EE16B
Designing Information Devices and Systems II

Lecture 14B
Last Lecture

Intro

- Last time:
  - Change of basis
  - Frequency analysis through projections onto complex harmonics
  - Discrete Fourier Transform

- Today
  - Wrap up DFT
  - How MRI works

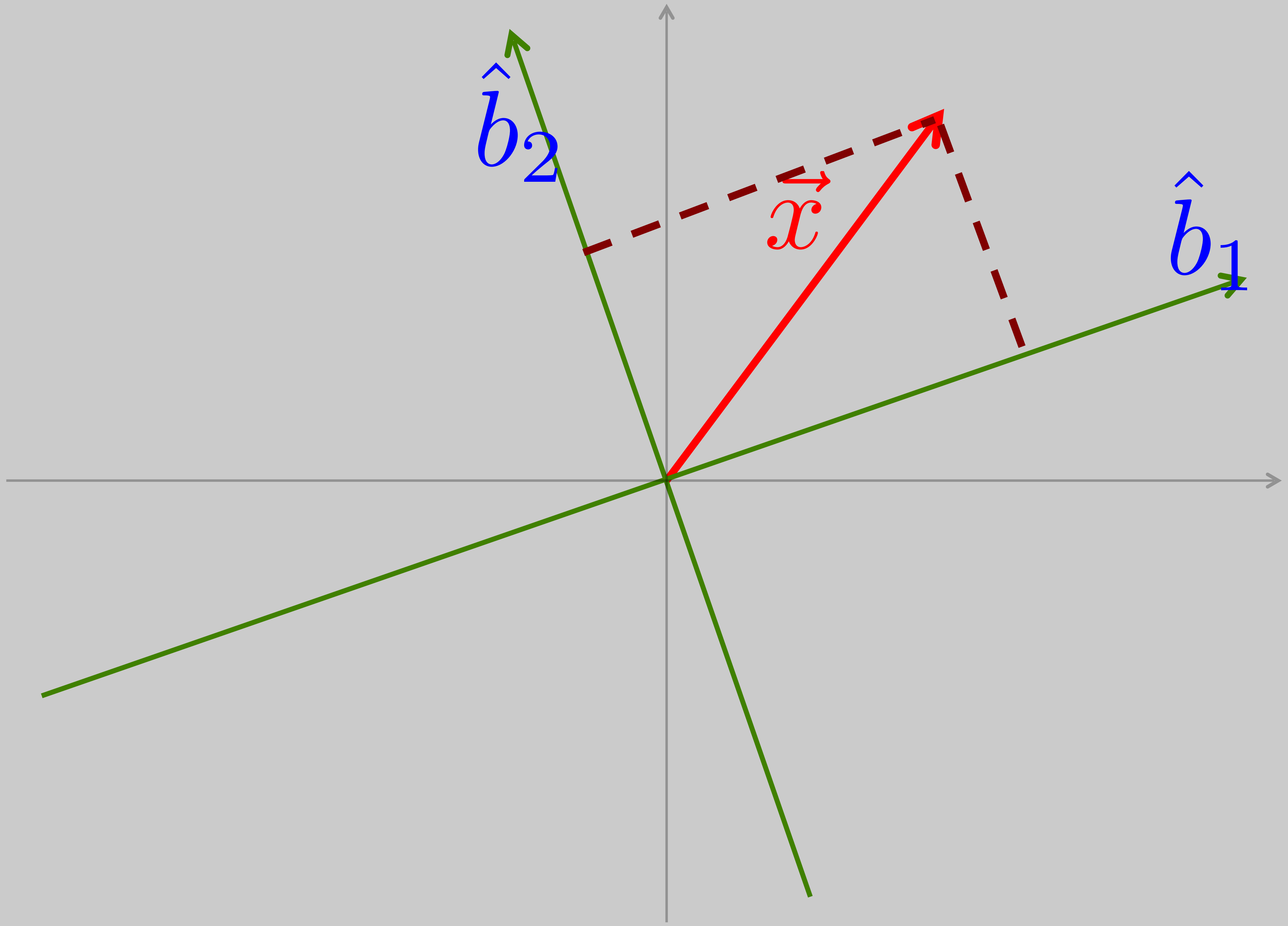
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

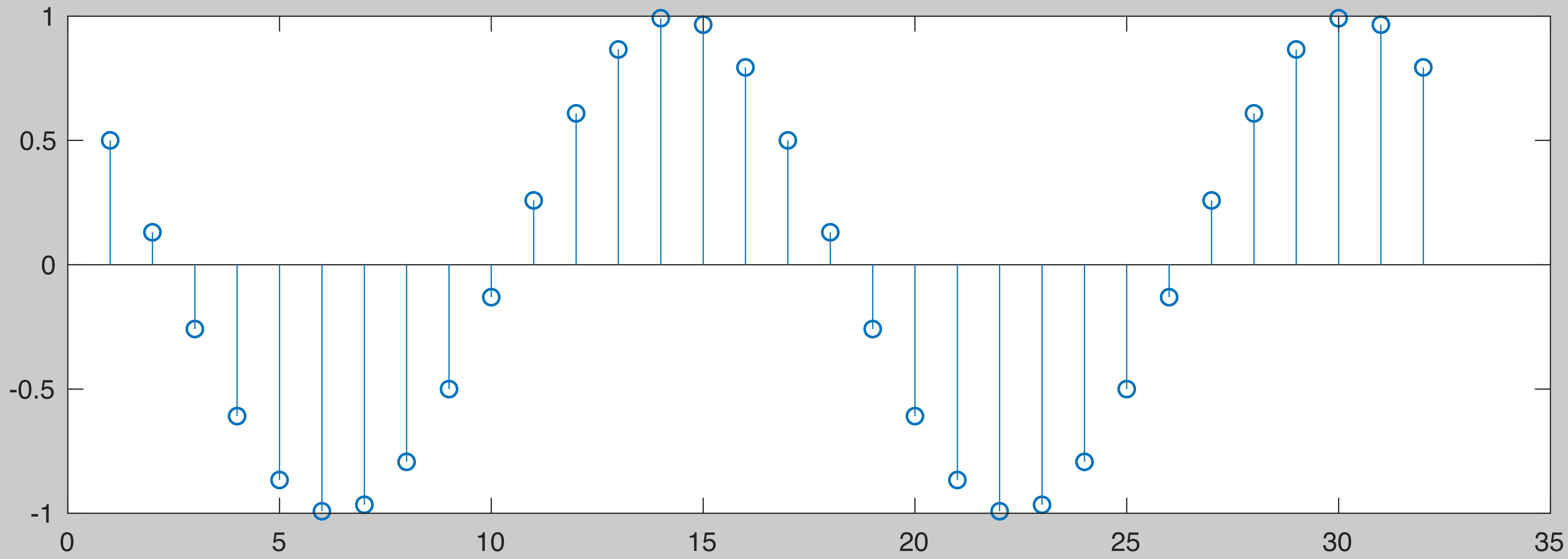
New coordinates:

$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b_1} & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



- How can we find the frequency of this N=32 length signal?



Project on unit sinusoidal vectors?

Frequency Analysis Through Projections

- N-length normalized discrete frequency:

$$u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n} \qquad 0 \leq n < N \qquad 0 \leq \omega < 2\pi$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix} \Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Also the DTFT of the finite sequence x

DFT vs DTFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix}$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X[k] = \vec{u}_k^* \vec{x}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

DFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \qquad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\vec{X} = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^*}_{\triangleq F^*} \vec{x}$$

DFT

- DFT Analysis

$$F = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} - & \vec{u}_0^* & - \\ - & \vec{u}_1^* & - \\ & \vdots & \\ - & \vec{u}_{N-1}^* & - \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\vec{X} = F^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

DFT

- DFT Synthesis

$$F = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = F\vec{X} \qquad = F(F^*\vec{x})$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{+nk}$$

Quiz

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix}$$

Compute a 2 point DFT of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{u}_0 =$

$\vec{u}_1 =$

$\vec{u}_0^* \vec{x} =$

$\vec{u}_1^* \vec{x} =$

$\vec{X} =$

Example cont

- DFT$_2$ matrix:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

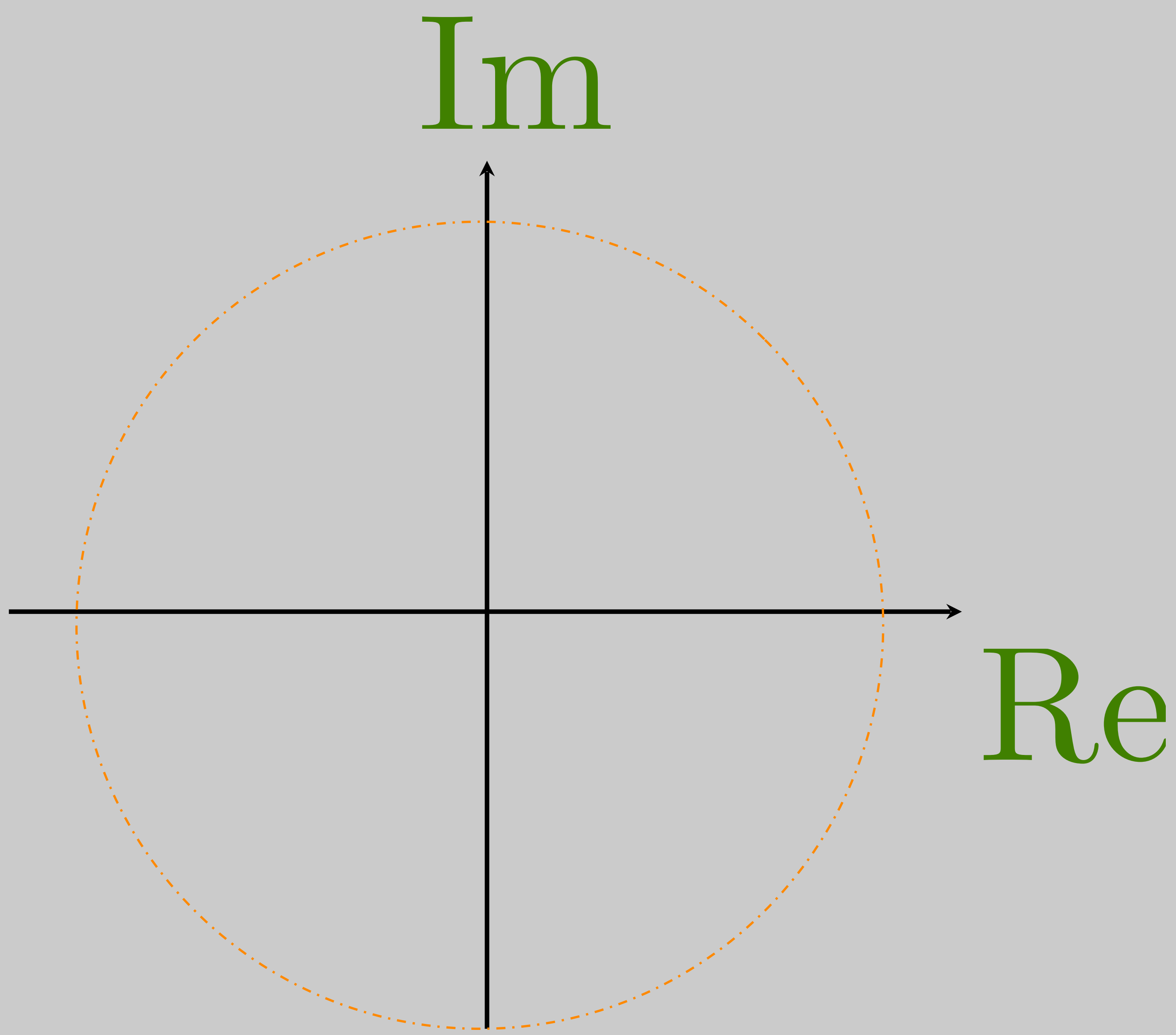
$$\vec{X} = F^* \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Example

$$\vec{u}_k = \frac{1}{\sqrt{N}}\begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}}\begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT$_4$ of: $\vec{X} = [1\ 1\ 1\ 1]^*$

$$\vec{x} = \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{W_4^{kn} \triangleq F} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT_4 of: $\vec{X} = [1\ 1\ 1\ 1]^*$

$$\vec{x} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT$_4$ of: $\vec{X} = [1\ 1\ 1\ 1]^*$

$$\vec{x} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Complexity computing thf DFT

- What's the complexity to compute the DFT?

$$\vec{X} = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^* \vec{x}$$

$$\triangleq F^*$$

A: Generally $O(N^2)$

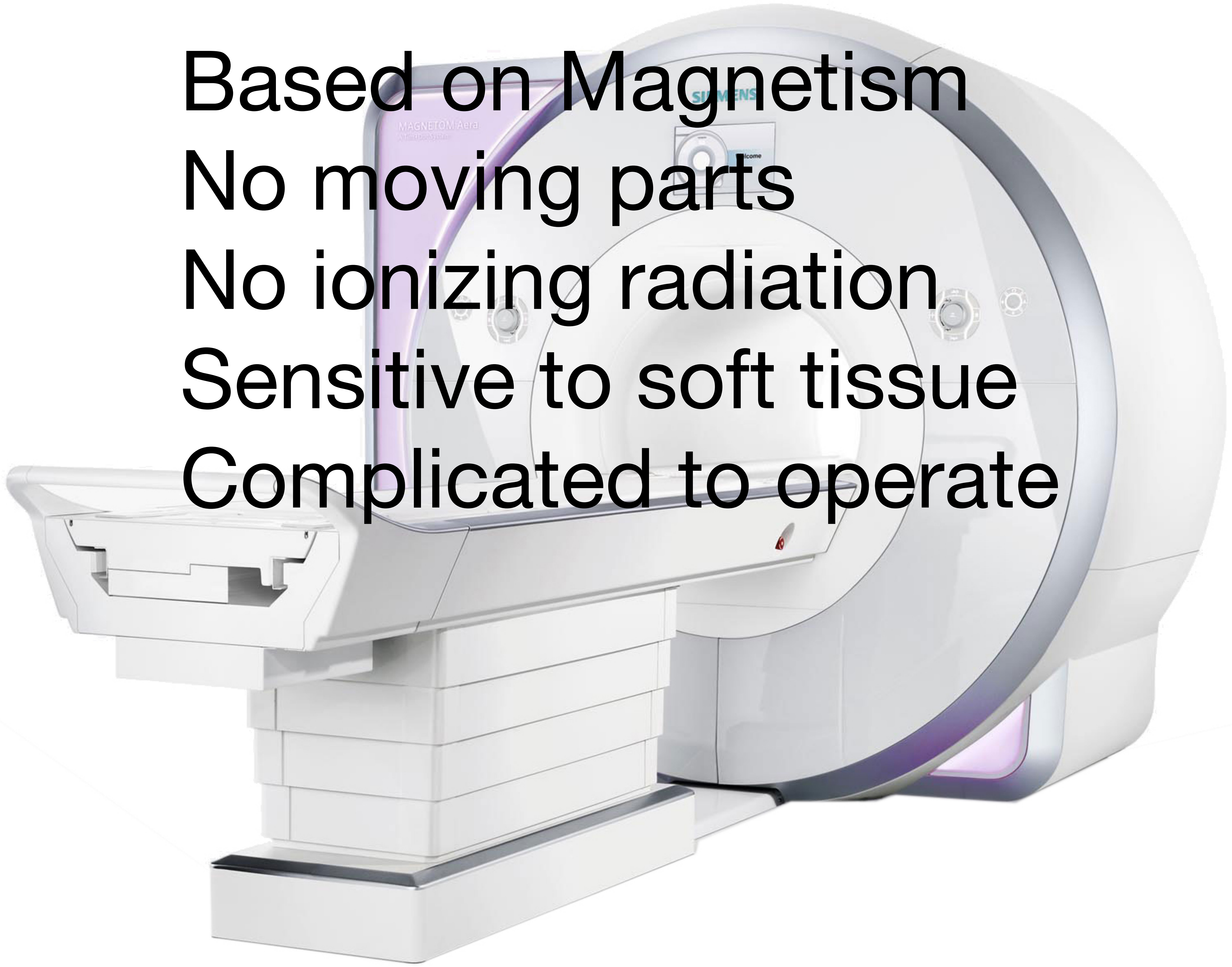
- Exploit structure in W_N^{nk} to speed up!
  - The Fast Fourier transform (FFT)

A: $O(N \log(N))$

MRI vs CT

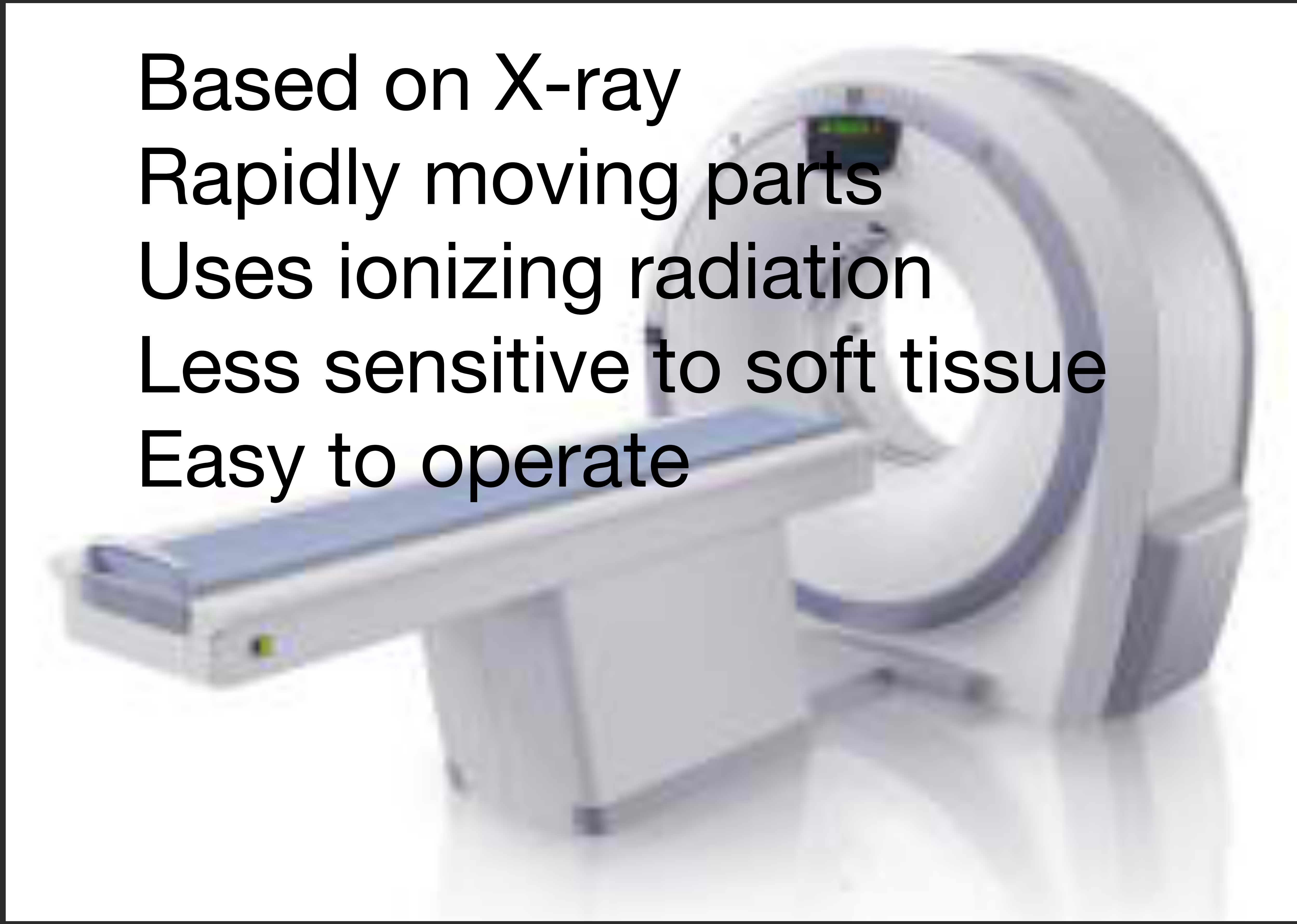
- MRI is VERY VERY different from CT

CT



Based on Magnetism
No moving parts
No ionizing radiation
Sensitive to soft tissue
Complicated to operate

MRI



Based on X-ray
Rapidly moving parts
Uses ionizing radiation
Less sensitive to soft tissue
Easy to operate

How Does MRI Work?

