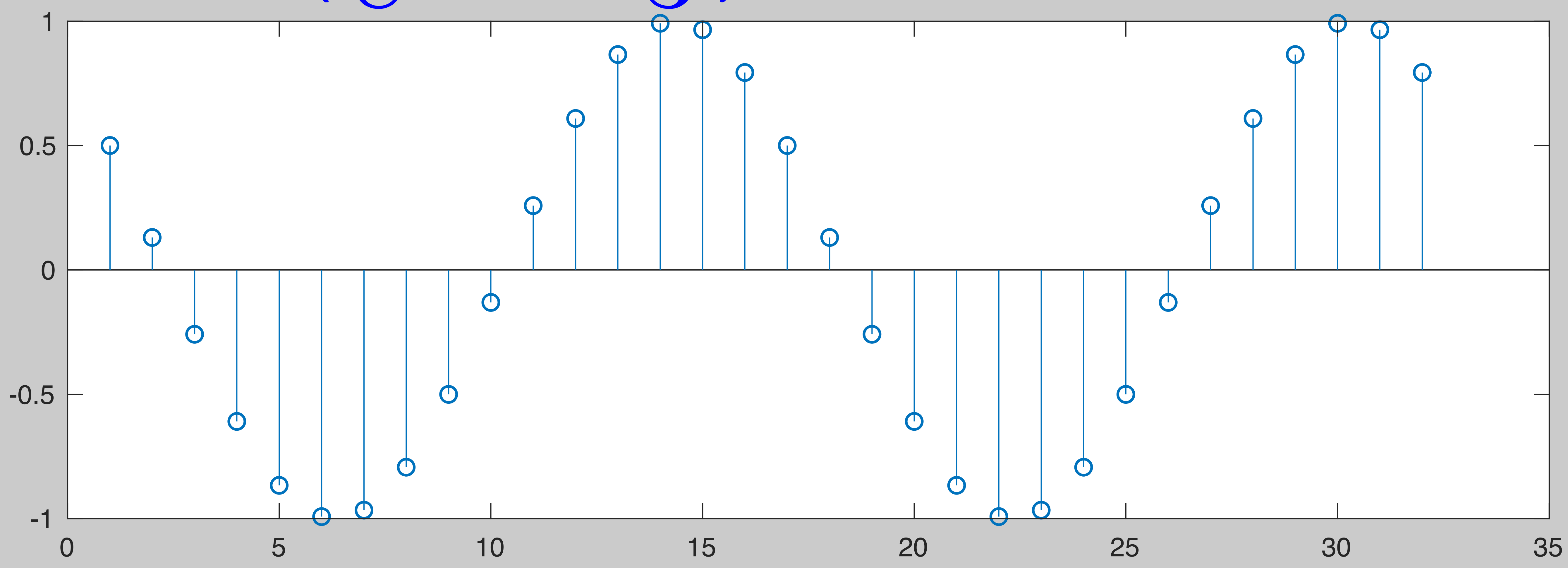
$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Also the DTFT of the finite sequence x

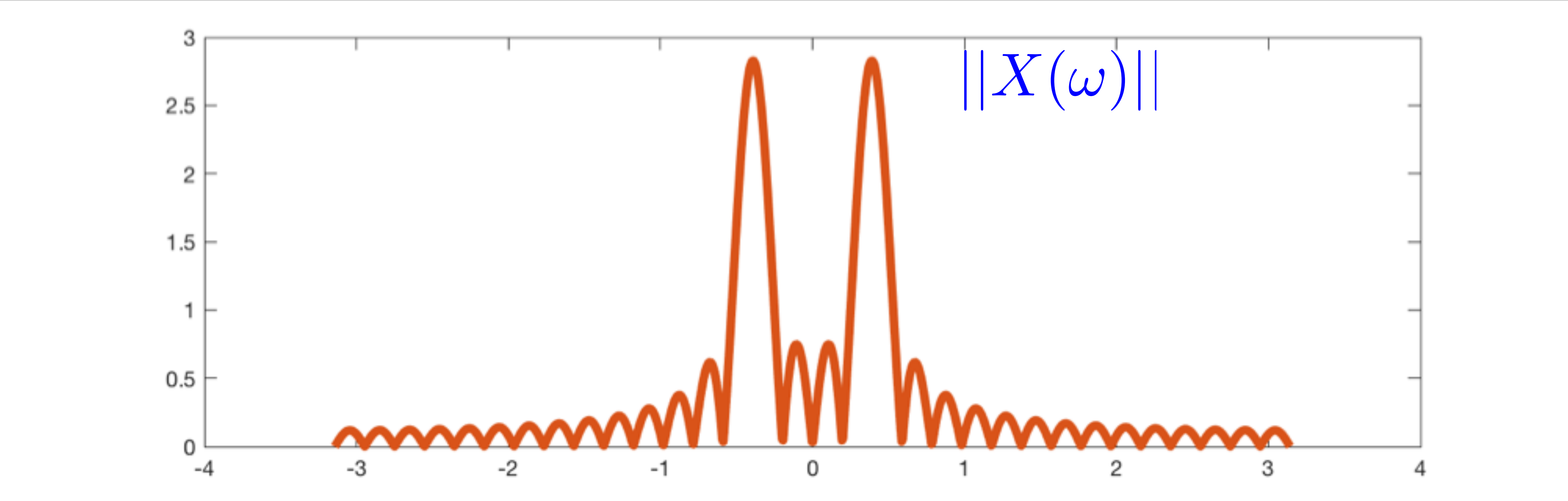
Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}\right)$

N = 32



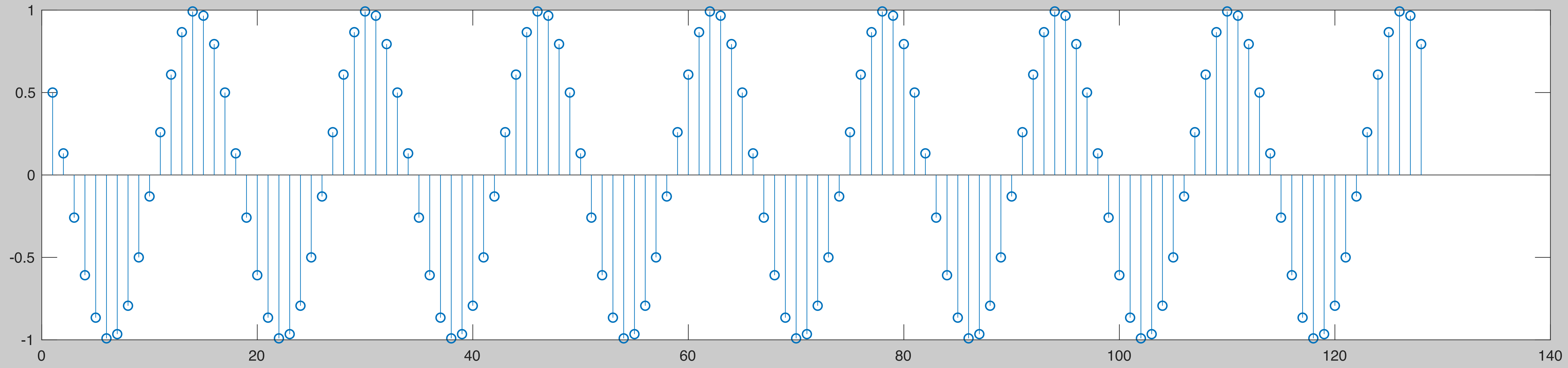
$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



