EE16B
Designing Information Devices and Systems II

Lecture 6B
Linearization

Intro

- Last time
  - Described systems with state-space model
  - Talked about linear systems

- Today
  - Change of variables
  - A bit on Linearization of non-linear systems

Revisiting Examples

Q: Is example 3 linear?

$$s(t+1) = s(t) + g(t) - w(t)$$
$$g(t+1) = r(t)$$
$$r(t+1) = u(t)$$

A: Linear

$$\underset{\vec{x}(t+1)}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \underset{A}{\begin{bmatrix} & & \\ & & \\ & & \\ \end{bmatrix}} \underset{\vec{x}(t)}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} + \underset{B_u}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} u(t) + \underset{W_u}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} w(t)$$

Revisiting Examples

Q: Is example 3 linear?

$$s(t+1) = s(t) + g(t) - w(t)$$
$$g(t+1) = r(t)$$
$$r(t+1) = u(t)$$

A: Linear

$$\underbrace{\begin{bmatrix} s(t+1) \\ g(t+1) \\ r(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s(t) \\ g(t) \\ r(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_u} u(t) + \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{W_u} w(t)$$

Changing State Variable

State variables are not unique!

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

Let T be an invertible matrix:

$$\vec{z} = T\vec{x}$$

Then,

$$\vec{z}(t+1) = T\vec{x}(t+1) = TA\vec{x}(t) + TB\vec{u}(t)$$

$$= TAT^{-1}\vec{z}(t) + TB\vec{u}(t)$$

Changing State Variables

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

Define:

$$\vec{z} = T\vec{x} \qquad A_{\text{new}} = TAT^{-1} \qquad B_{\text{new}} = TB$$

Can be written as,

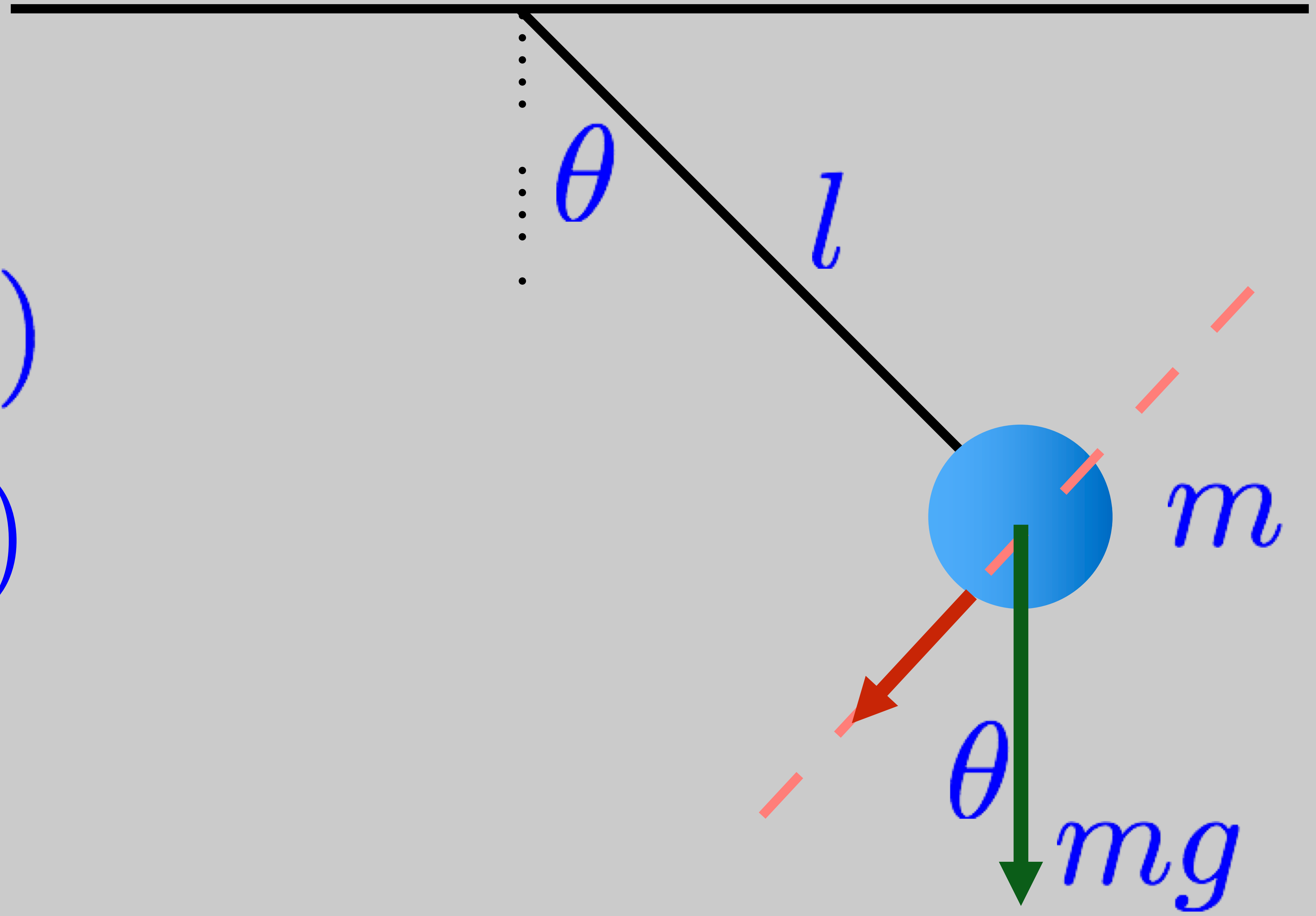
$$\vec{z}(t+1) = A_{\text{new}}\vec{z}(t) + B_{\text{new}}\vec{u}(t)$$

Similarly for continuous systems!