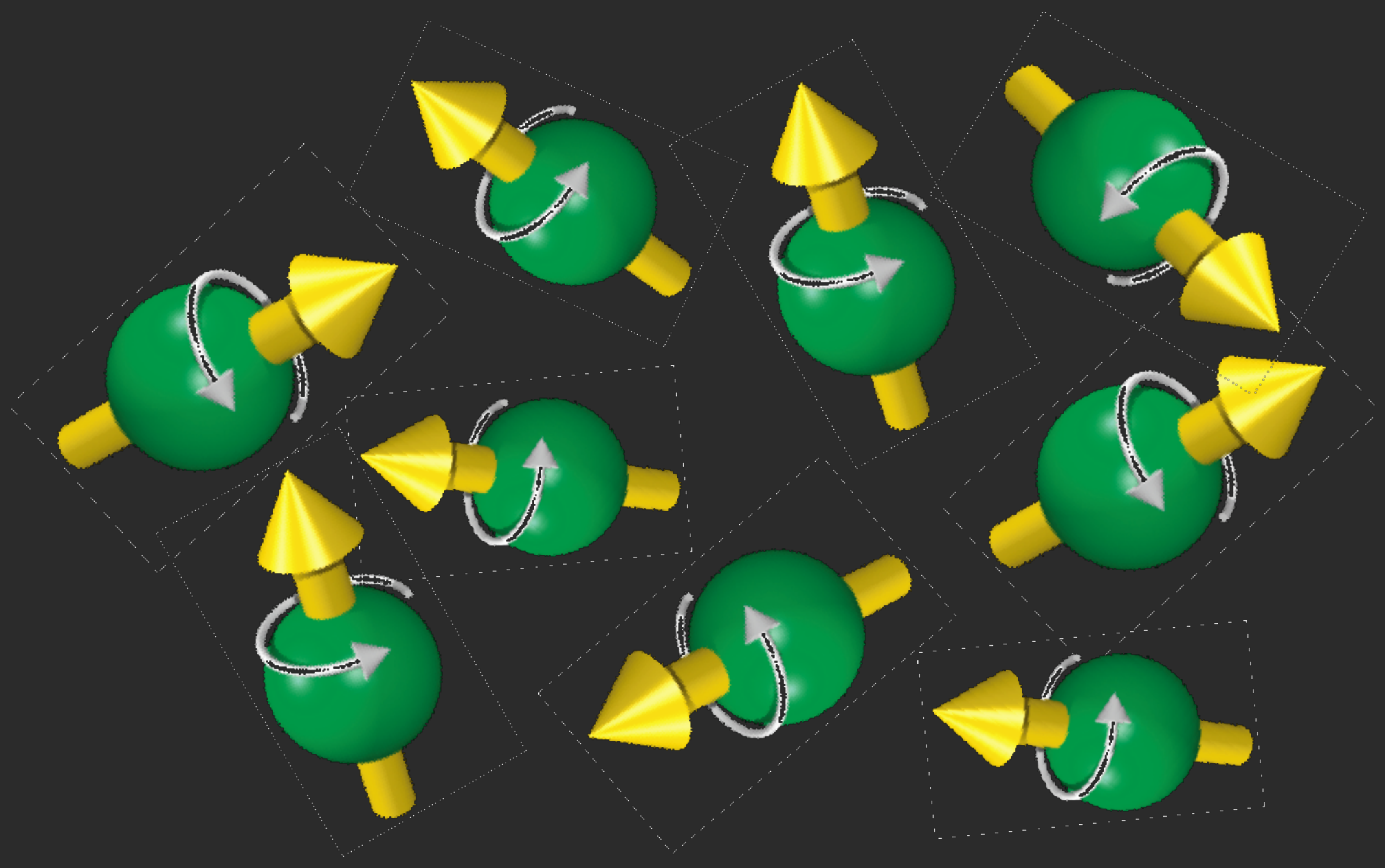
- Magnetic Polarization
  -- Very strong uniform magnet
- Excitation
  -- Very powerful RF transmitter
- Acquisition
  -- Location is encoded by gradient magnetic fields
  -- Very powerful audio amps

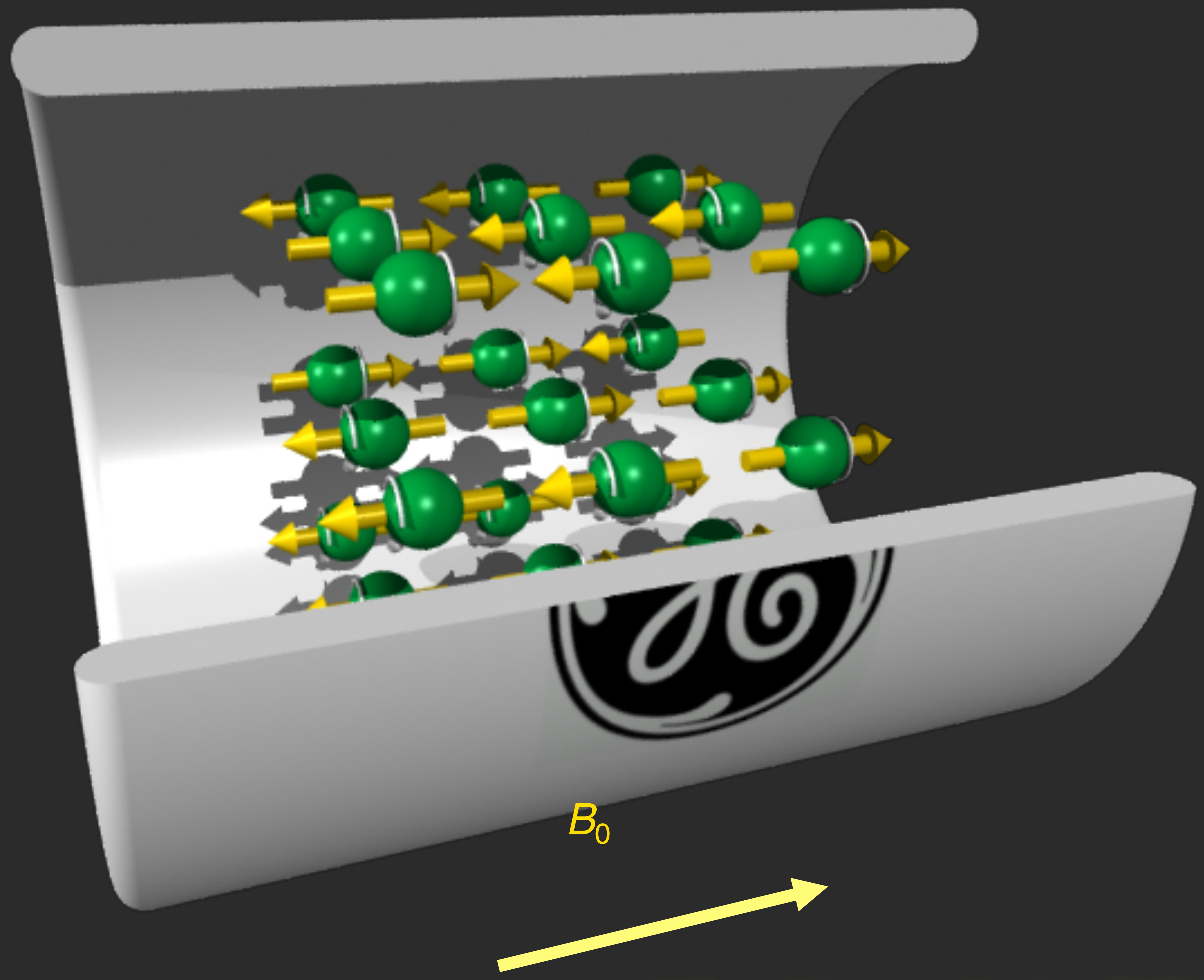
Polarization

- Protons have a magnetic moment
- Protons have spins
- Like rotating magnets

Polarization

- Body has a lot of protons
- In a strong magnetic field B0, spins align with B0 giving a net magnetization