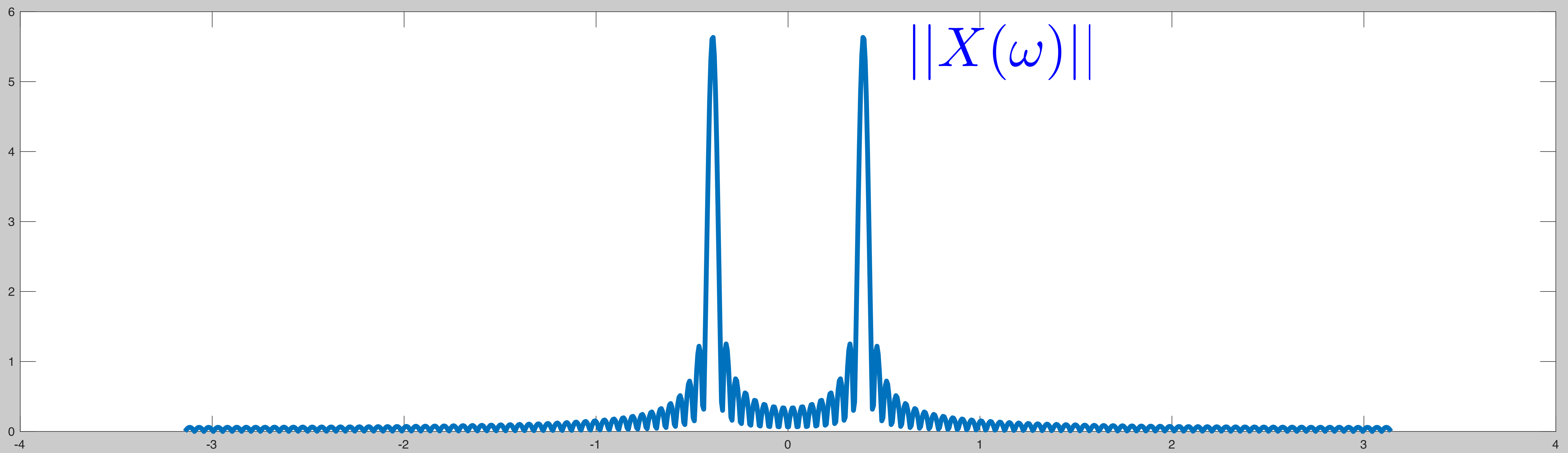
Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}\right)$

N = 128

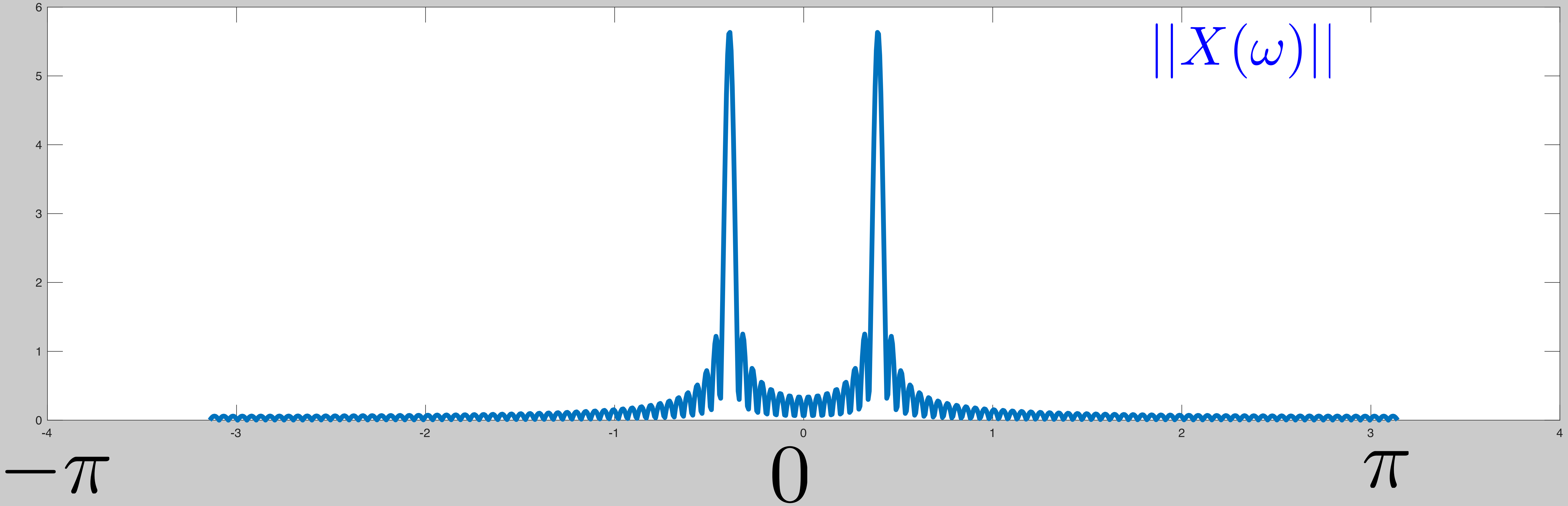


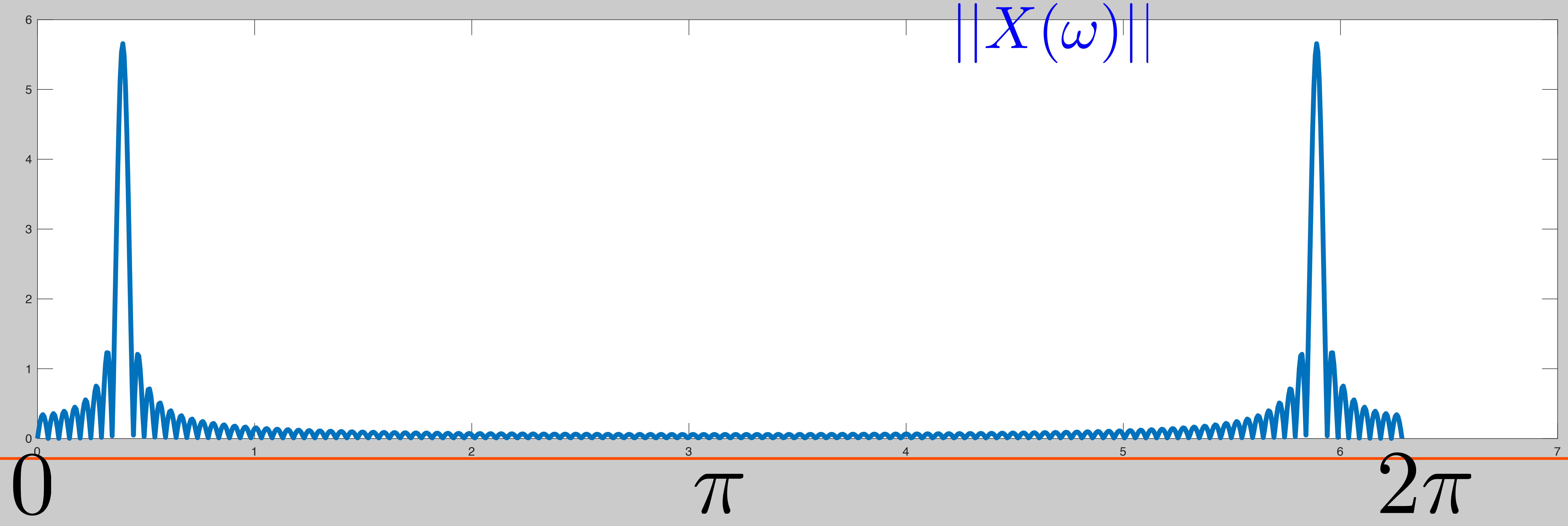
$$\Rightarrow X(\omega) = \vec{u}_w^{*}\vec{x}$$



Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}\right)$

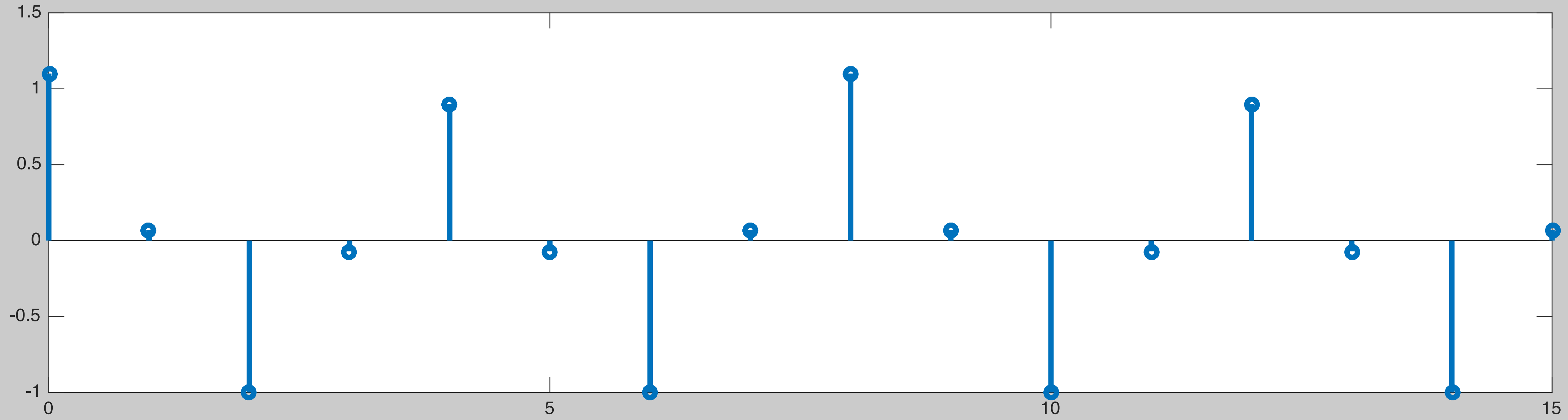




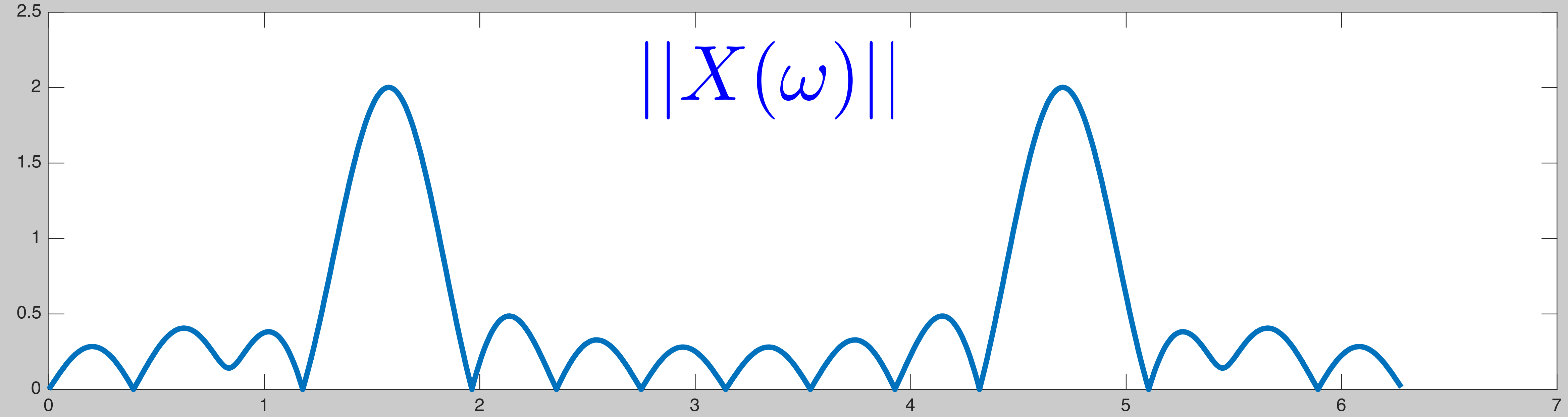
Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$

N = 16



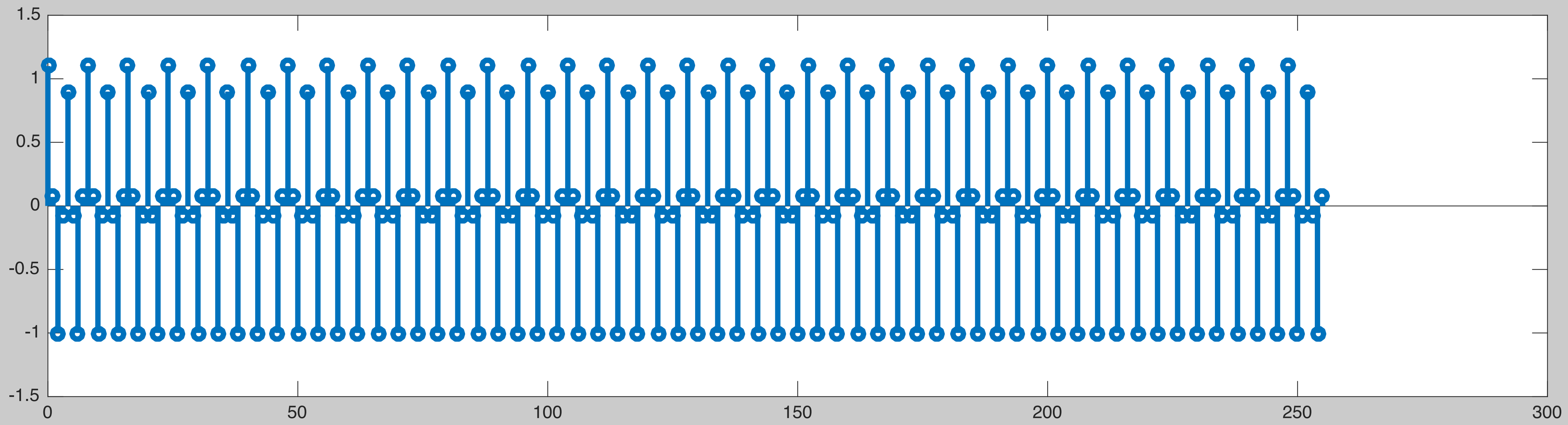
$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



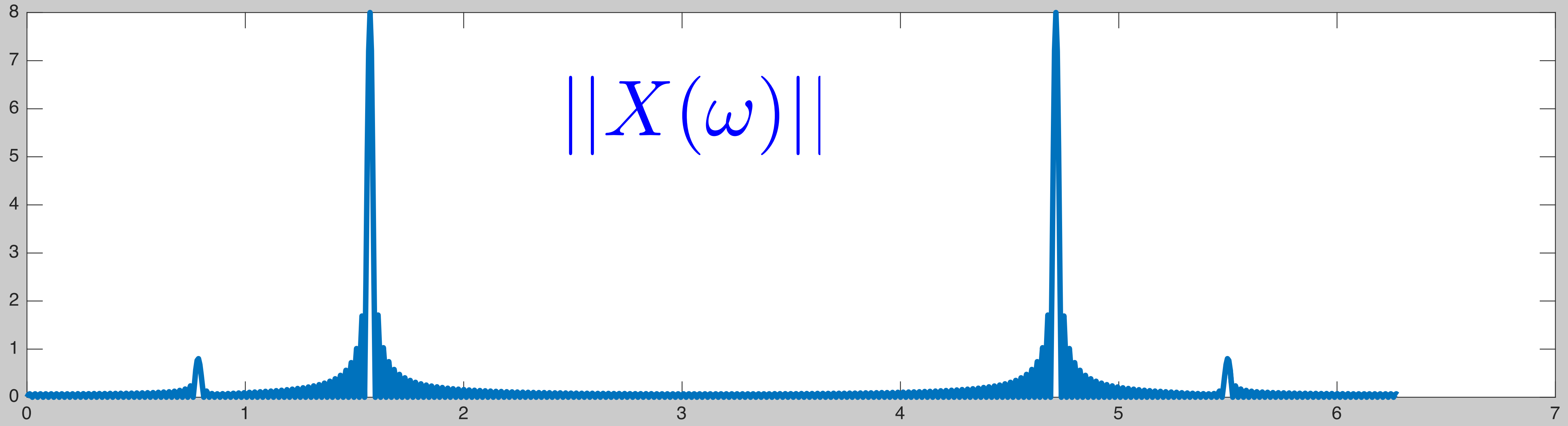
Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$