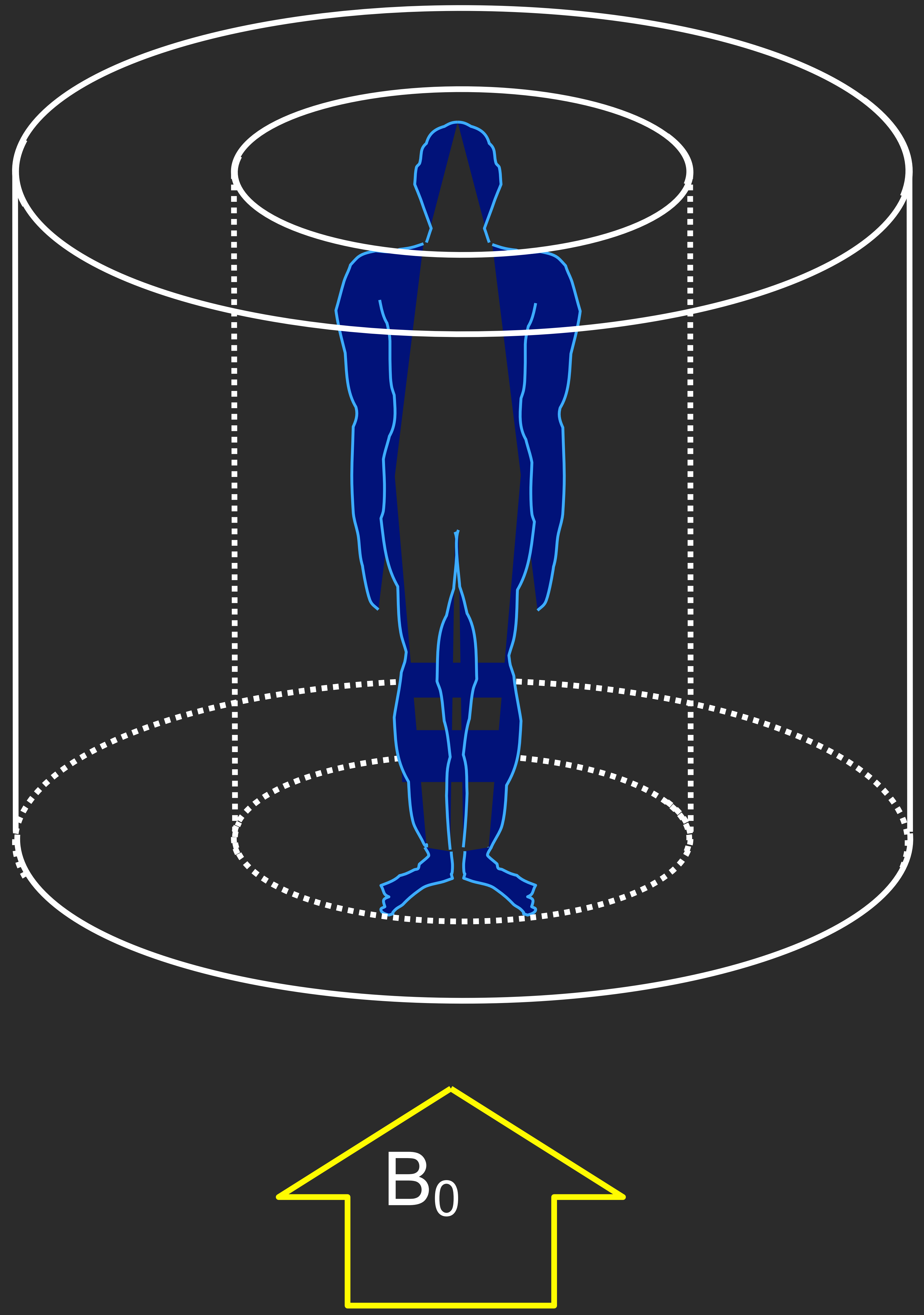




*Graphic rendering Bill Overall

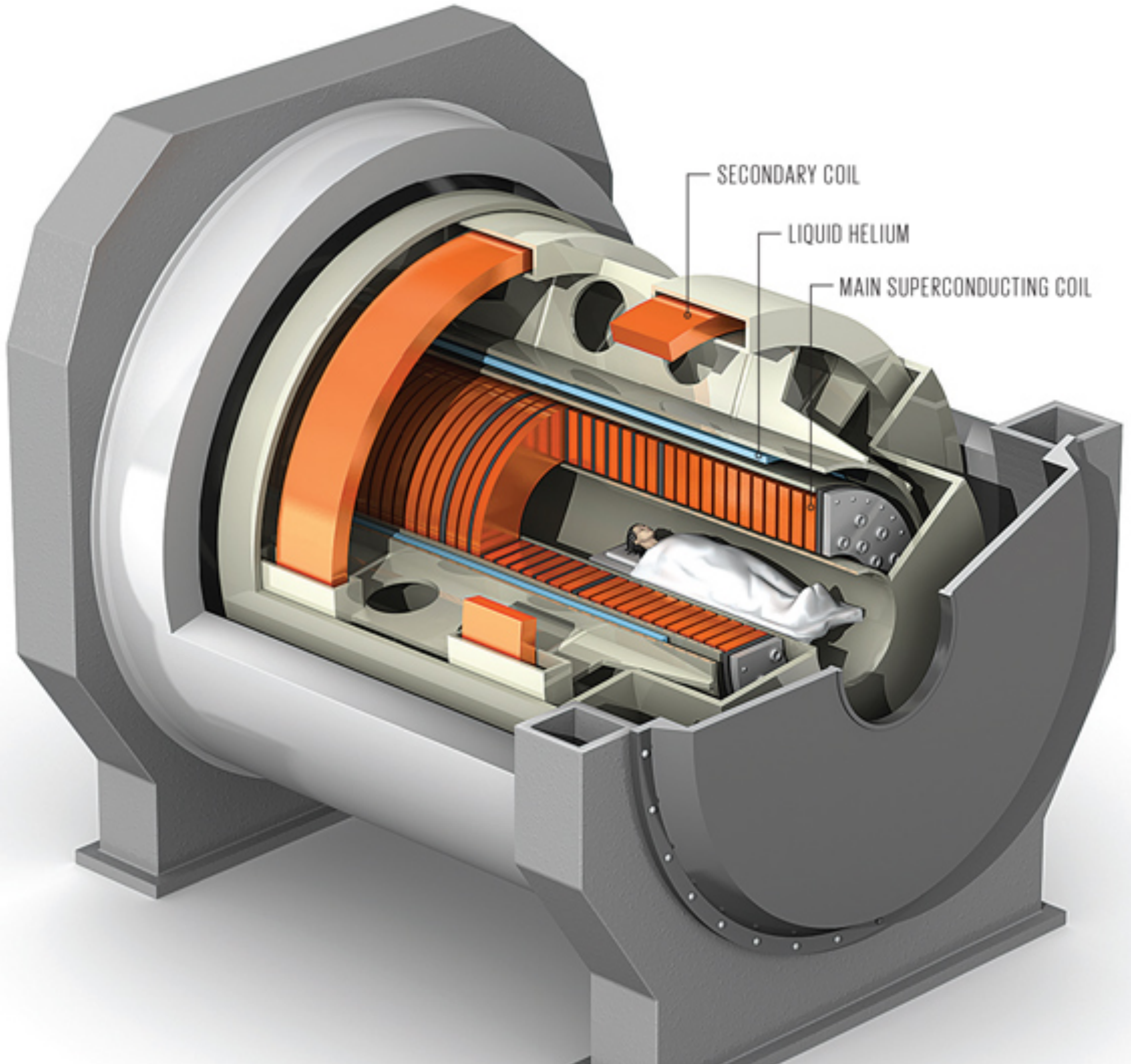
Polarizing Magnet

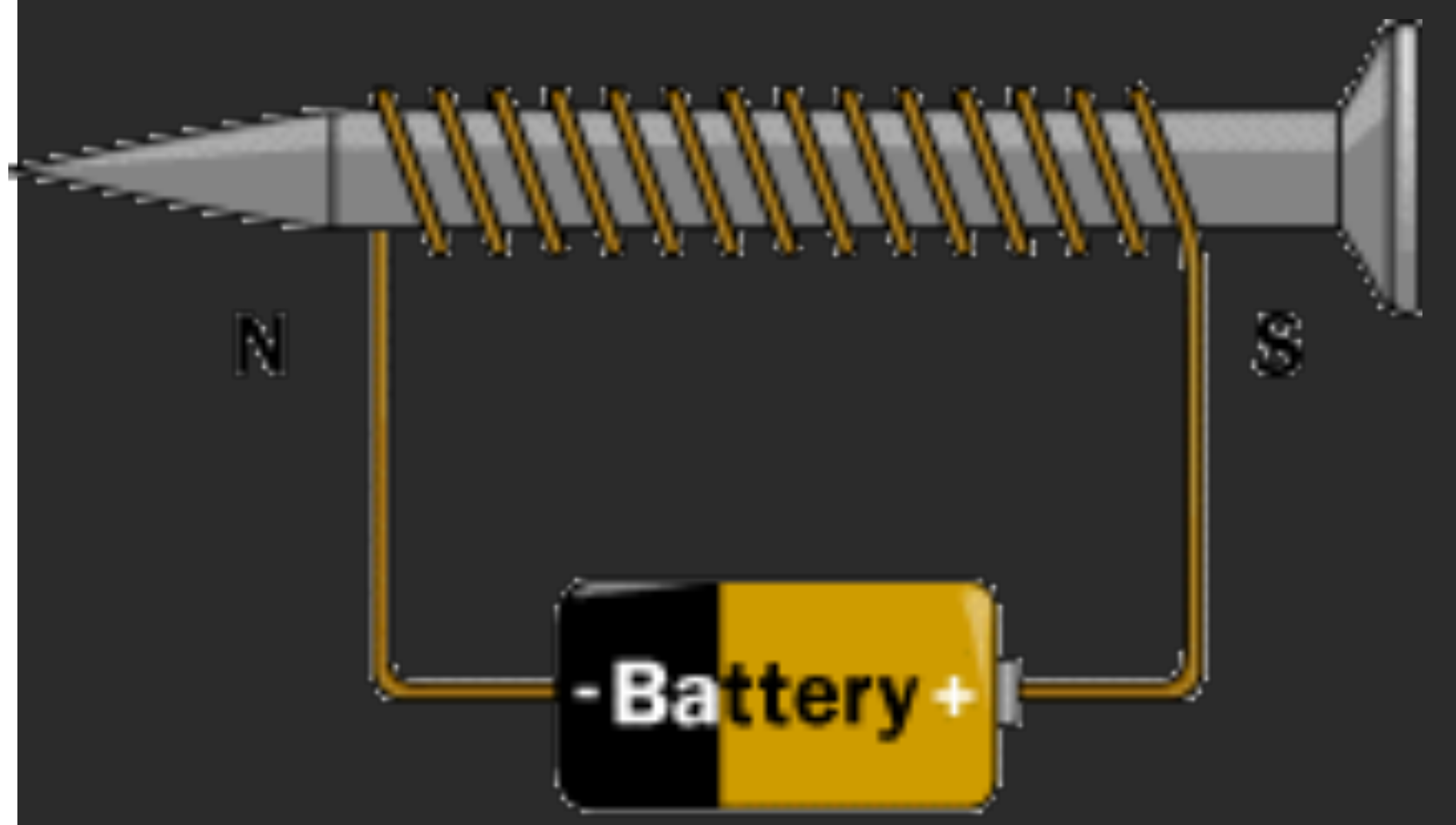
- 0.1 to 12 Tesla
- 0.5 to 3 T common
- 1 T is 10,000 Gauss
- Earth's field is 0.5G
- Typically a superconducting magnet



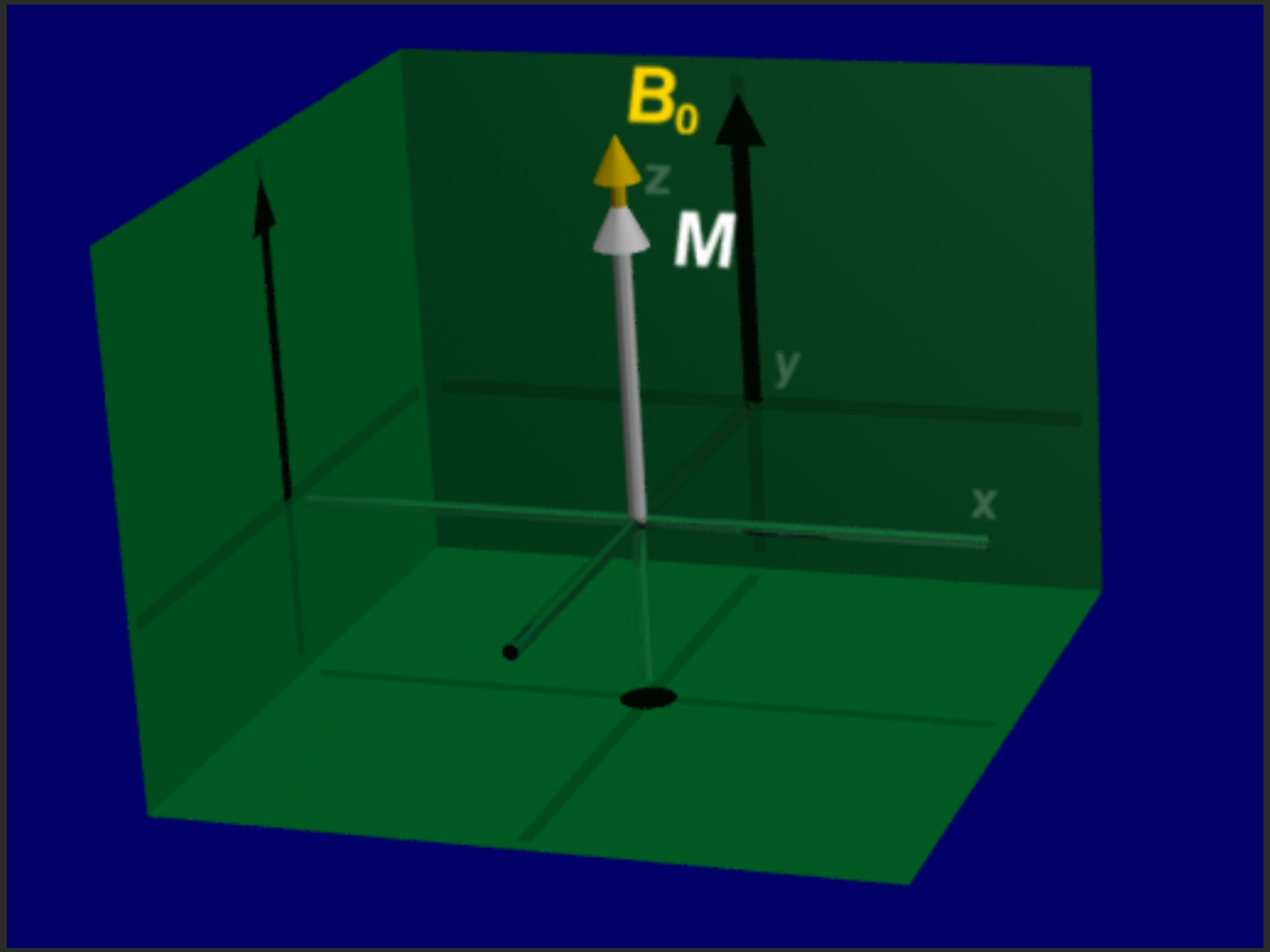
Typical MRI Scanner

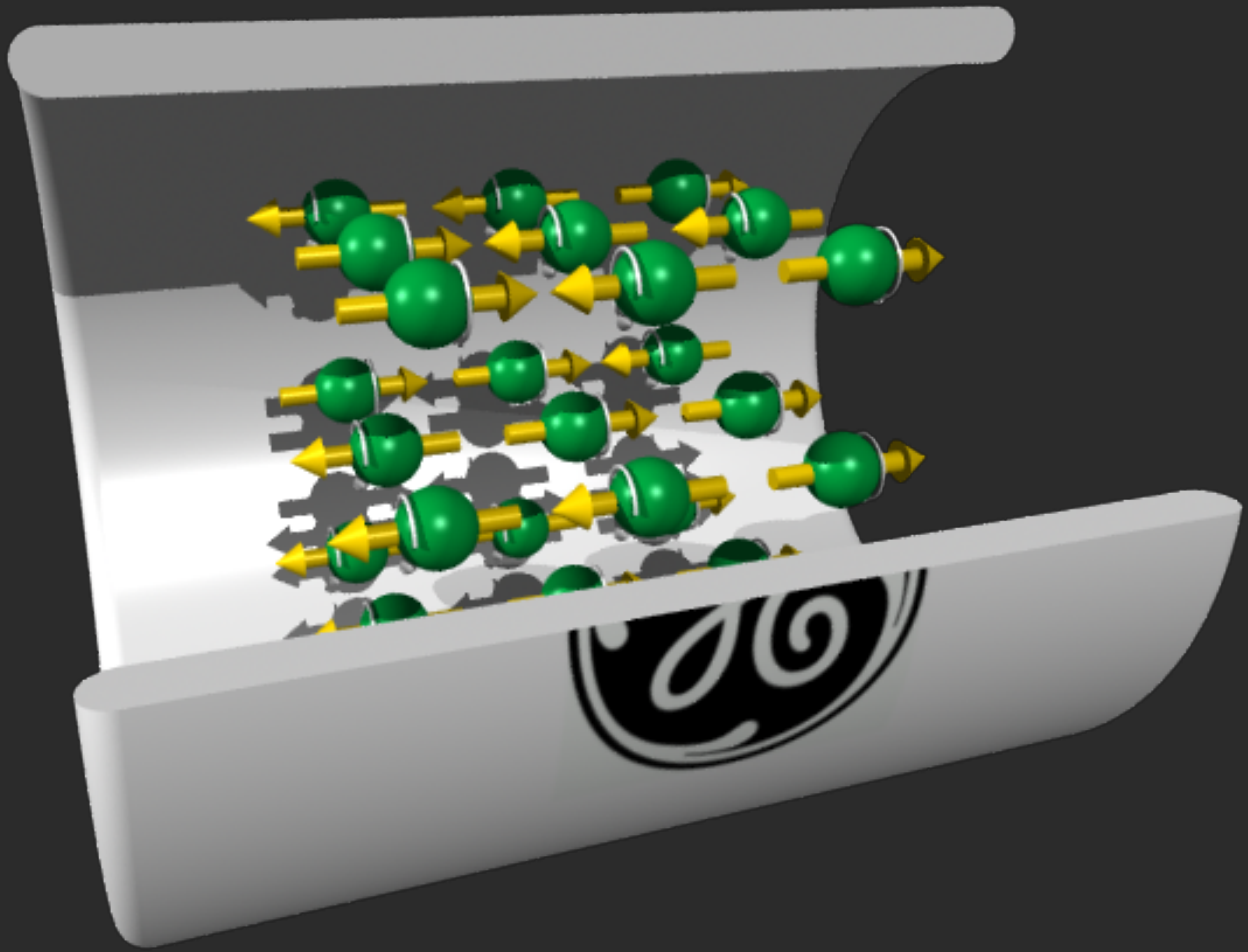

A magnetic object


SECONDARY COIL
LIQUID HELIUM
MAIN SUPERCONDUCTING COIL


N
S
-Battery+

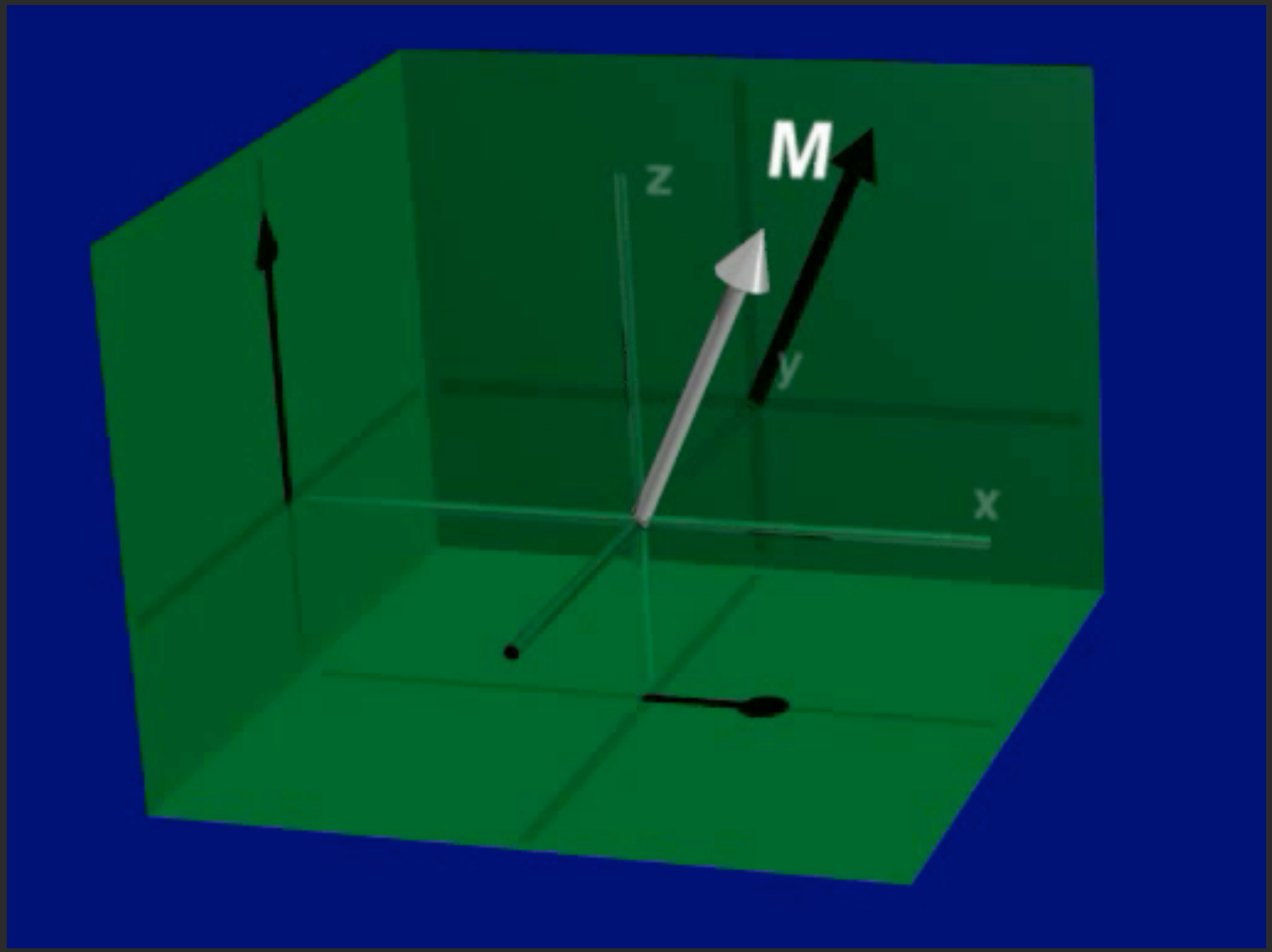
Polarizaion





Free Precession

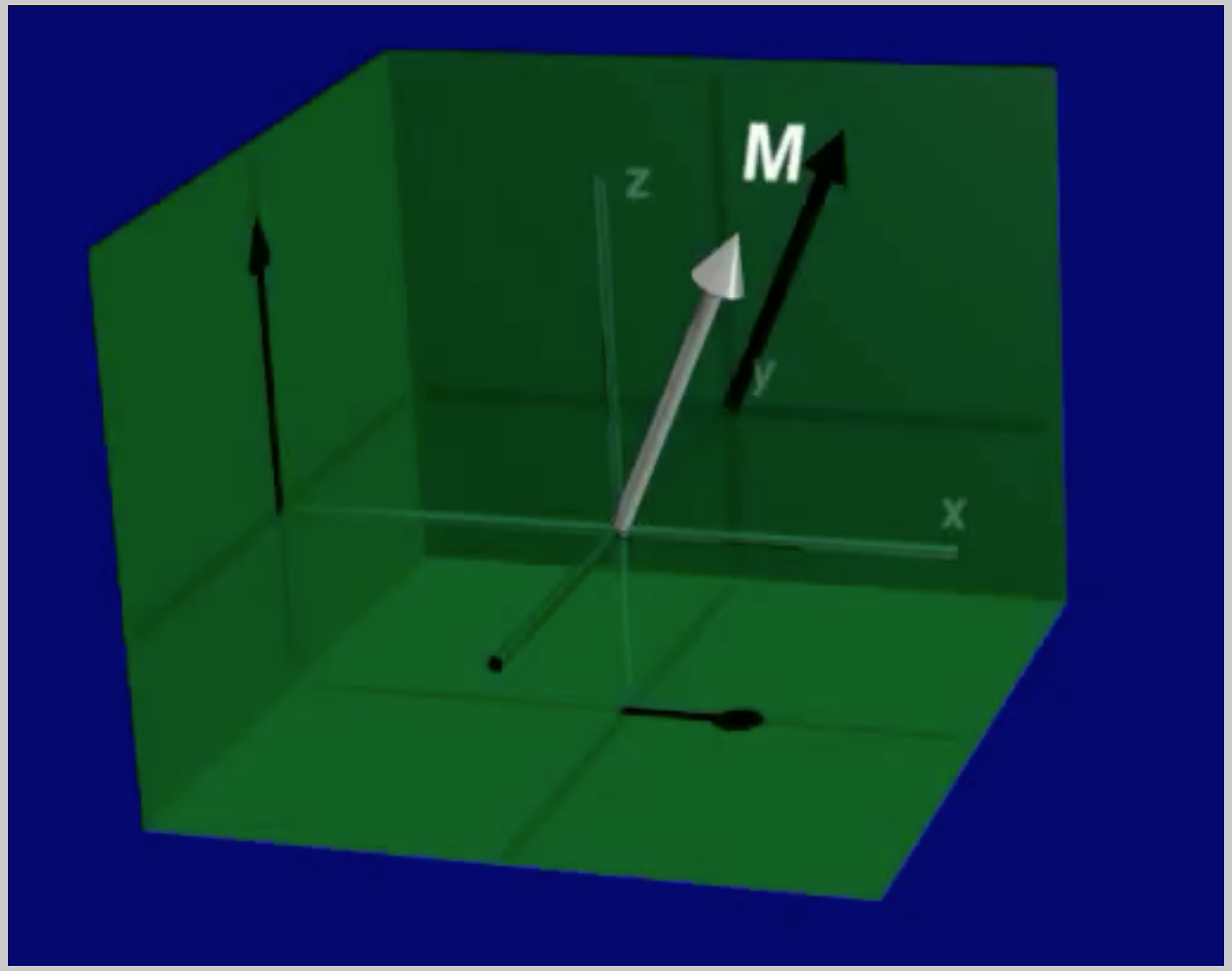
- Much like a spinning top
- Frequency proportional to the field
- f = 127MhZ @ 3T



MIT physics demos

Free precession

$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} 0 & \gamma B_0 & 0 \\ -\gamma B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix}$$

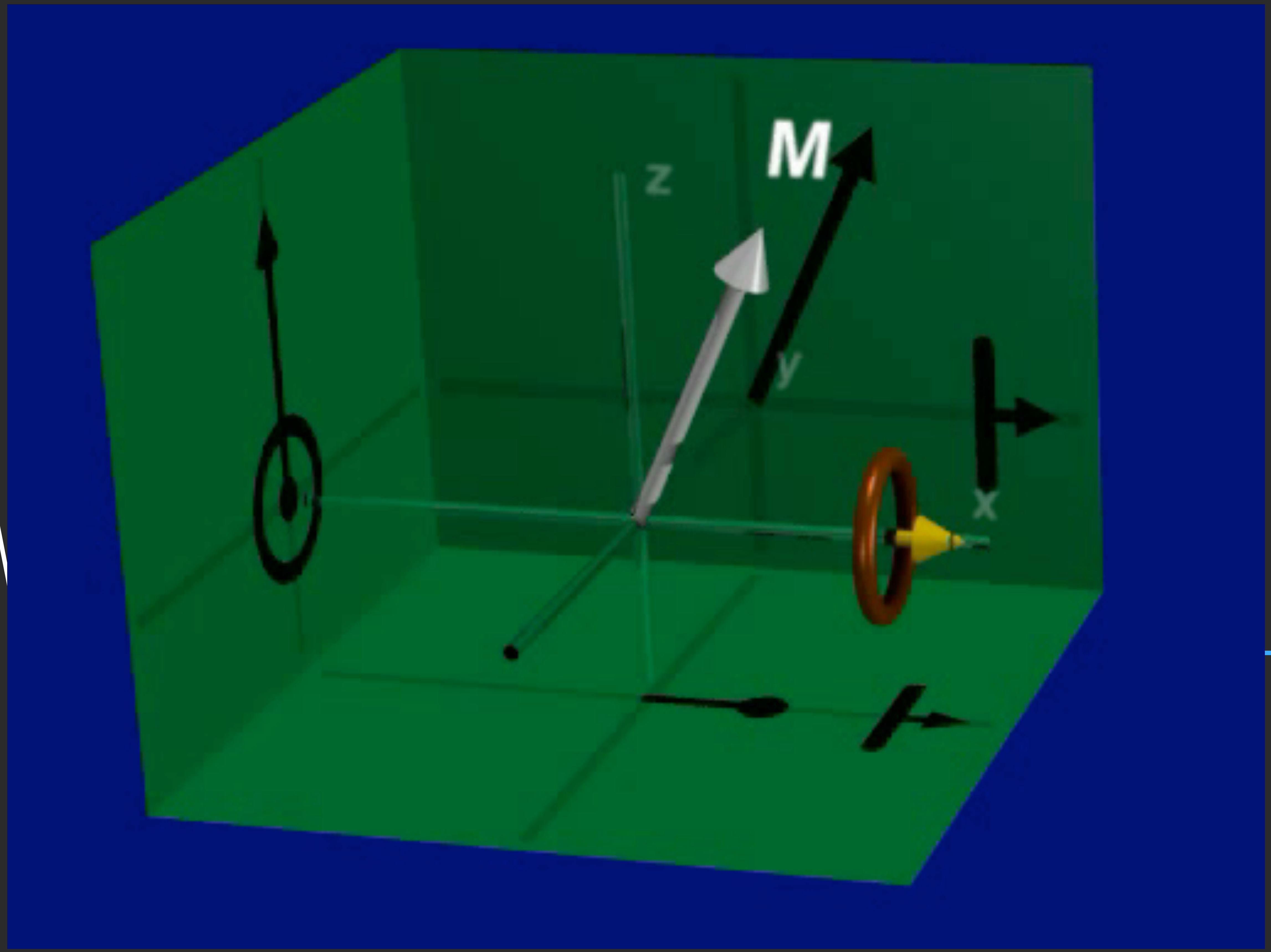


Free Precession

- Precession induces magnetic flux
- Flux induces voltage in a coil

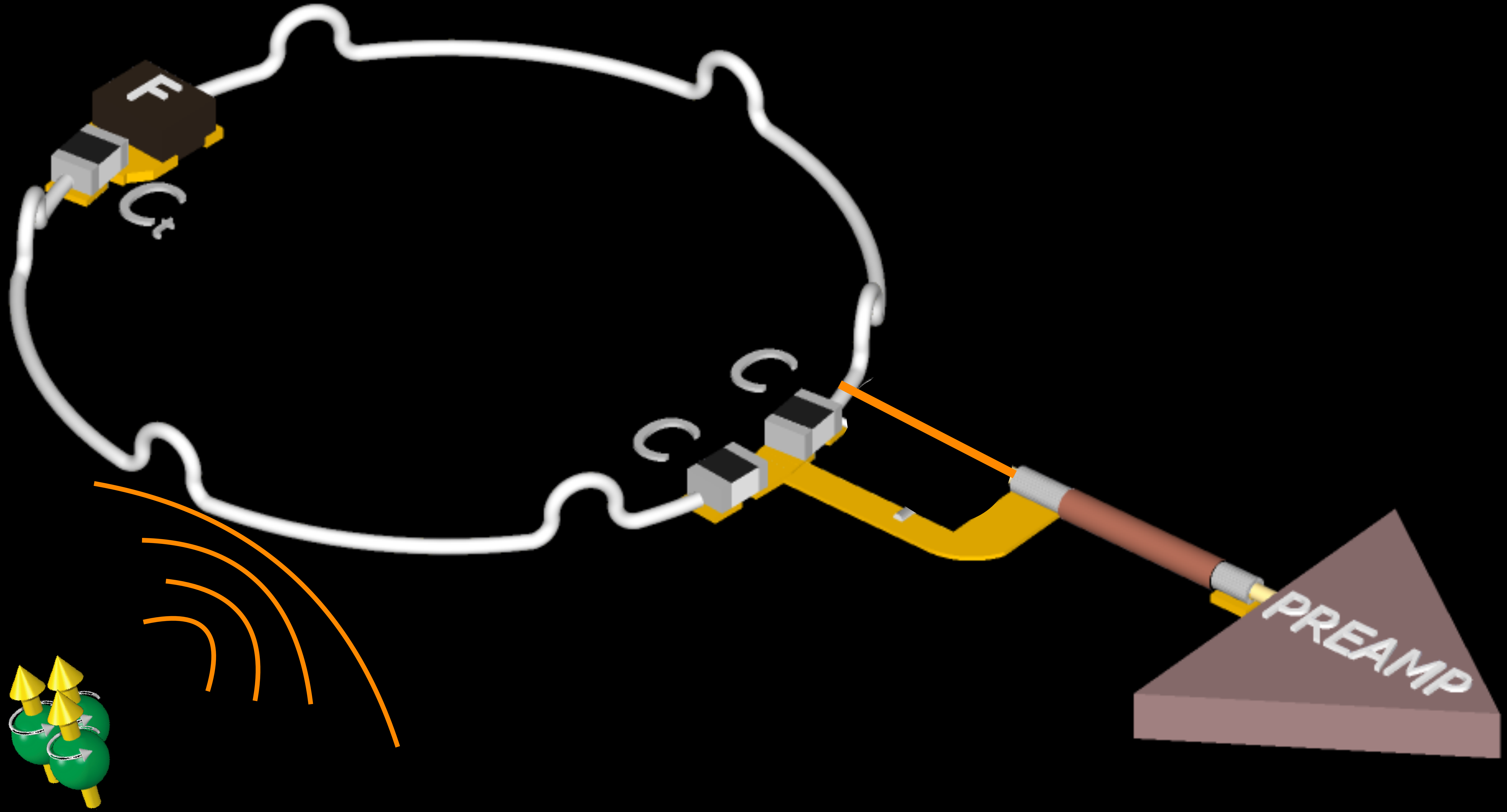
$$\vec{y}(t) = \int_{\vec{R}} C M(\vec{r}, t) d\vec{r}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Signal

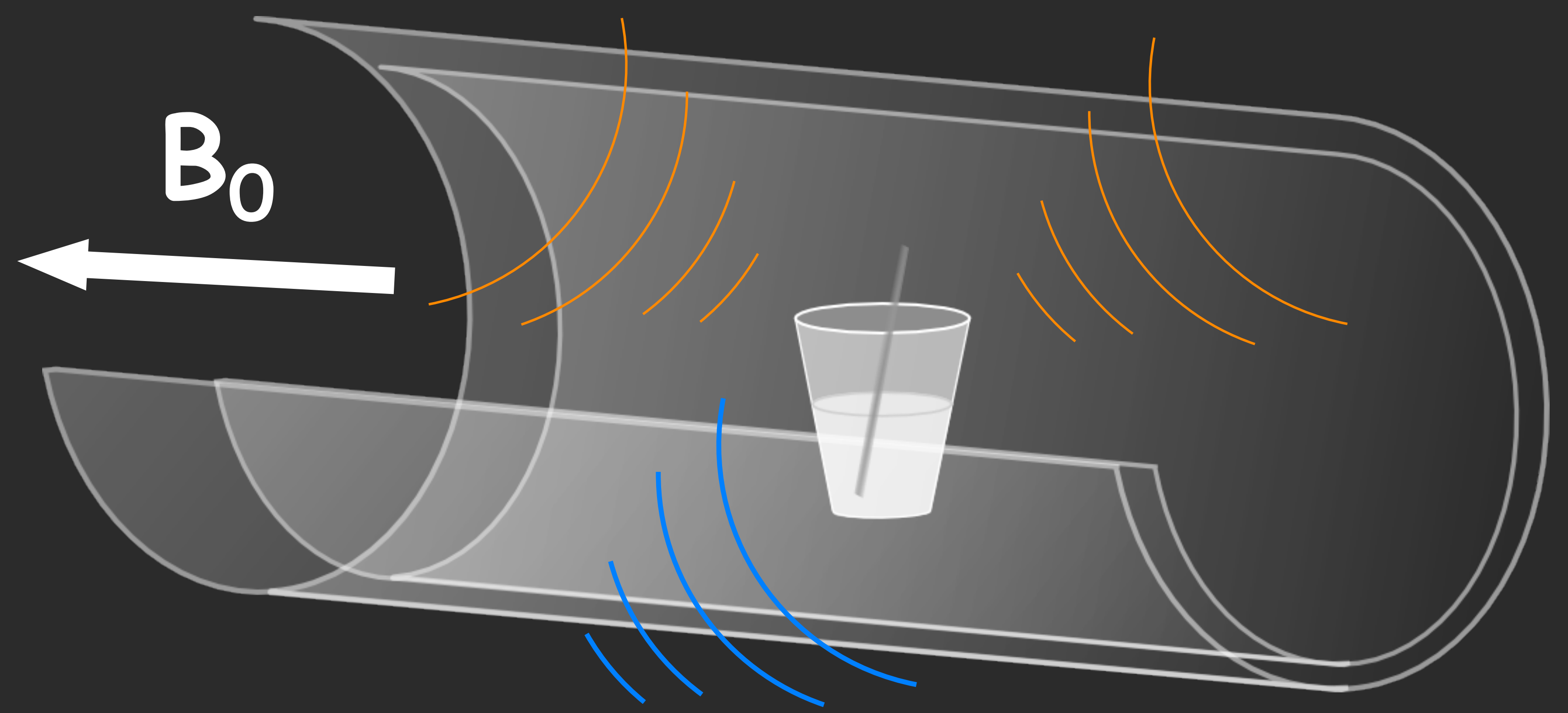

Audacity
Default Input Source
-21 0
-21 0
15.0 16.0 17.0 18.0 19.0 20.0 21.0 22.0 23.0 24.0 25.0 26.0 27.0 28.0
Audio Trac
tereo, 8192Hz
2-bit float
Mute
Solo
0.05
0.00
0.05
0.00
isk space remains for recording 761 hours and 21 minutes
oject rate: 8192
Cursor: 0:00.000000 min:sec [Snap-To Off]