N = 256



$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



Discrete-Time-Fourier-Transform

- DTFT (not DFT)

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Discrete Fourier Transform (DFT)

- For $u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n}$, pick a set of N frequencies, which will result in an orthogonal basis

- Choose: $\omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N} n}$

$$k \in [0, N-1]$$

$$n \in [0, N-1]$$

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

DFT vs DTFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix}$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X[k] = \vec{u}_k^* \vec{x}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

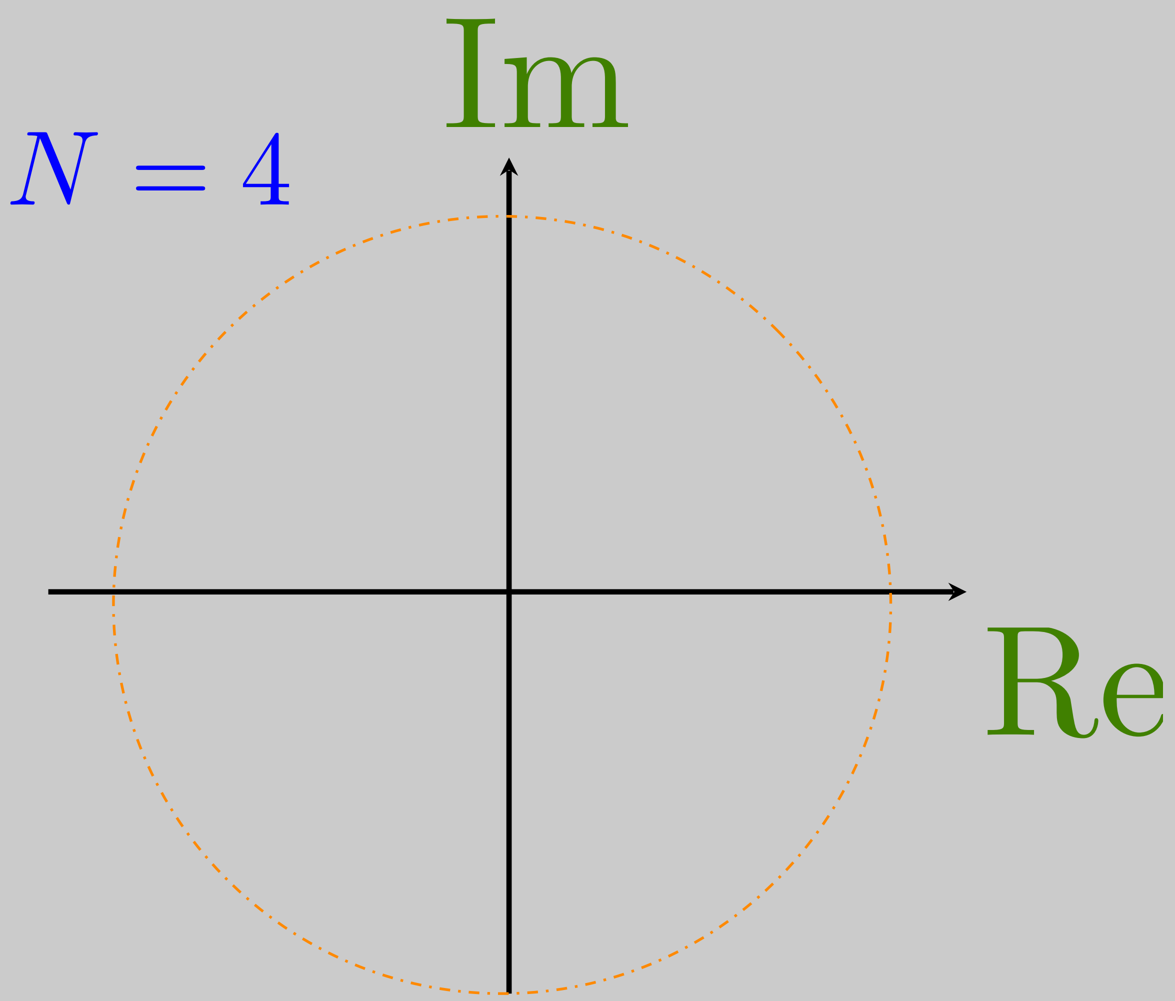
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



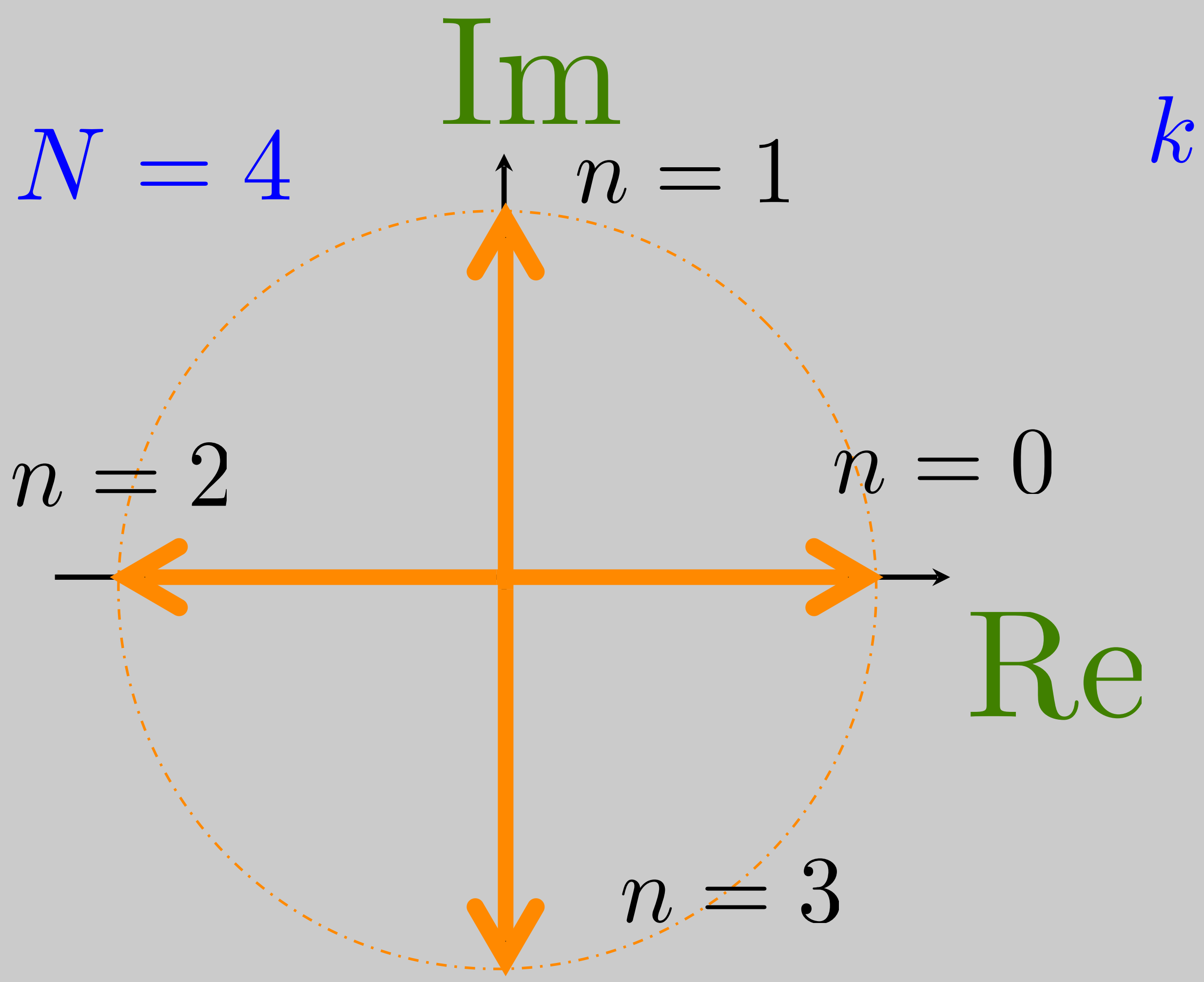
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

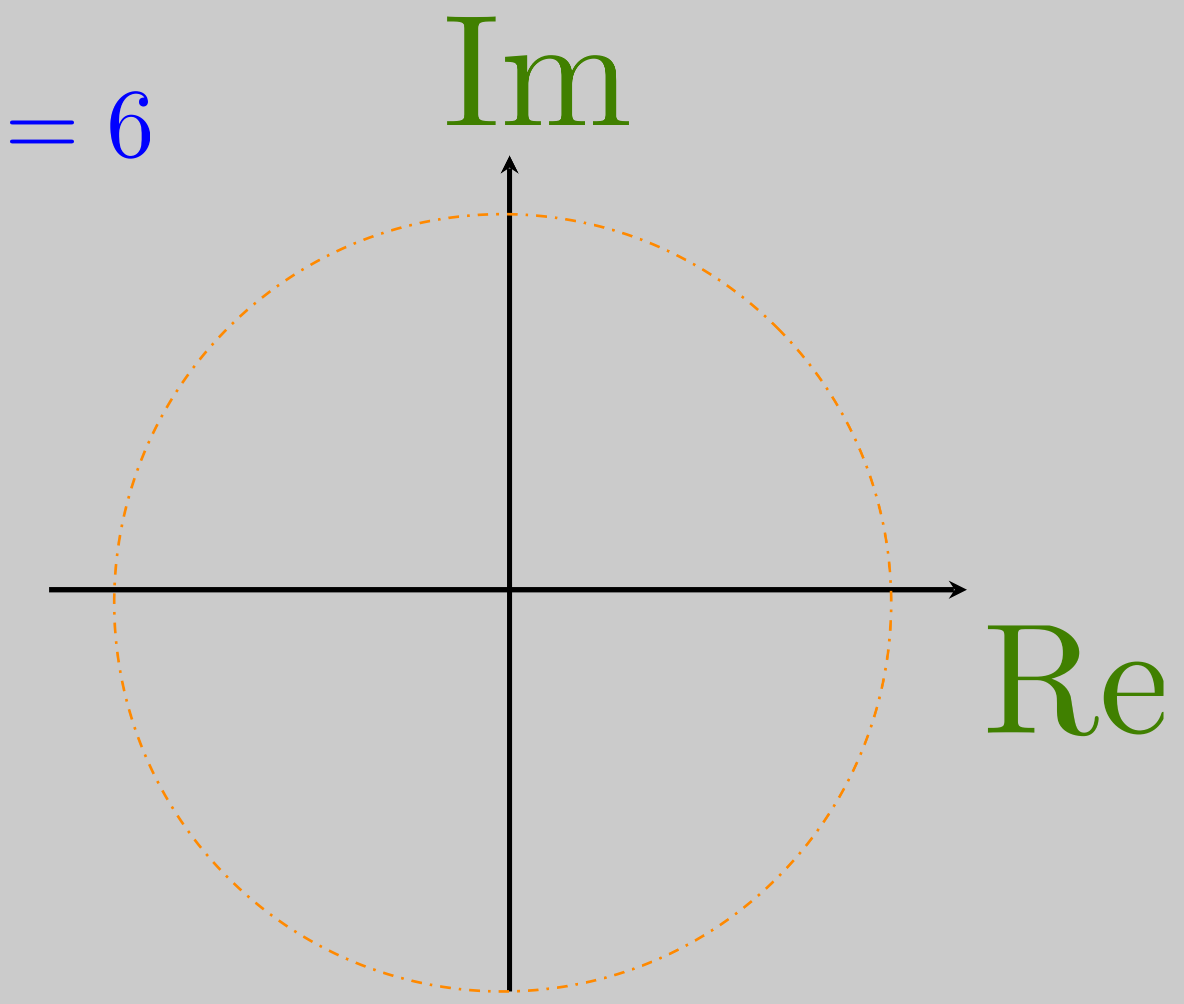
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$



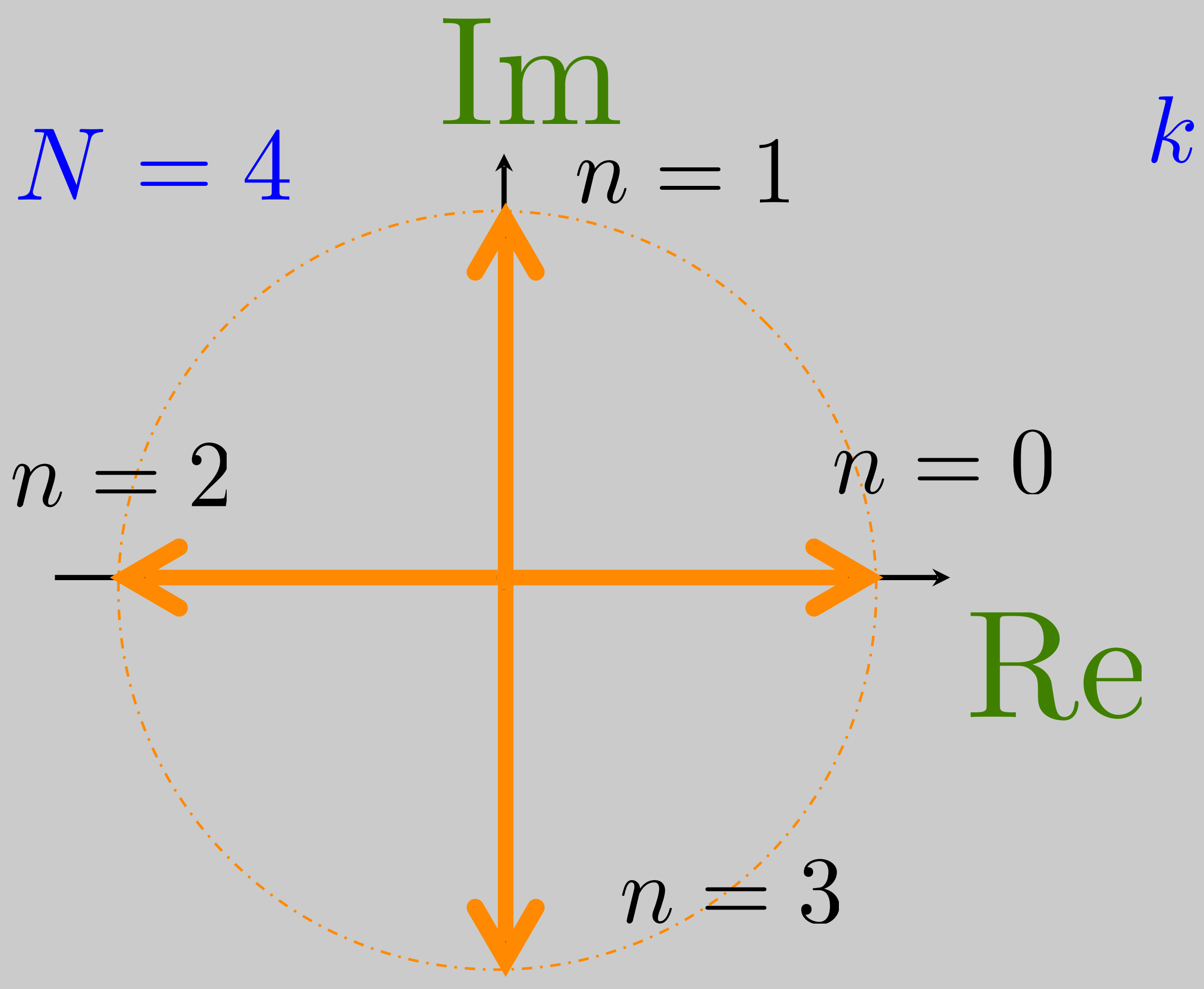
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