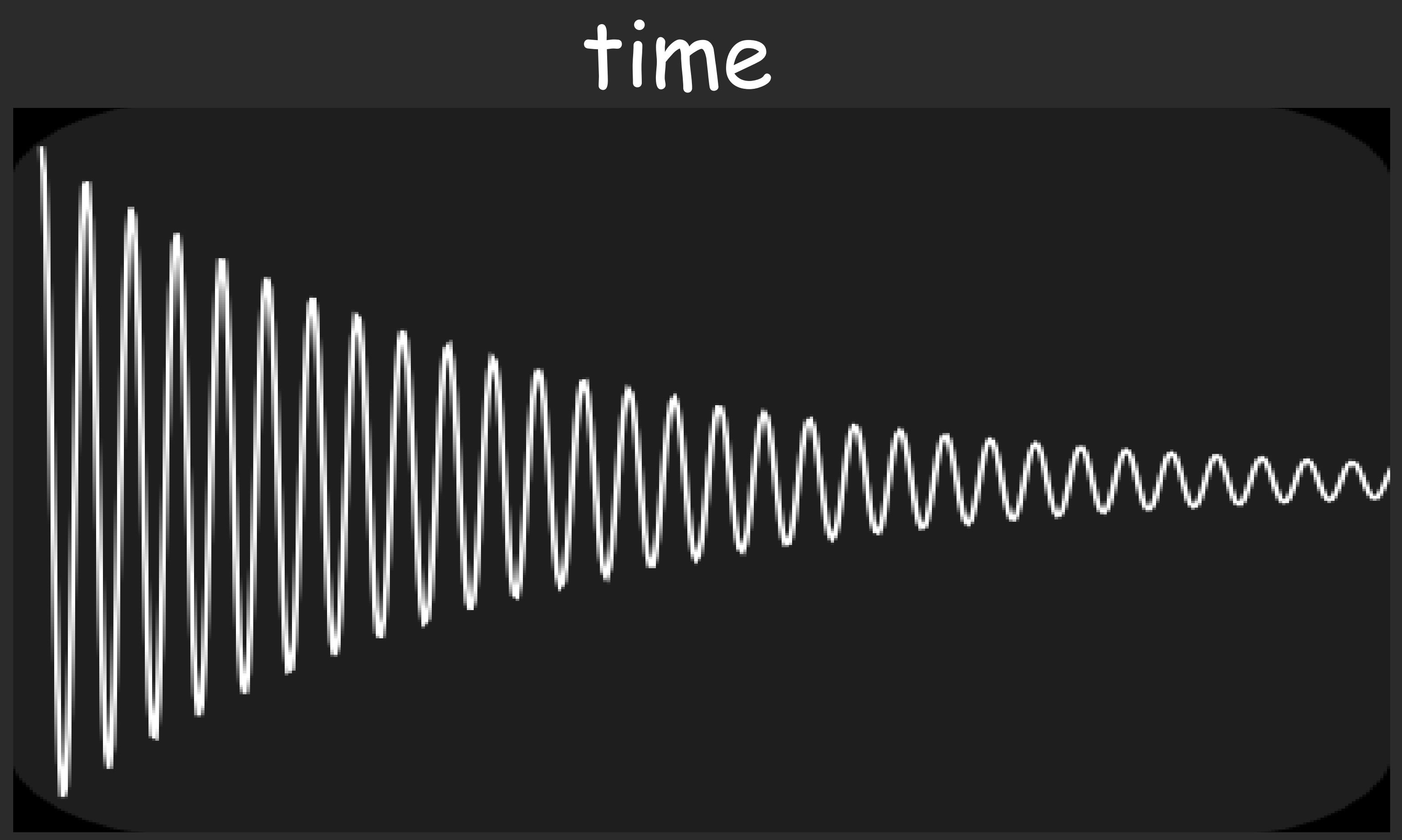


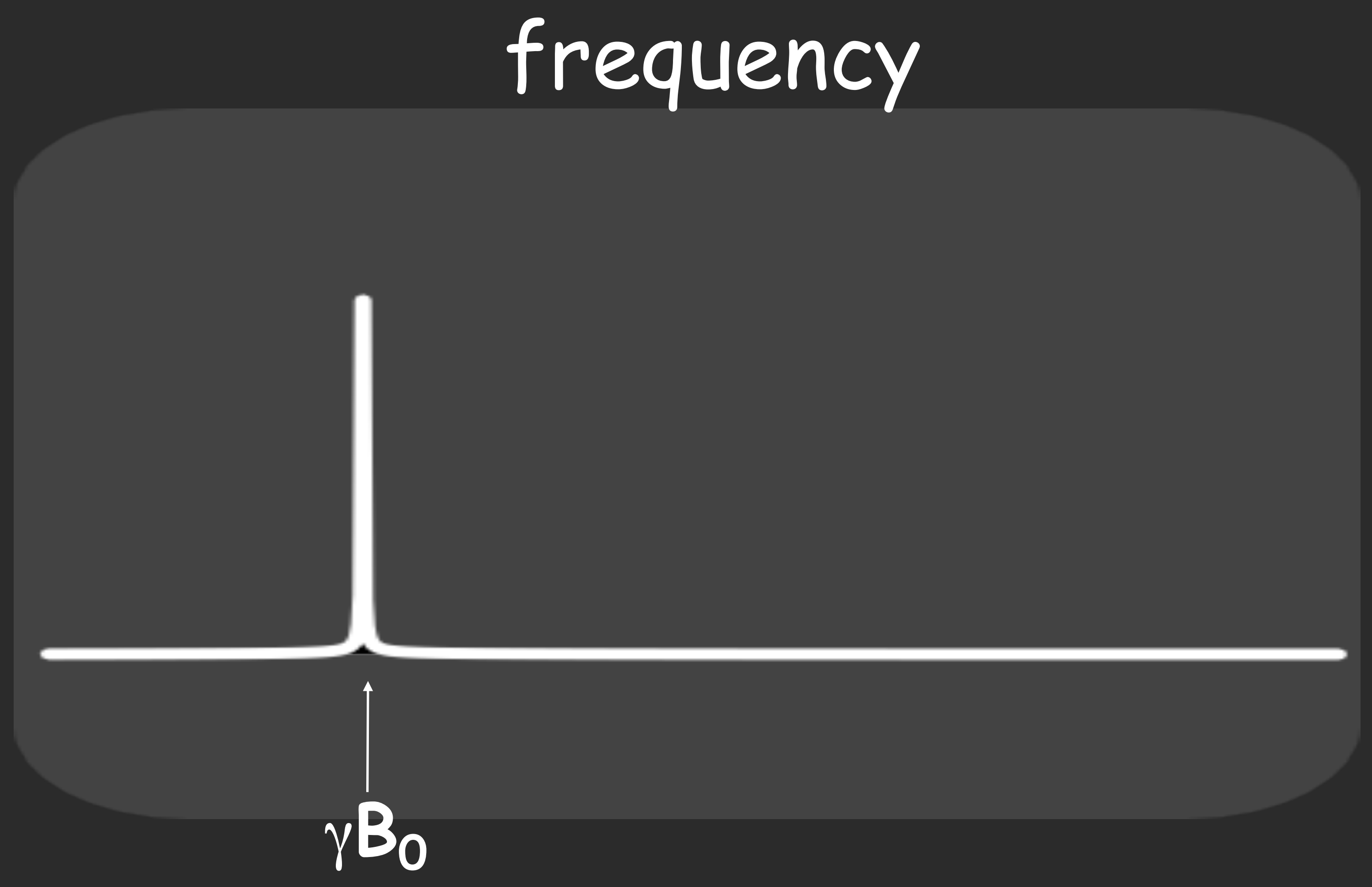
courtesy Boris Keil, Larry Wald, MGH

Intro to MRI - The NMR signal

- Signal from ^1H (mostly water)
- Magnetic field ⇒ Magnetization
- Radio frequency ⇒ Excitation
- Frequency ∝ Magnetic field







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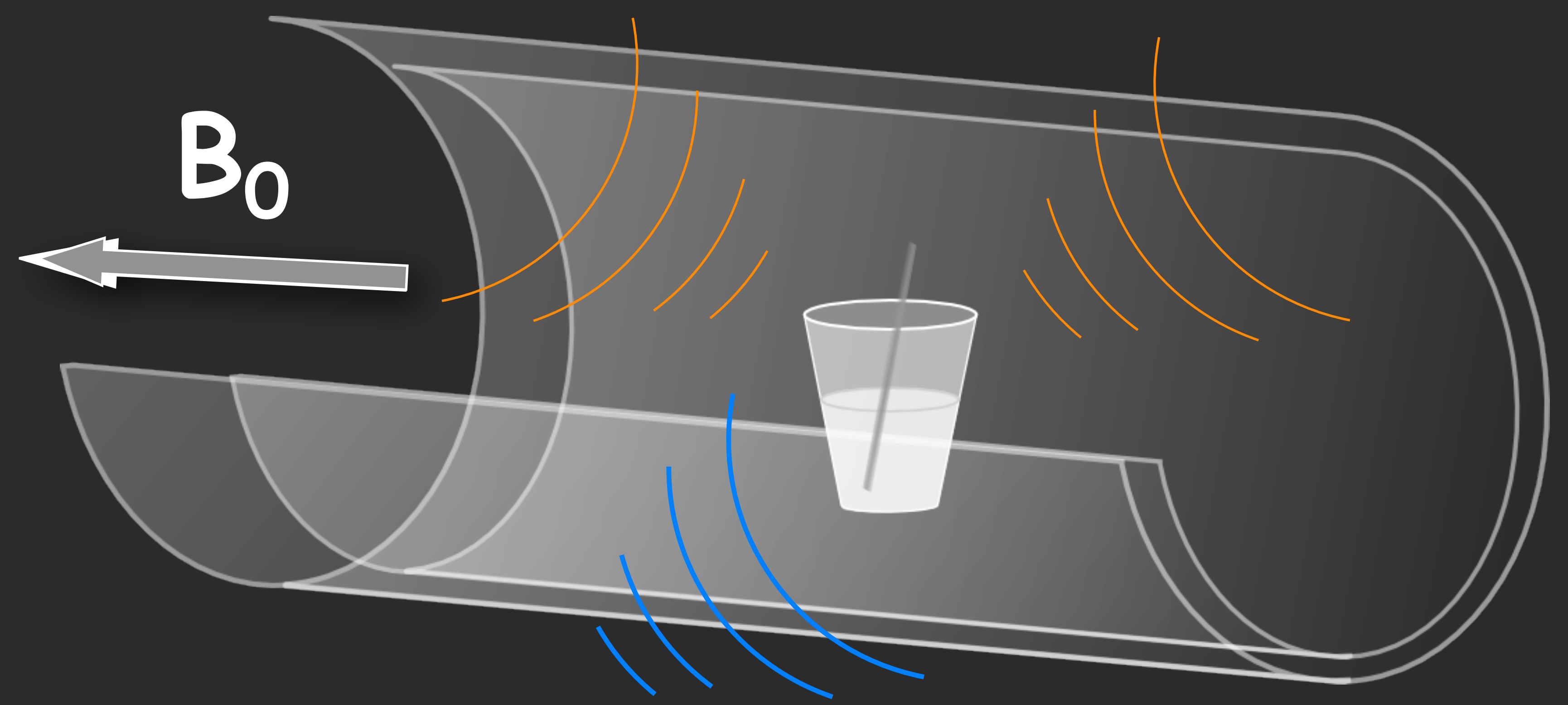
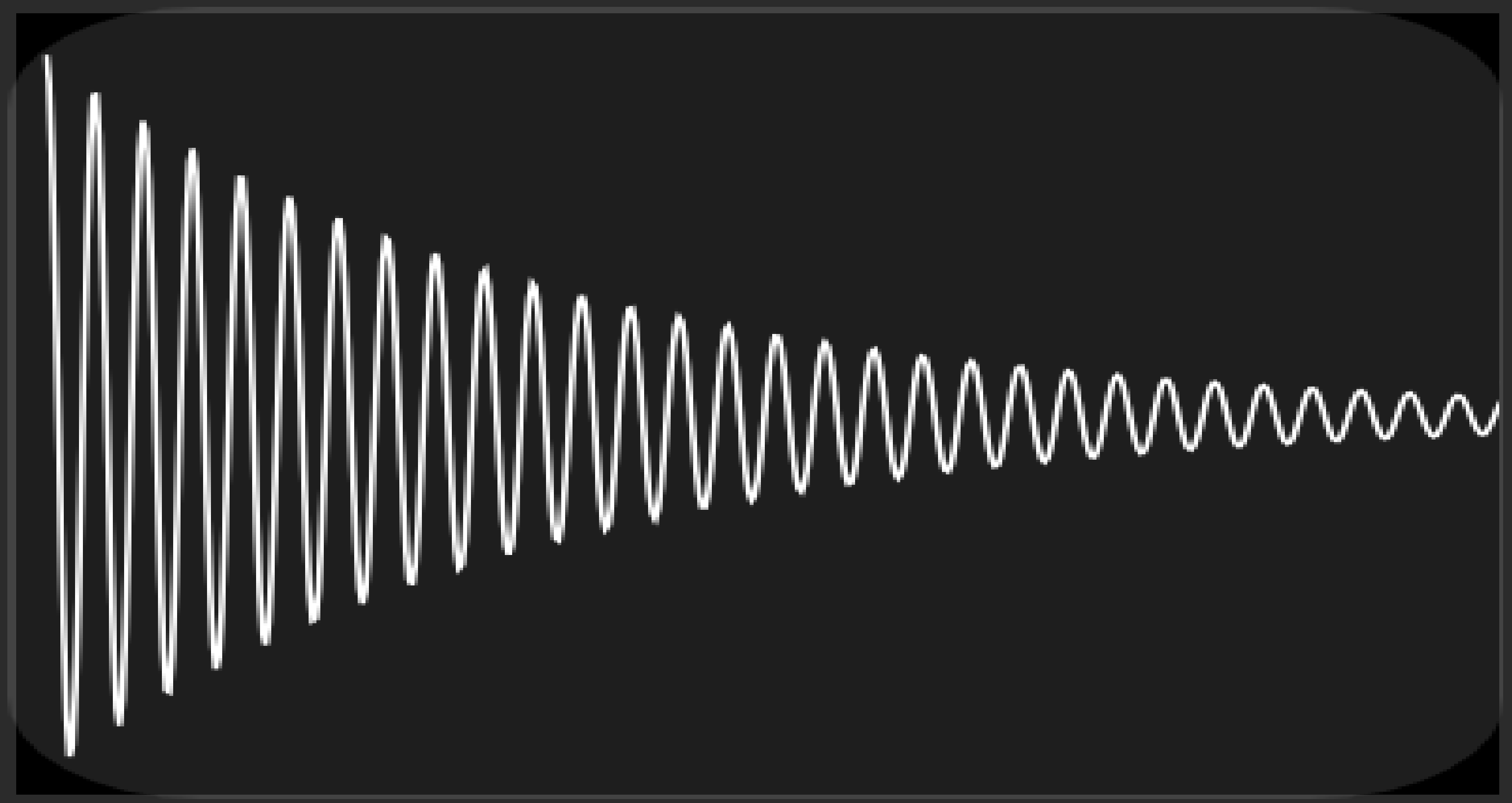
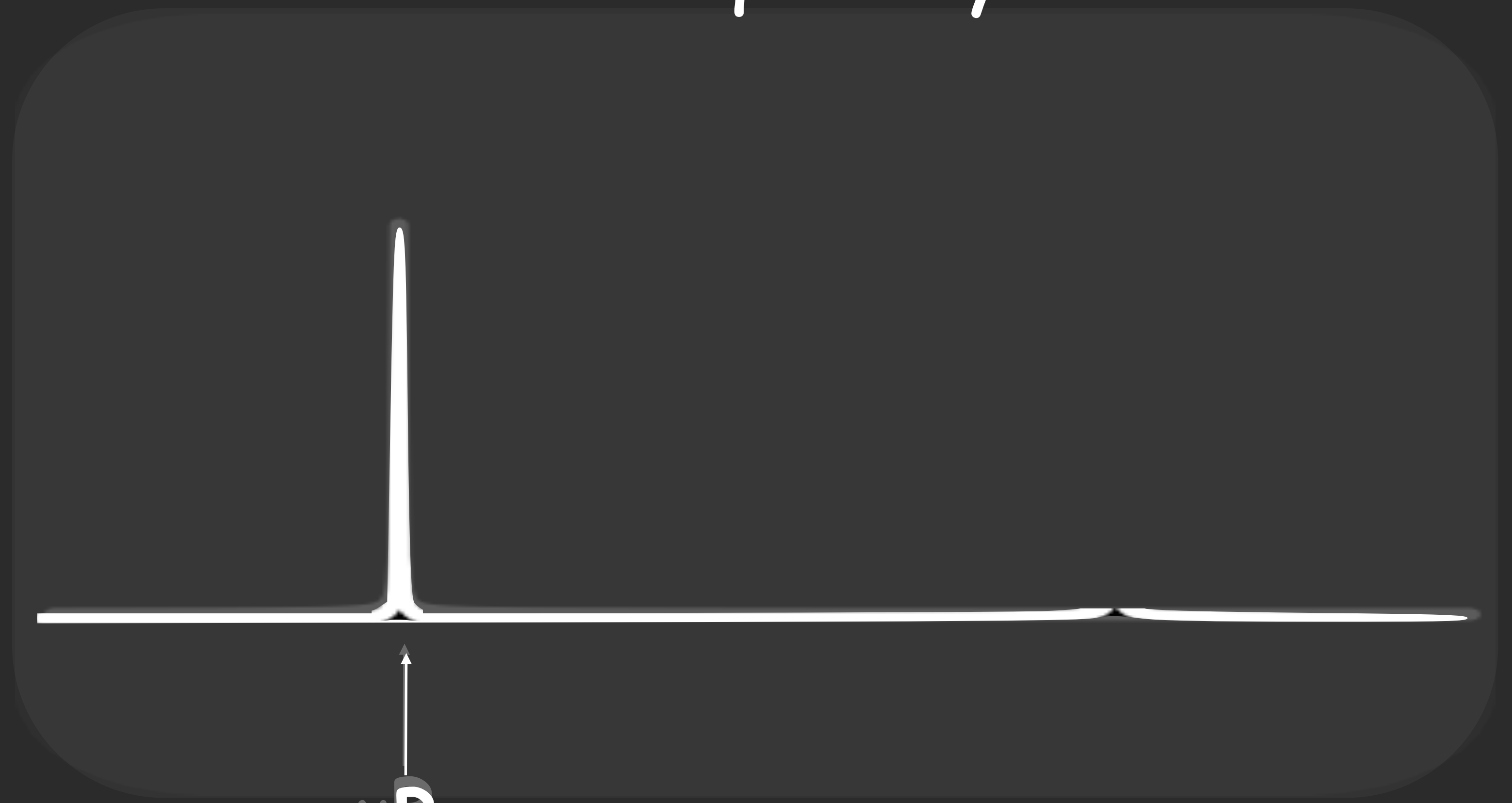