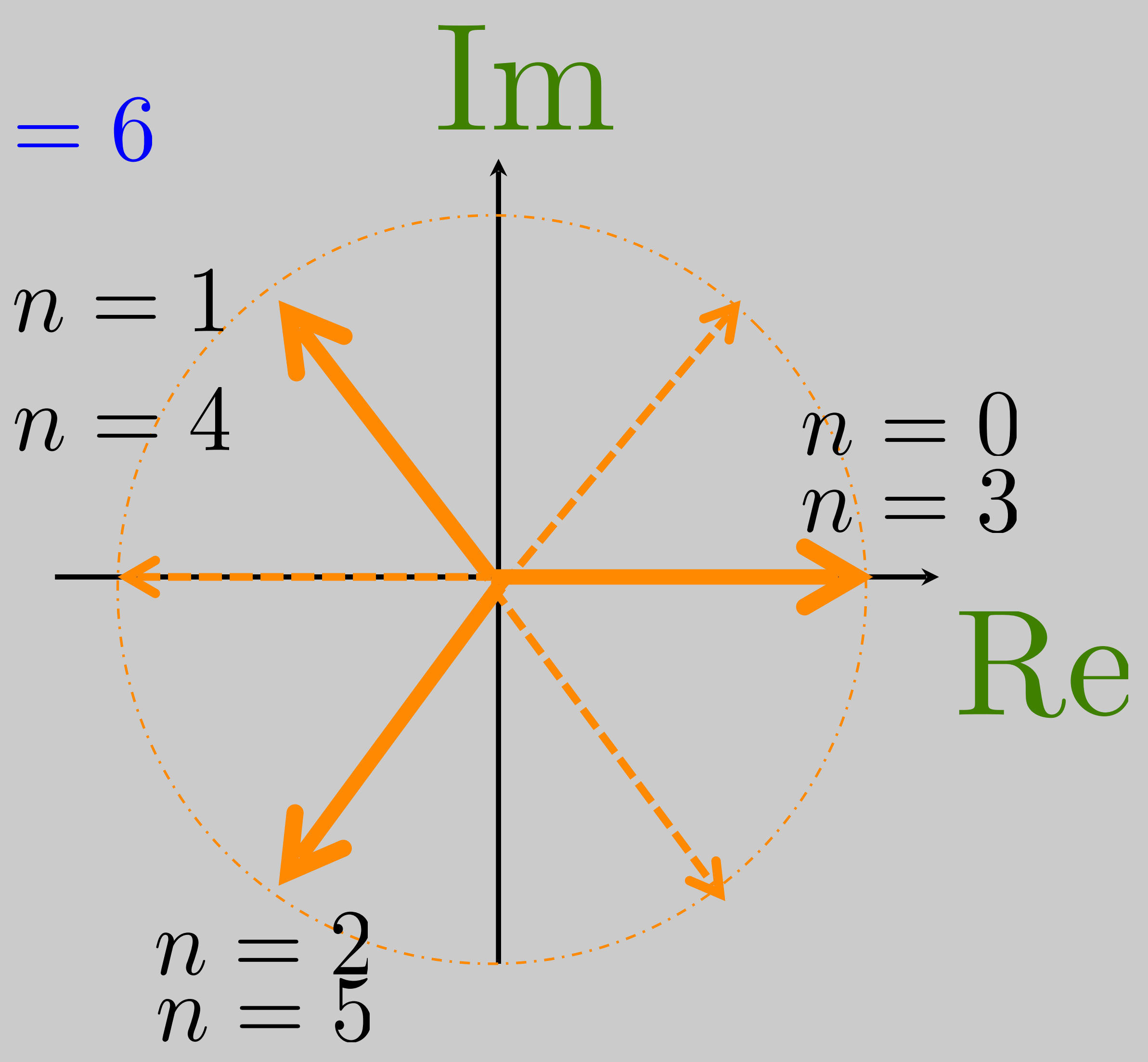
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$



$$\sum_{n=0}^{N-1} W_N^{nk} =?= \begin{cases} N & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Orthonormality of DFT Basis

- DFT basis vectors are orthonormal. Proof:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$\vec{u}_k^* \vec{u}_m = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{-nk} W_N^{nm} = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{n(m-k)} = \begin{cases} 1 & k=m \\ 0 & k \neq m \end{cases}$$

Example

$$N = 16$$

$$\vec{u}_k = \frac{1}{\sqrt{16}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{16}} \\ e^{j\frac{2\pi k \cdot 1}{16}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (15)}{16}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_{16}^{k\cdot 0} \\ W_{16}^{k\cdot 1} \\ \vdots \\ W_{16}^{k\cdot 15} \end{bmatrix}$$

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n) \quad = 0.5(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}} + 0.1e^{j\frac{\pi n}{4}} + 0.1e^{-j\frac{\pi n}{2}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{-j\frac{2\pi 4n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{-j\frac{2\pi 2n}{16}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{j\frac{2\pi 12n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{j\frac{2\pi 14n}{16}})$$

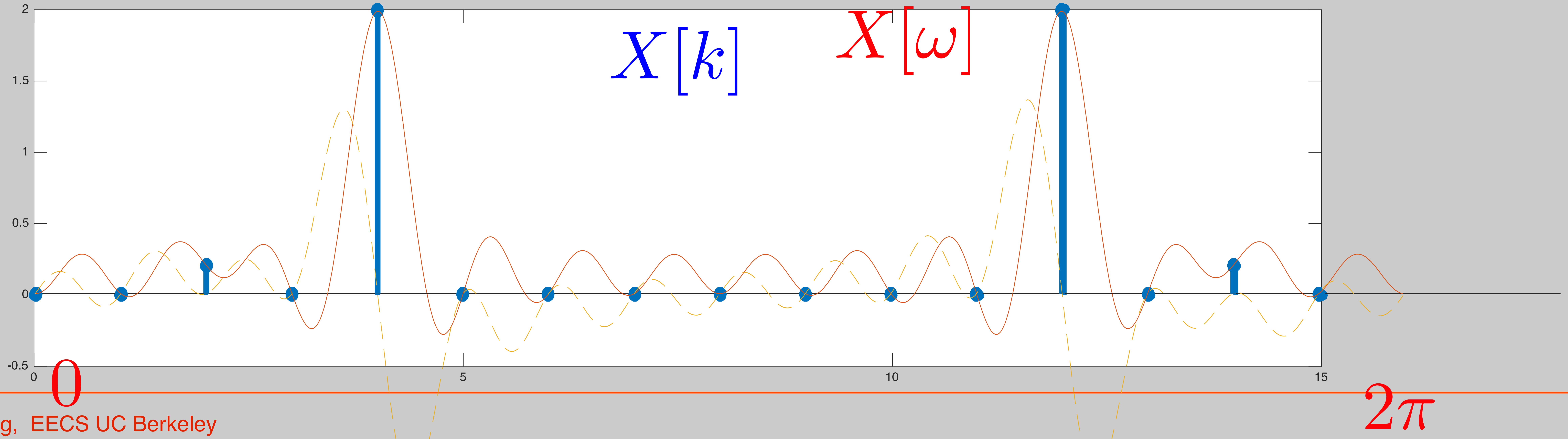
$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