time

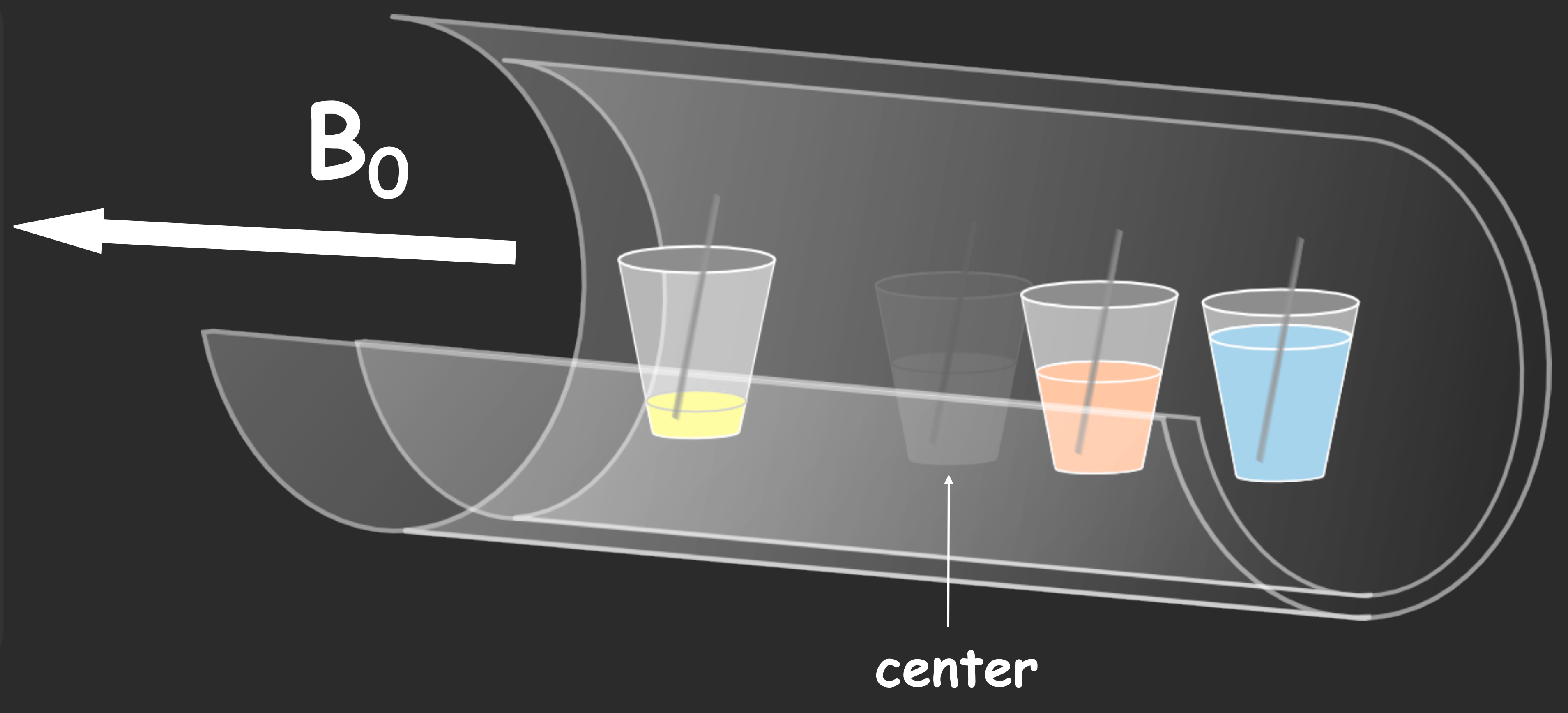


frequency

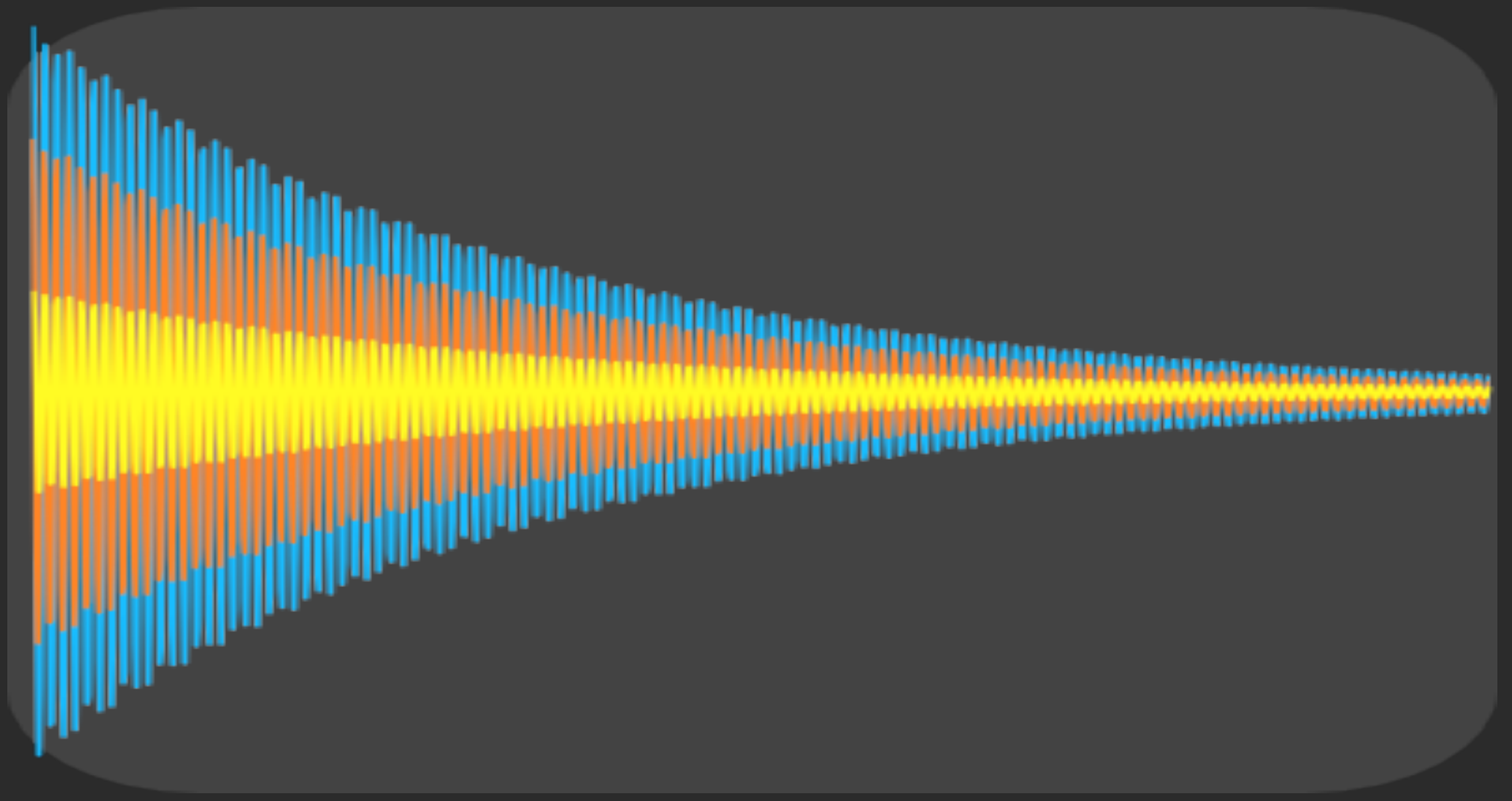


Intro to MRI - Imaging

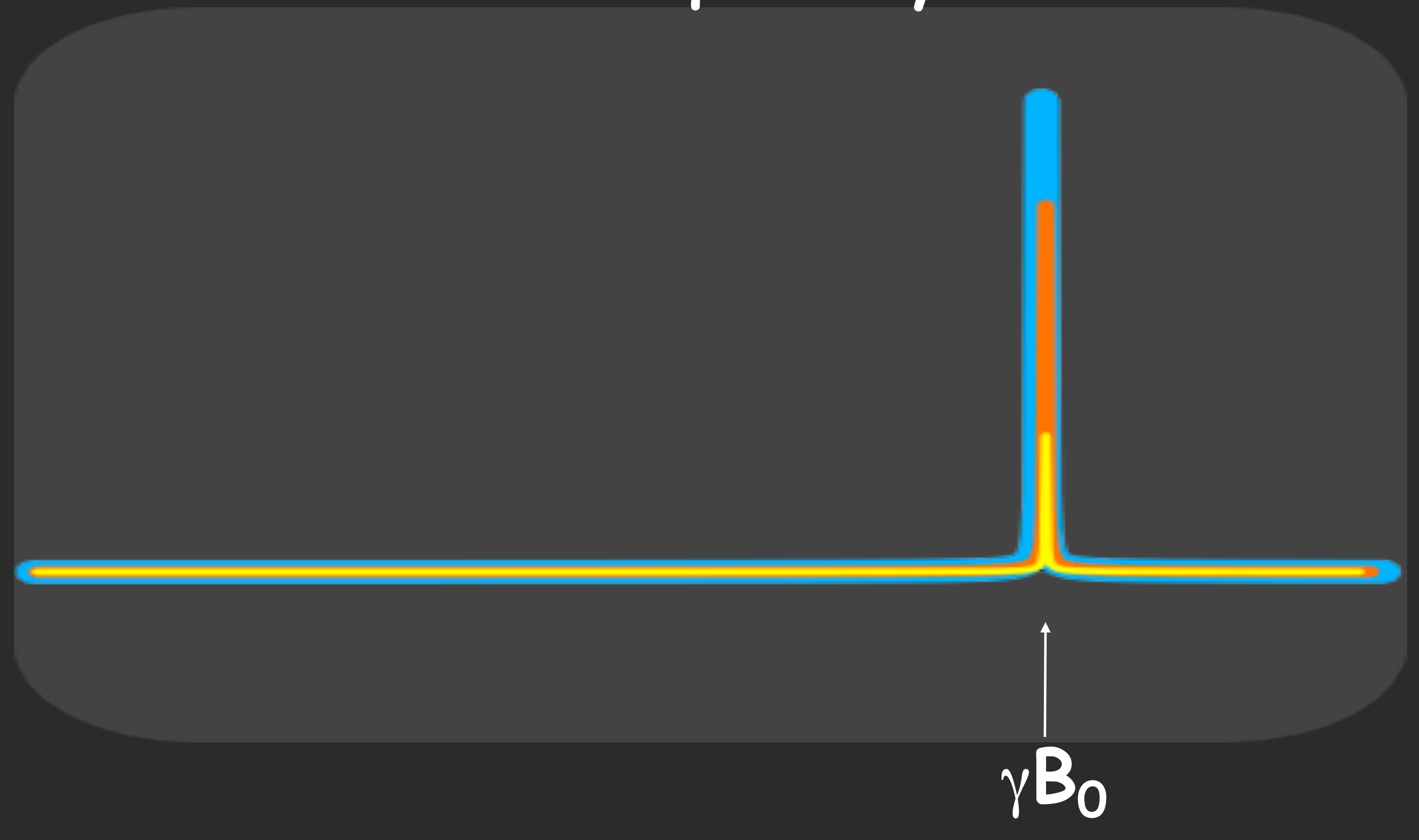
- B_0 Missing spatial information



time

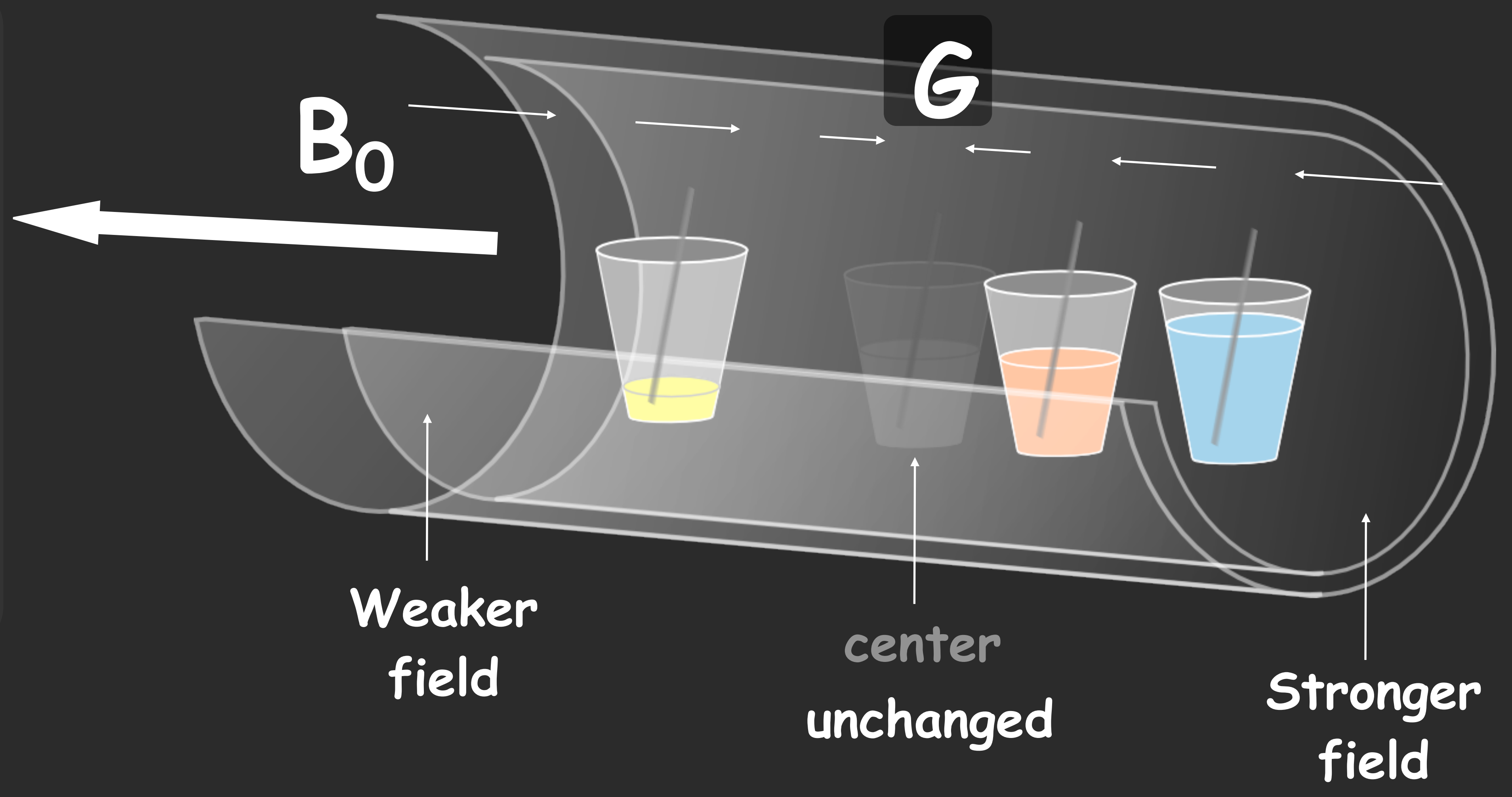


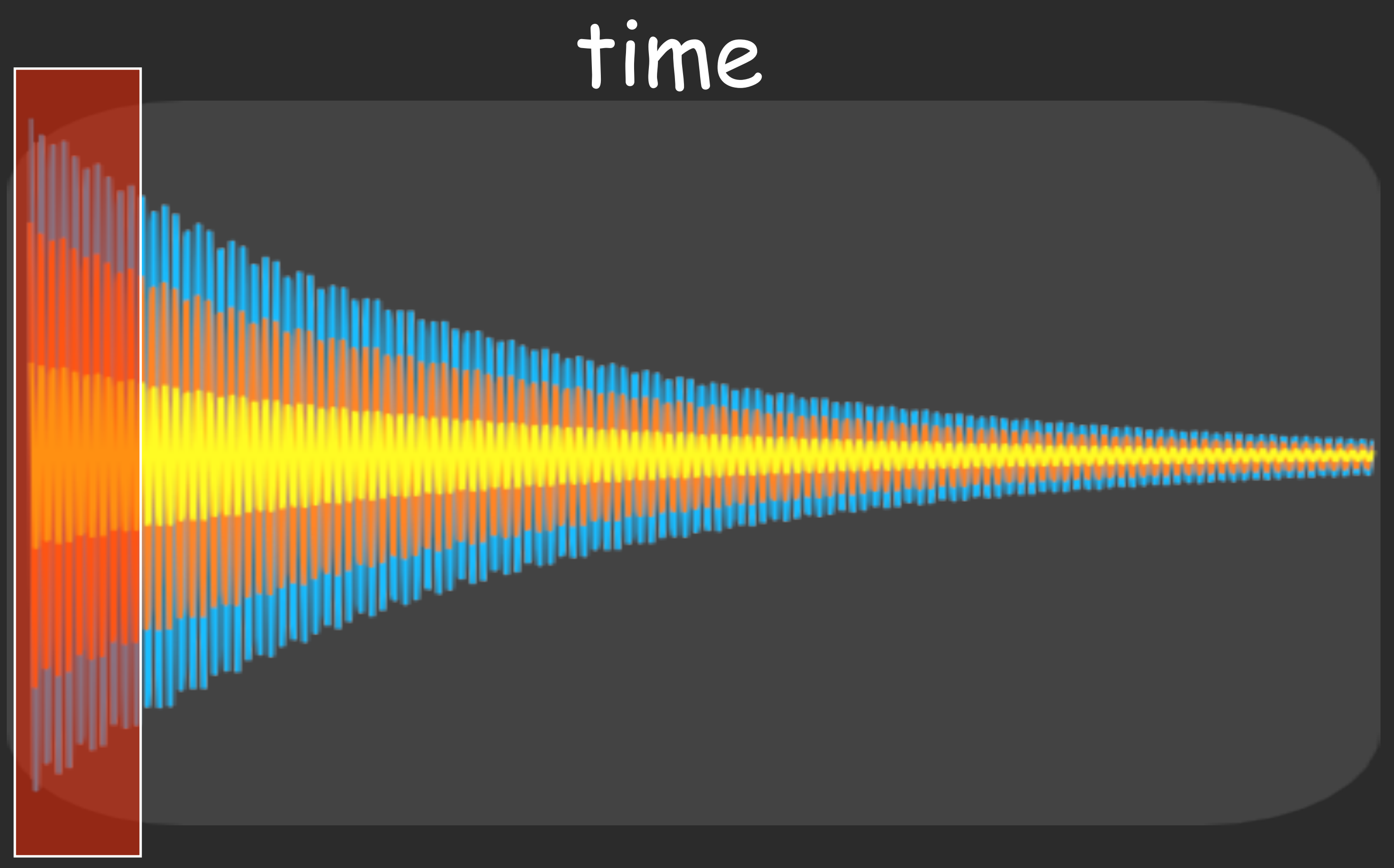
frequency

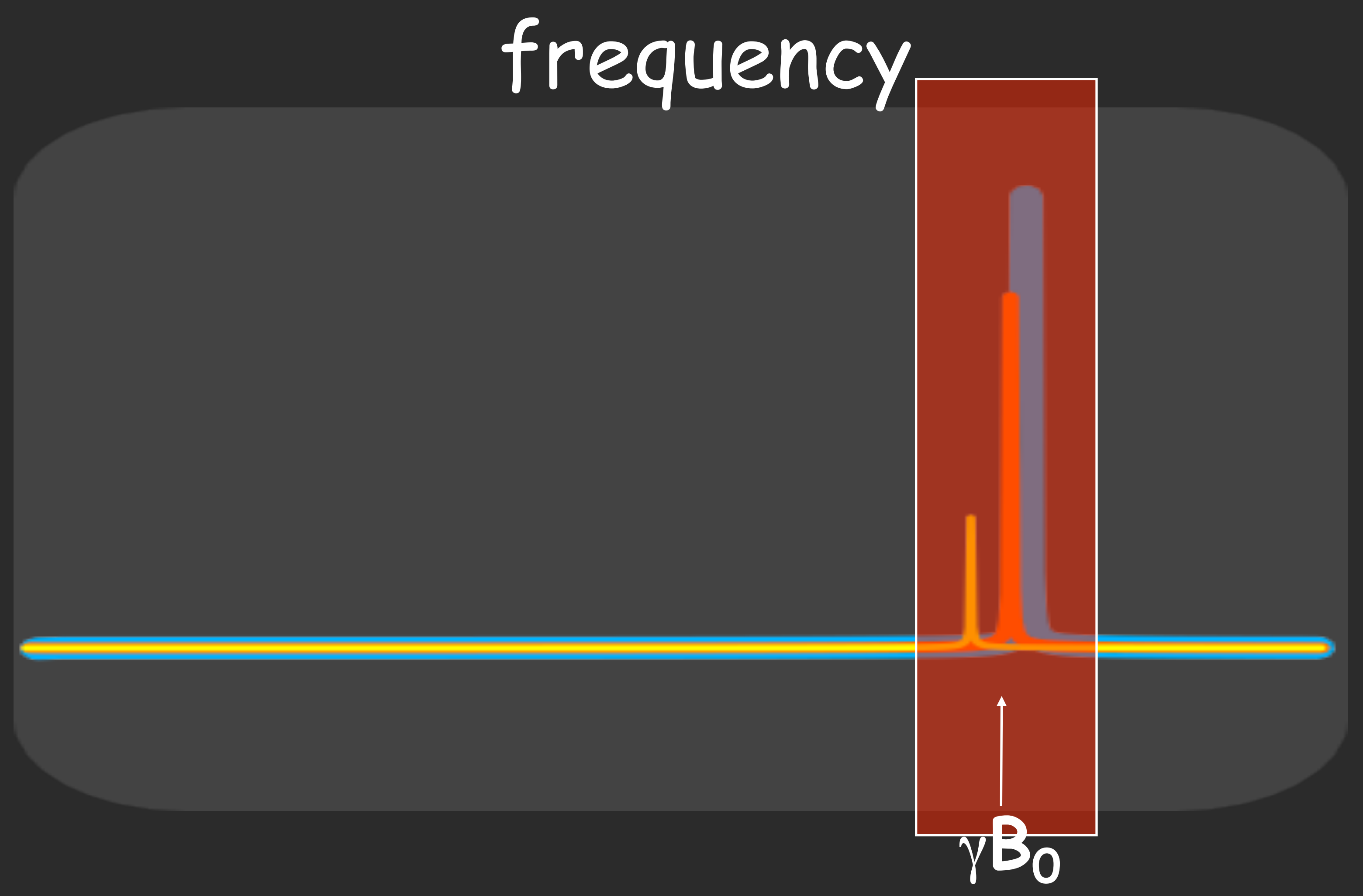


Intro to MRI - Imaging

- B_0 Missing spatial information
- Add gradient field, G

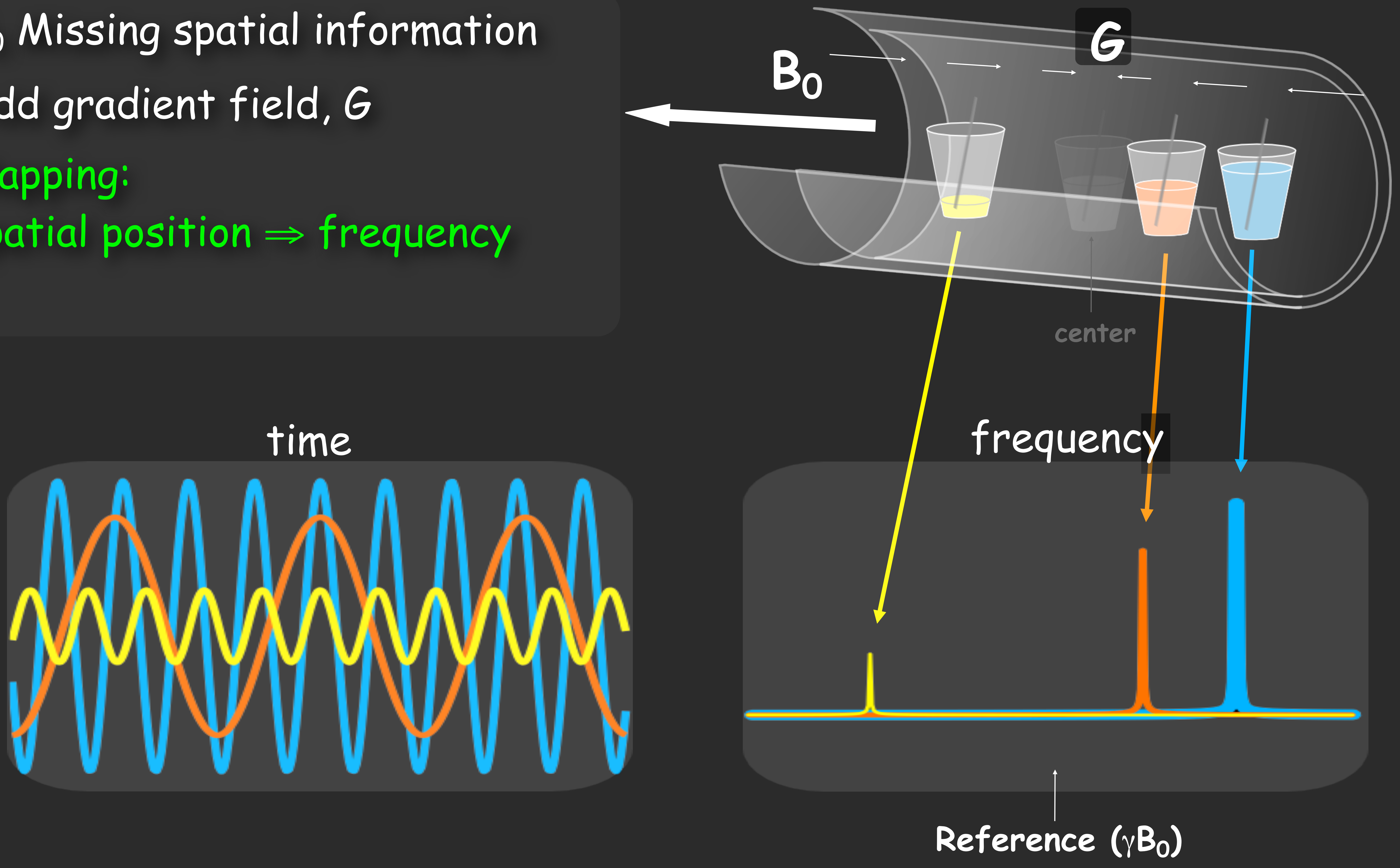




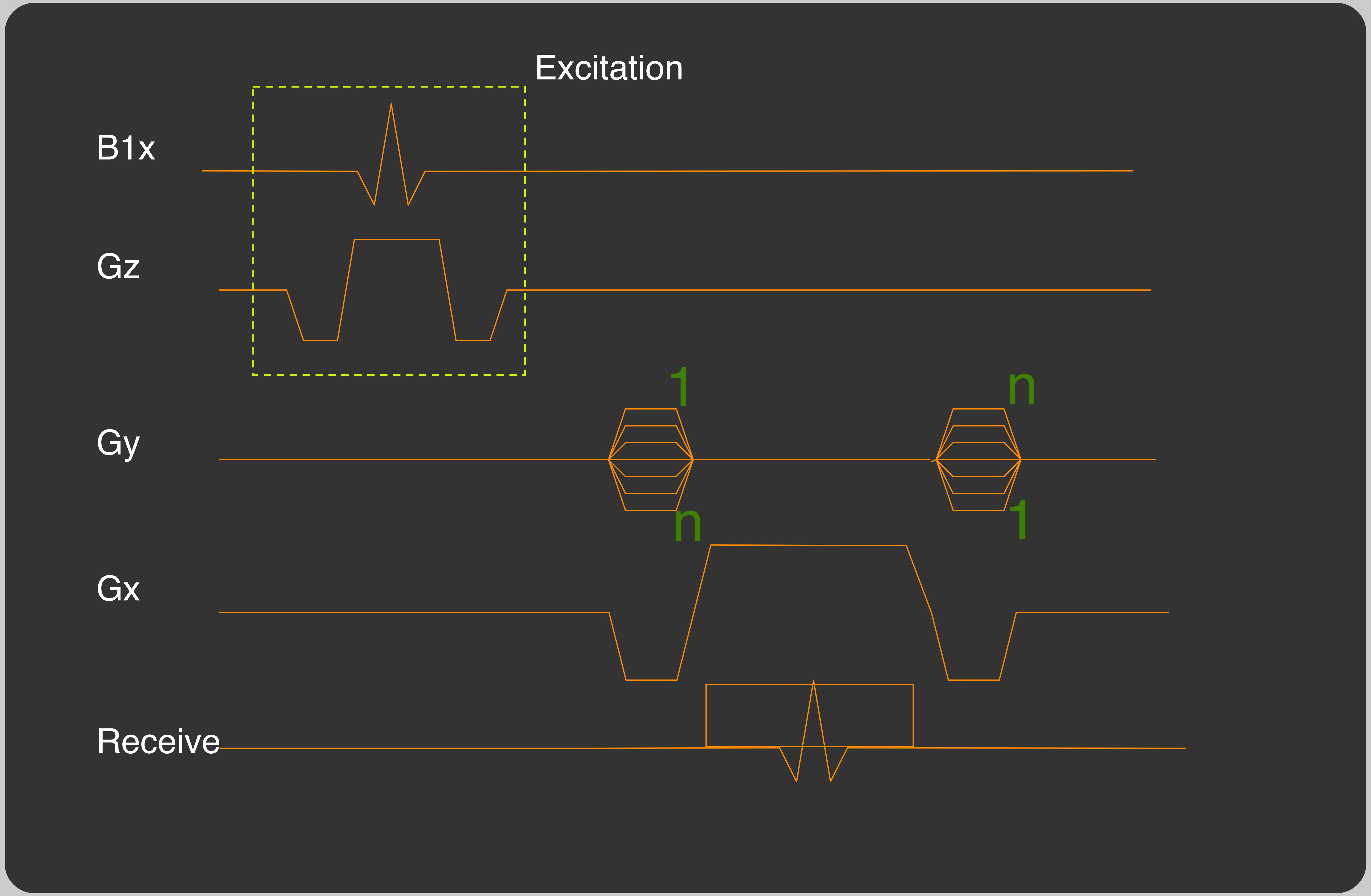


Intro to MRI - Imaging