Example

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$$

$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

$$= 0.2\vec{u}_2 + 2\vec{u}_4 + 2\vec{u}_{12} + 0.2\vec{u}_{14}$$

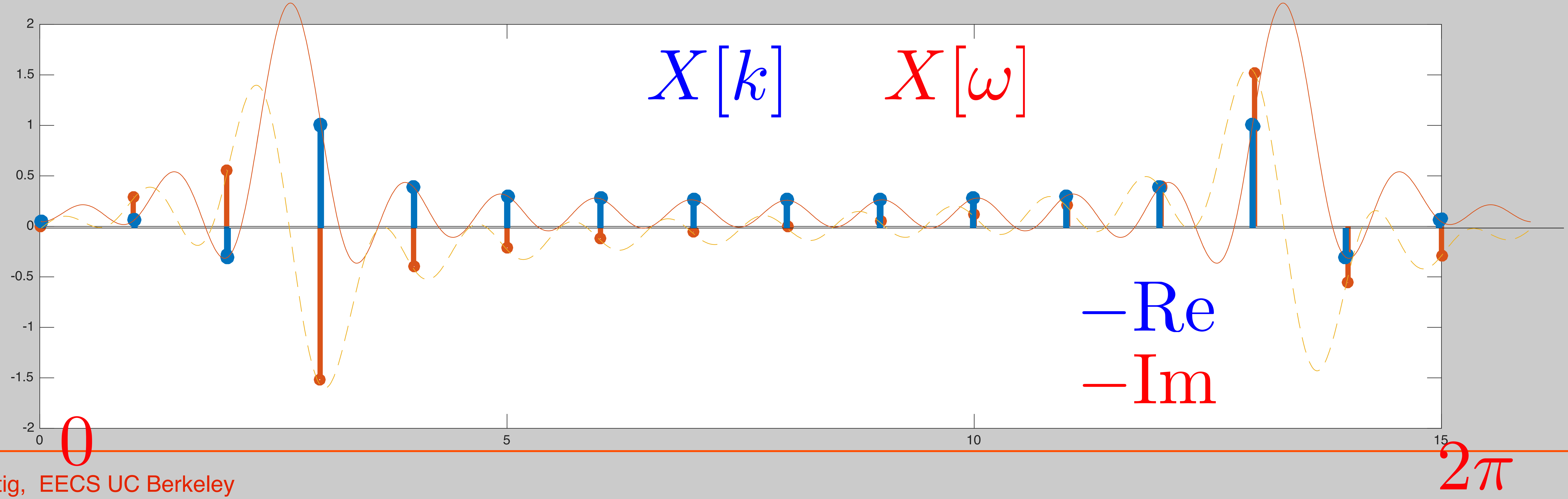


Example 2

What if there is no integer k to fit the frequency

$$\omega_k = \frac{2\pi k}{N}$$

$$x[n] = \cos(\frac{\pi}{3}n) + 0.1\cos(\frac{\pi}{6}n)$$



DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \; k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

DFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} \qquad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\vec{X} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^*}_{\triangleq F^*} \vec{x}$$

DFT

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

- DFT Analysis

$$\begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} - & \vec{u}_0^* & - \\ - & \vec{u}_1^* & - \\ & \vdots & \\ - & \vec{u}_{N-1}^* & - \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\vec{X} = F^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

DFT

- DFT Synthesis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = F\vec{X} \qquad = F(F^*\vec{x})$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{+nk}$$

Quiz

Compute a 2 point DFT of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix}$$

$\vec{u}_0 =$ $\vec{u}_1 =$

$\vec{u}_0^* \vec{x} =$ $\vec{u}_1^* \vec{x} =$

$\vec{X} =$

Example cont

- DFT_2 matrix:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

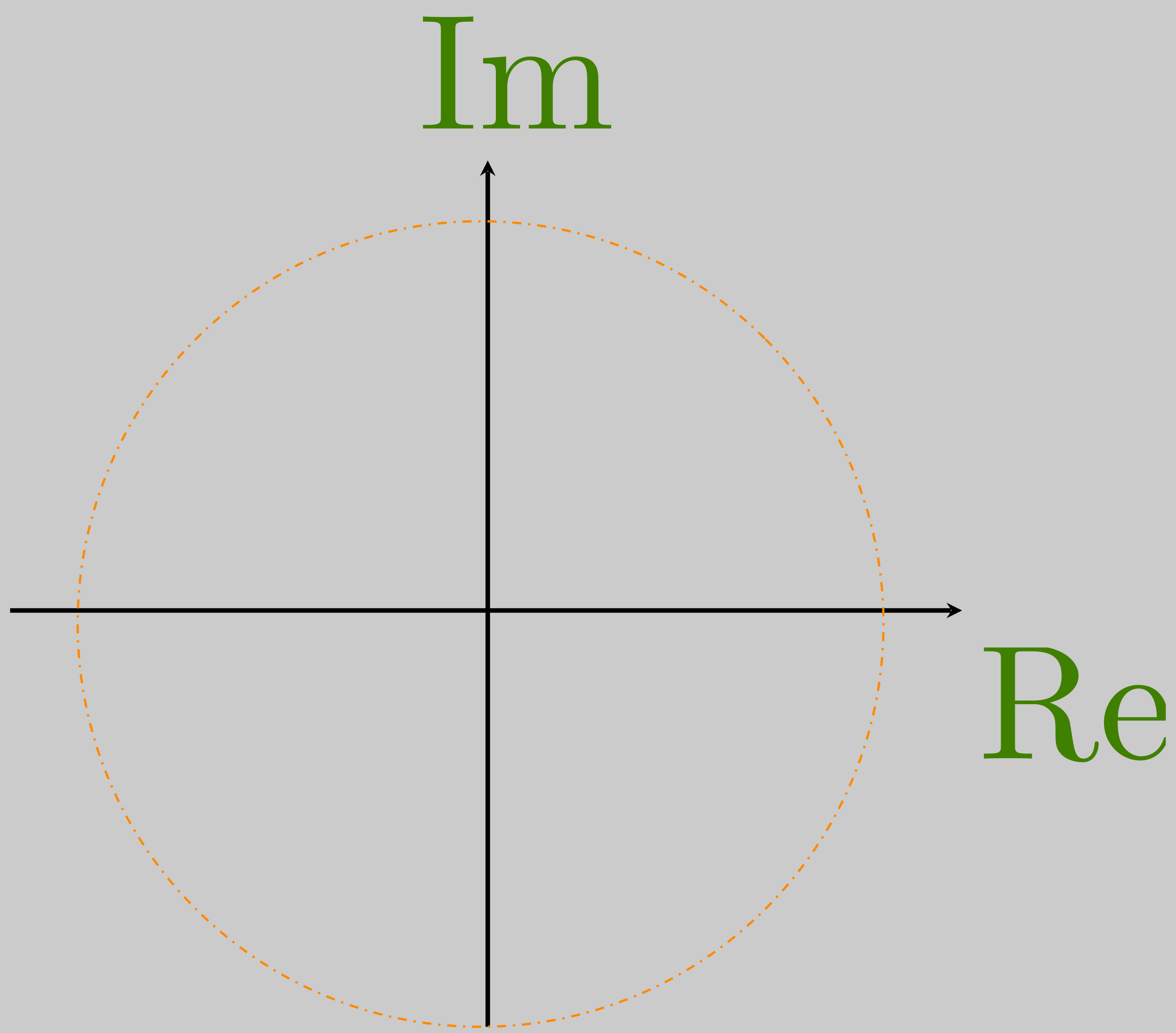
$$\vec{X} = F^* \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT_4 of: $\vec{X} = [1\ 1\ 1\ 1]^*$

$$\vec{x} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{W_4^{kn} \triangleq F} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT_4 of: $\vec{X} = [1\ 1\ 1\ 1]^*$

$$\vec{x} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT_4 of: $\vec{X} = [1\ 1\ 1\ 1]^*$