- B_0 Missing spatial information
- Add gradient field, G
- Mapping:
  spatial position ⇒ frequency



MRI Pulse Sequence



Repeat n times
rate = TR seconds

MR Imaging

Fourier

magnitude k-space (Raw Data)

Image

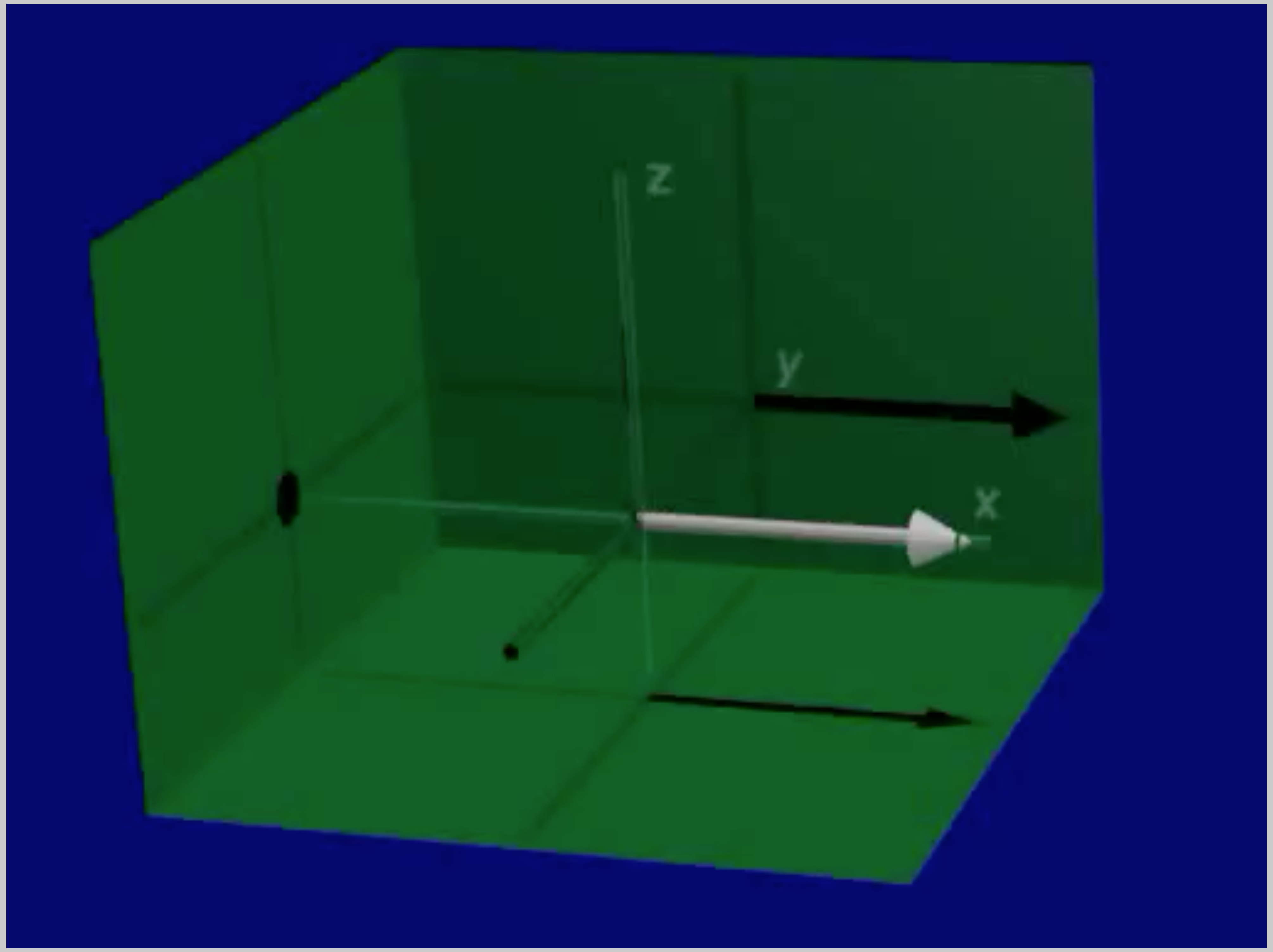
Discrete Fourier transform →

Video courtesy Brian Hargreaves


MRI is all about contrast......
http://thundafunda.com

Relaxation

$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0(\vec{r})$$



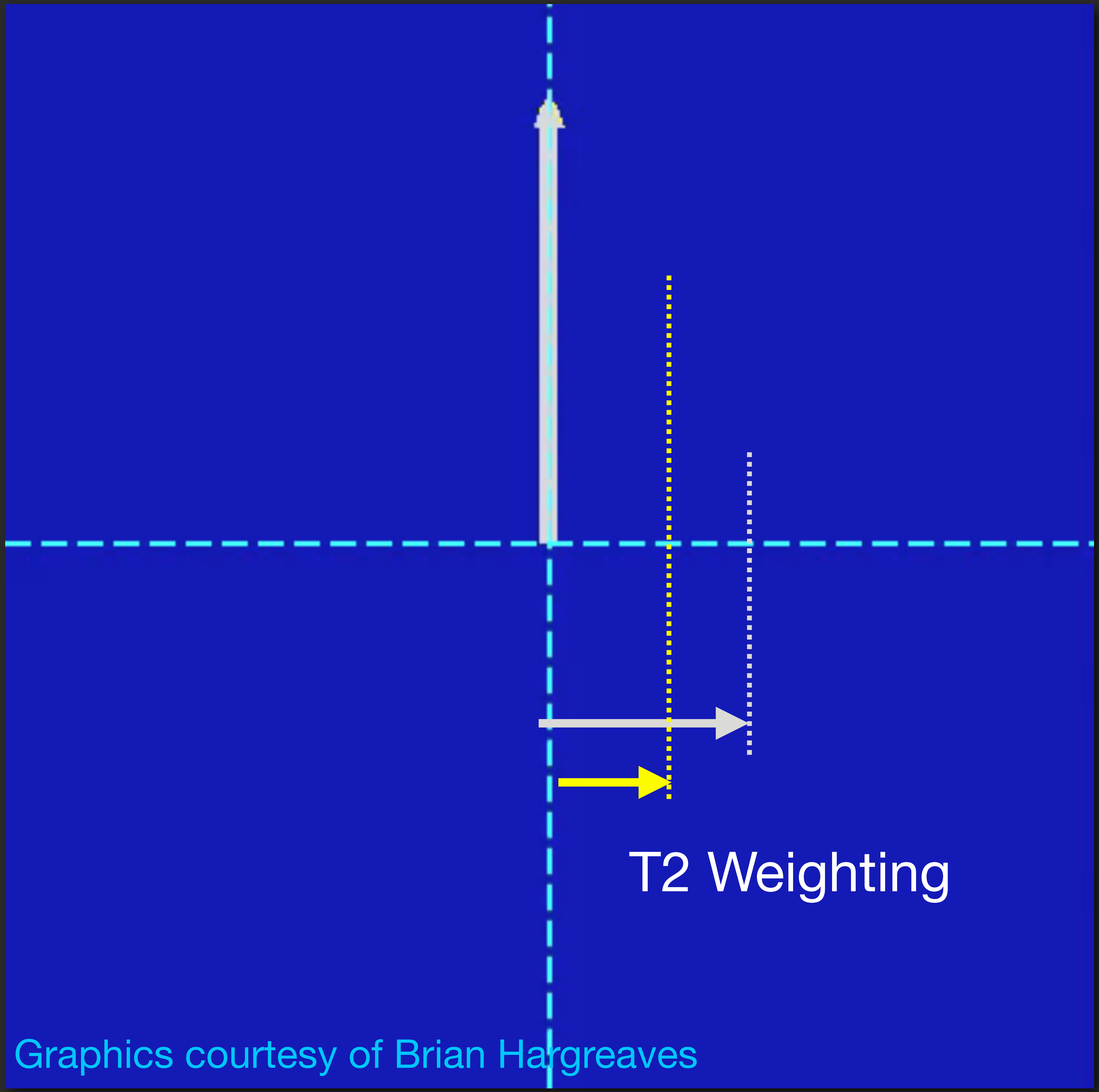
The Toilette Analogy (©2009 Al Macovski)