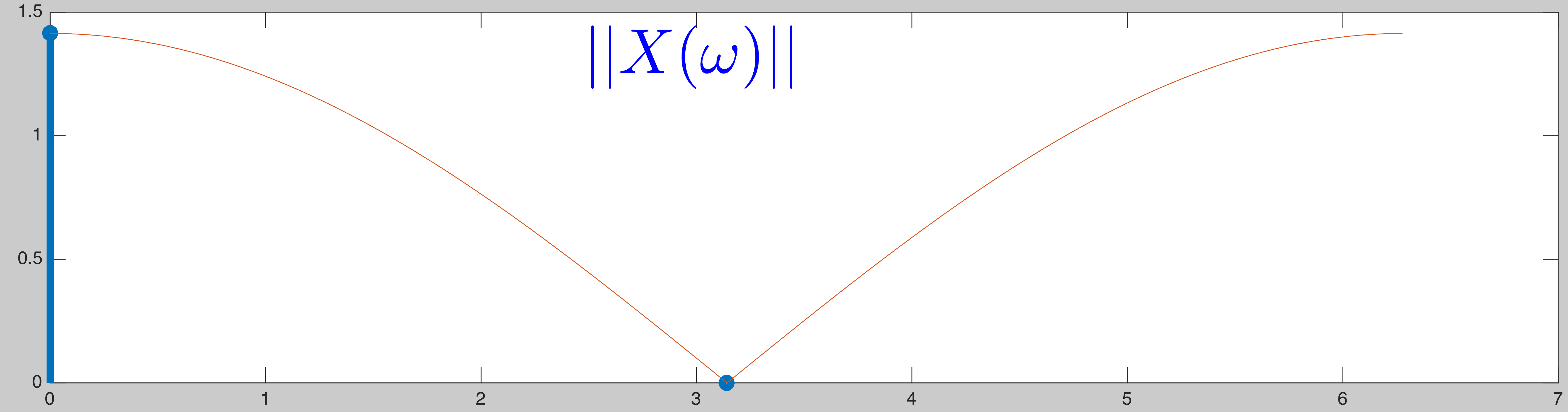
$$\vec{x} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Spectral Analysis with DFT

- Recall:

$$\vec{X} = F^*\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

$$\Rightarrow X(\omega) = \vec{u}_w^*\vec{x}$$



$$k \in [0, N-1] \qquad \omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$$

Zero-Padding For Frequency Analysis

- What does it mean to compute a DFT_4 of an N=2 sequence?
  - Assume sequence is zero elsewhere

Example: Compute DFT_4 of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

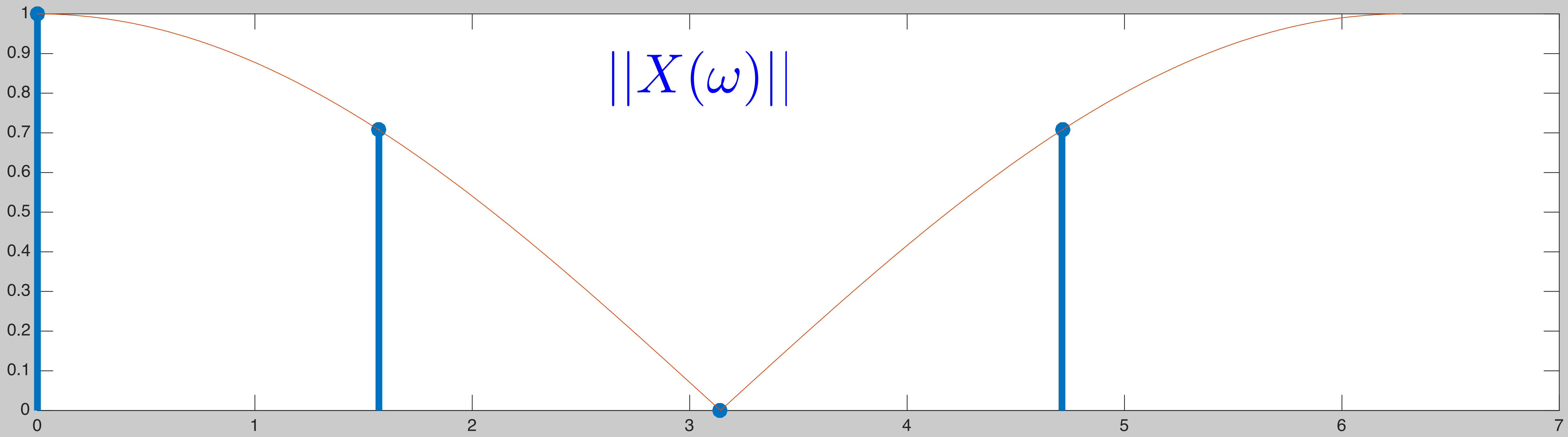
Zeropad:

$$\vec{X} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Zero-Padding For Frequency Analysis

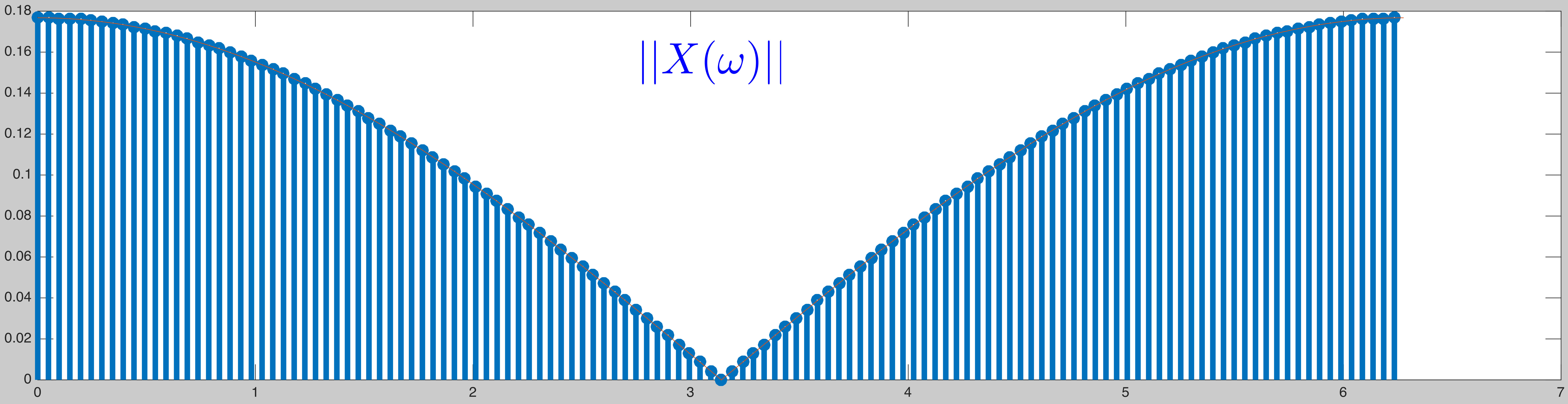
$$\vec{X} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cdots & W_4^{-n\cdot 0} & \cdots \\ \cdots & W_4^{-n\cdot 1} & \cdots \\ \cdots & W_4^{-n\cdot 2} & \cdots \\ \cdots & W_4^{-n\cdot 3} & \cdots \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -j \\ 1 & -1 \\ 1 & +j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & W_2^{-1\cdot 0} \\ 1 & W_2^{-1\cdot 0.5} \\ 1 & W_2^{-1\cdot 1} \\ 1 & W_2^{-1\cdot 1.5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Zeropadding

- Zero-pad to 128 – evaluate w at more points!



- Note that result should be scaled by $\frac{\sqrt{N_{\text{zp}}}}{\sqrt{N}}$