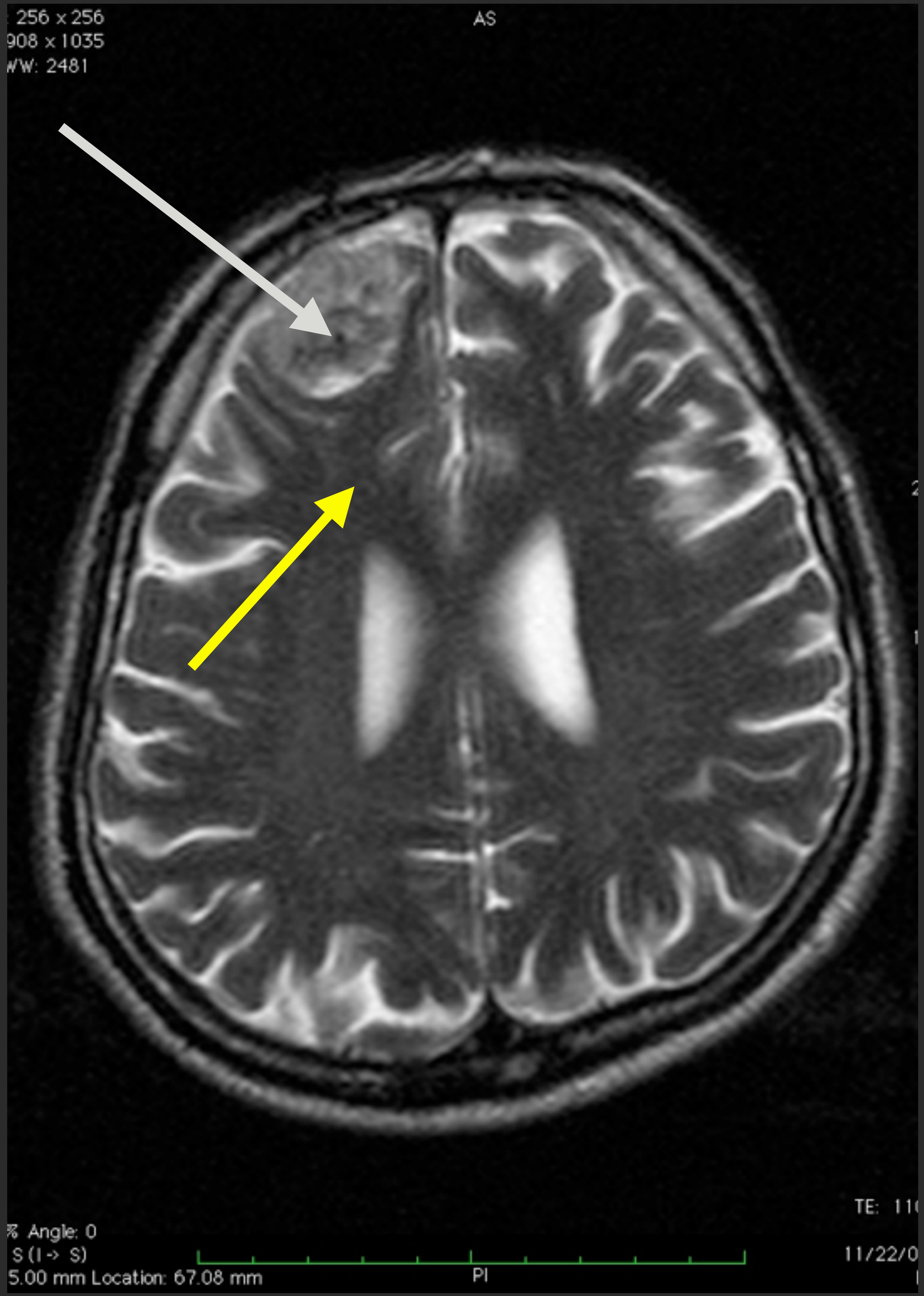
- Excitation = Flush

- Dynamics:
  - Water drains = signal decays, equivalent to T2
  - Tank refills = Magn. recovers, equivalent to T1

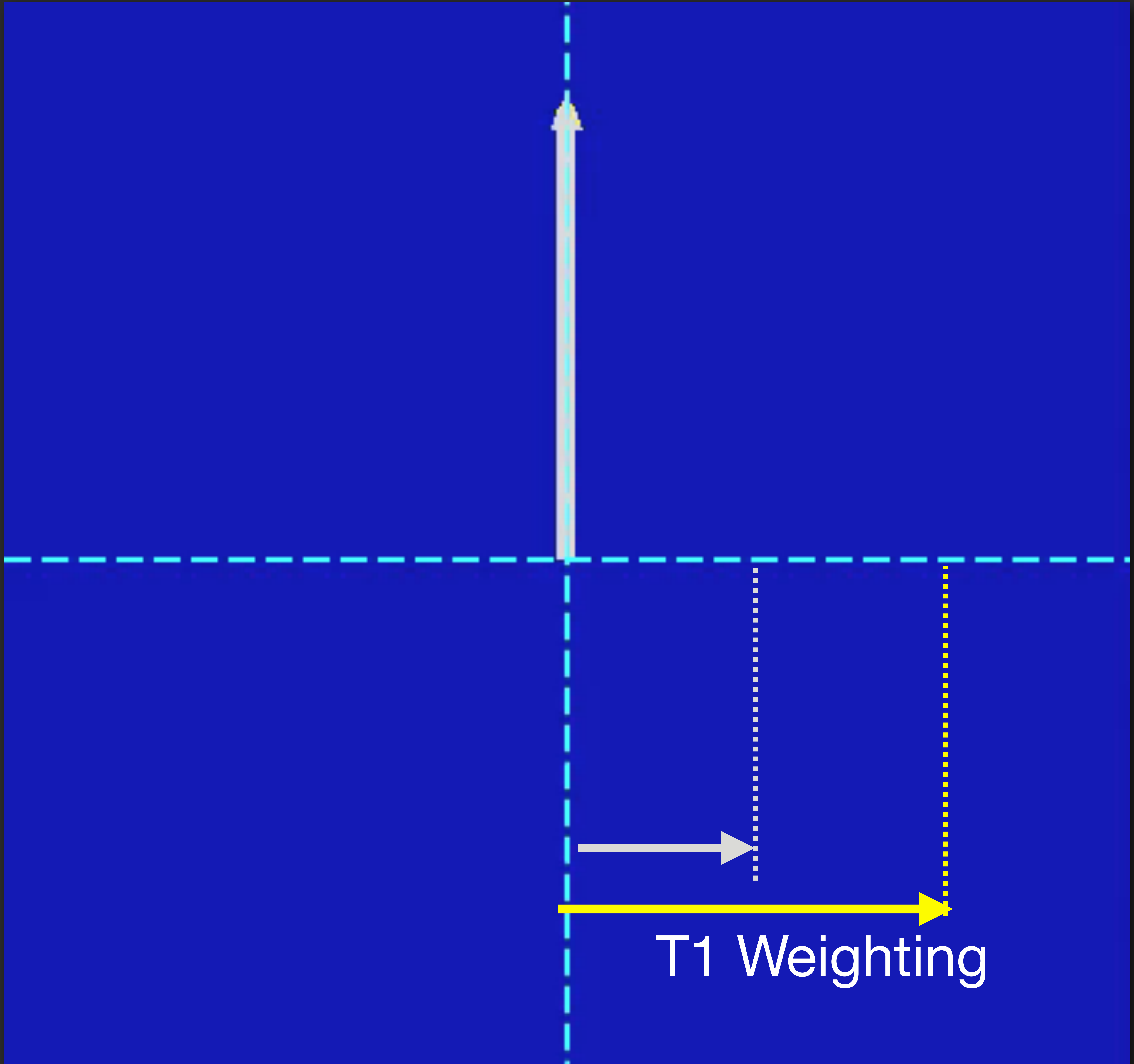
- Observed signal = water in the bowl
- Different toilettes = different tissues

The Toilette Analogy (©2009 Al Macovski)

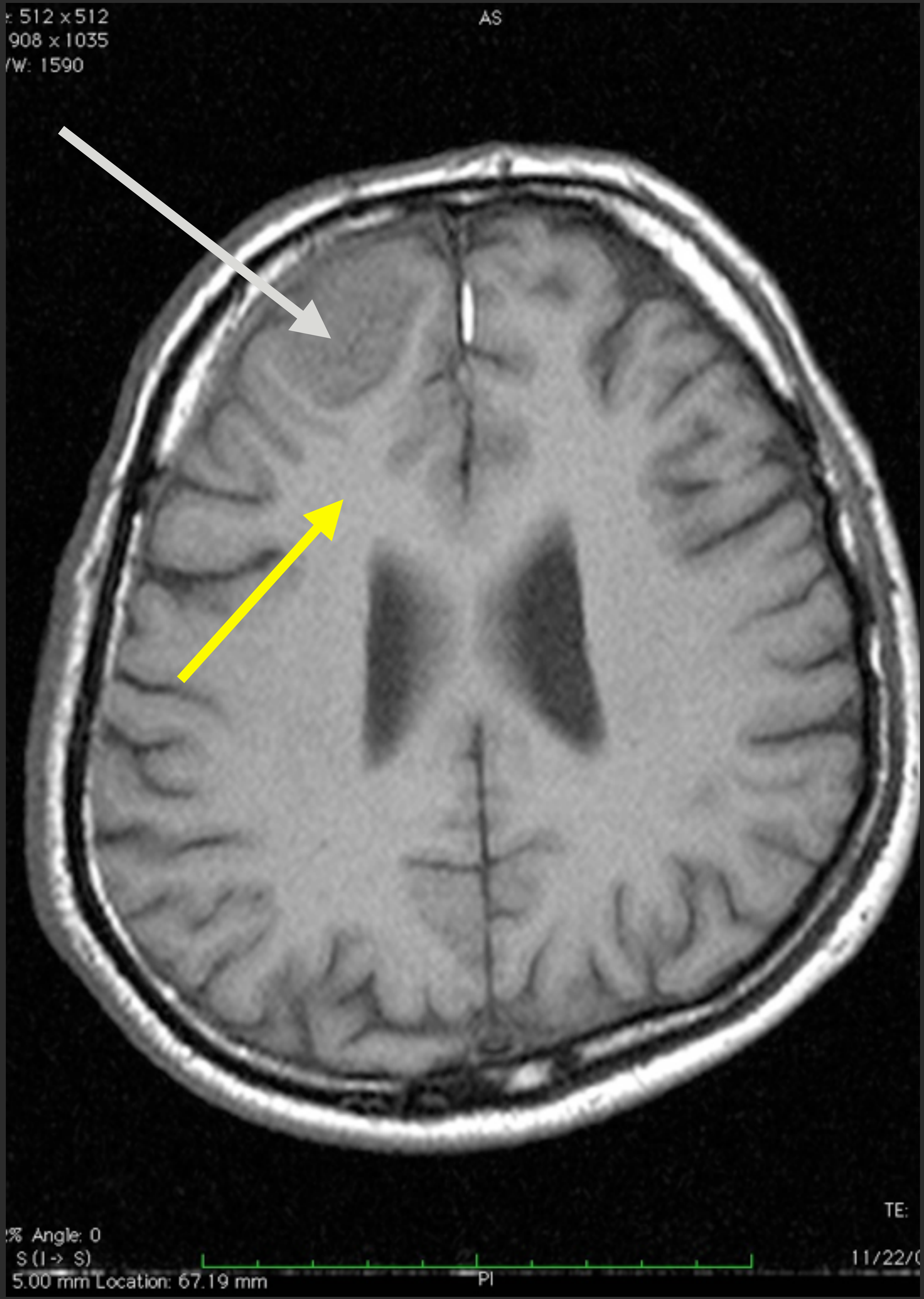




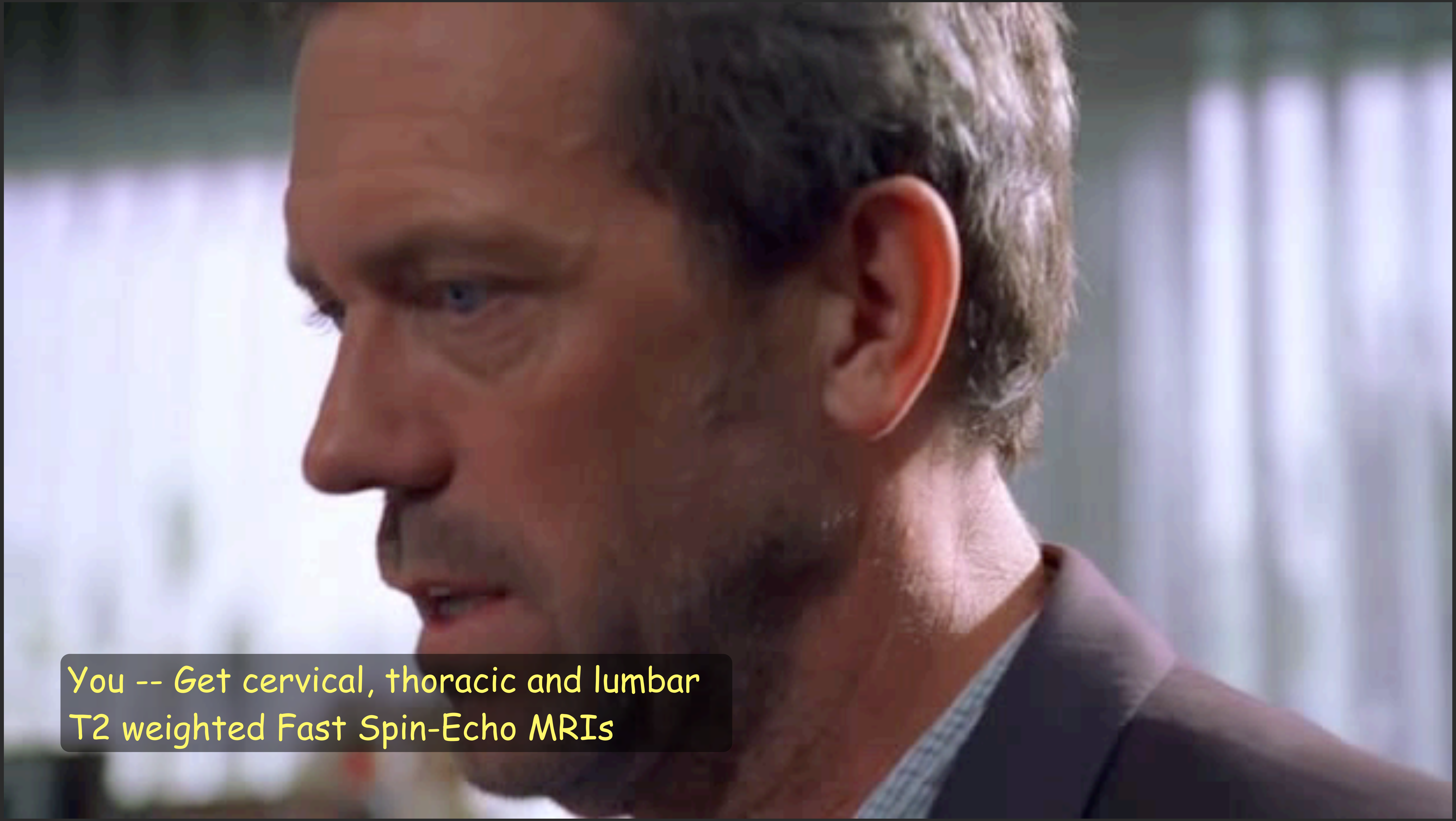
The Toilette Analogy (©2009 Al Macovski)



Graphics courtesy of Brian Hargreaves

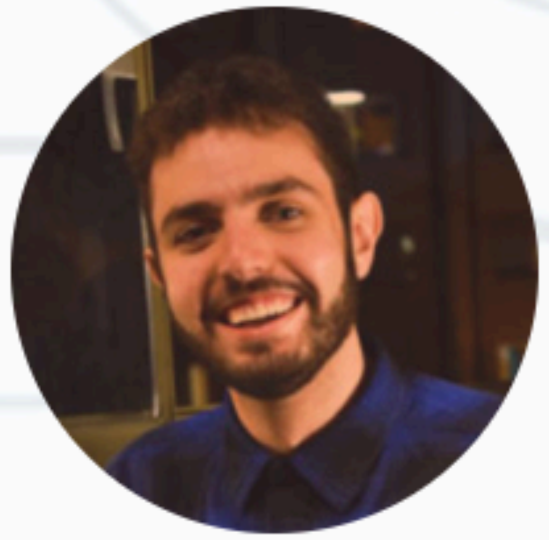


..... House Prefers T2



Summary

- MRI is about the interactions of magnetic fields with Nuclear spins
  - Governed by a linear dynamical system!
- Dynamics result in rotations, which frequency depend on position
  - Decode via DFT!
- Damping causes exponential relaxation — which we use to set the image contrast!
- To maximize signal in antenna — we use LC resonance!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Head Admin
ramseymardini@


Maxwell Chen
Head Admin
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Mikaela Frichtel
Head Lab
mfrichtel@


Steven Lu
Head Lab
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Kaitlyn Chan
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Sravya Basvapatri
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Caleb Kuo
Lab
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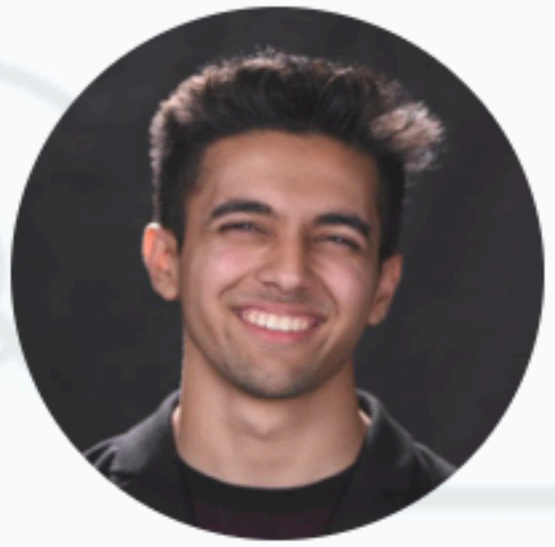
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