Next: We will see how a special choice of T will make it easy to analyze system properties like *stability*, and *controllability*

https://www.youtube.com/watch?v=SPO9pVwoxVg

Linearization



State variables:

$$x_1(t) = \theta(t)$$
$$x_2(t) = \dot{\theta}(t)$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{g}{l}\overbrace{\sin(x_1(t))}^{\approx x_1} - \frac{k}{m}x_2(t)$$

Linearization:

$$\frac{dx_2(t)}{dt} = -\frac{g}{l}x_1(t) - \frac{k}{m}x_2(t)$$

Linearization

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{g}{l}x_1(t) - \frac{k}{m}x_2(t)$$

$$\Rightarrow \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$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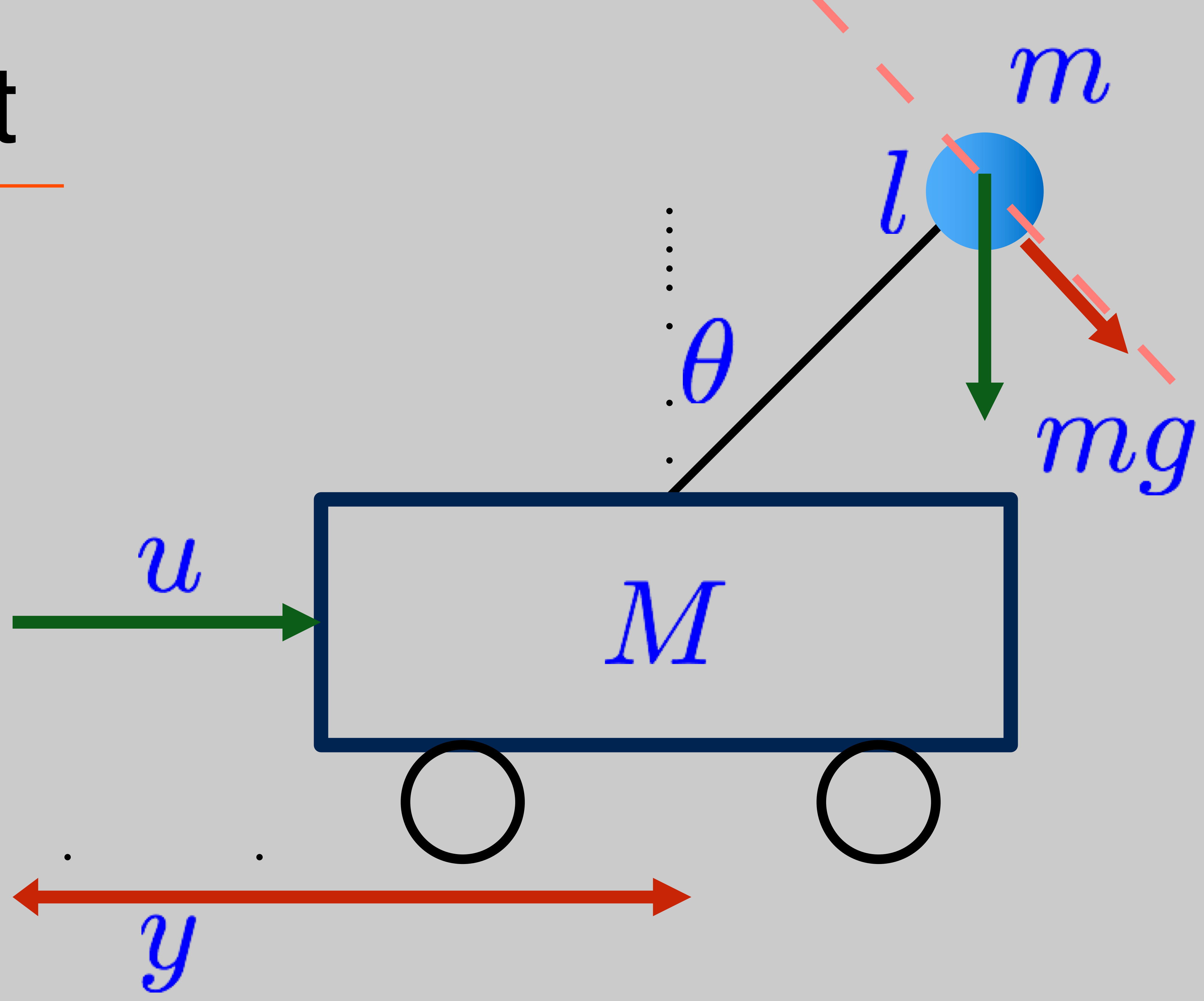
Scary Example: Pole on a Cart

How many state variables?

How to systematically linearize?



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g \sin\right.$$

Linearization- teaching approach

Start with scalar

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Continue to Vector to scalar functions

$$f : \mathbb{R}^N \rightarrow \mathbb{R}$$

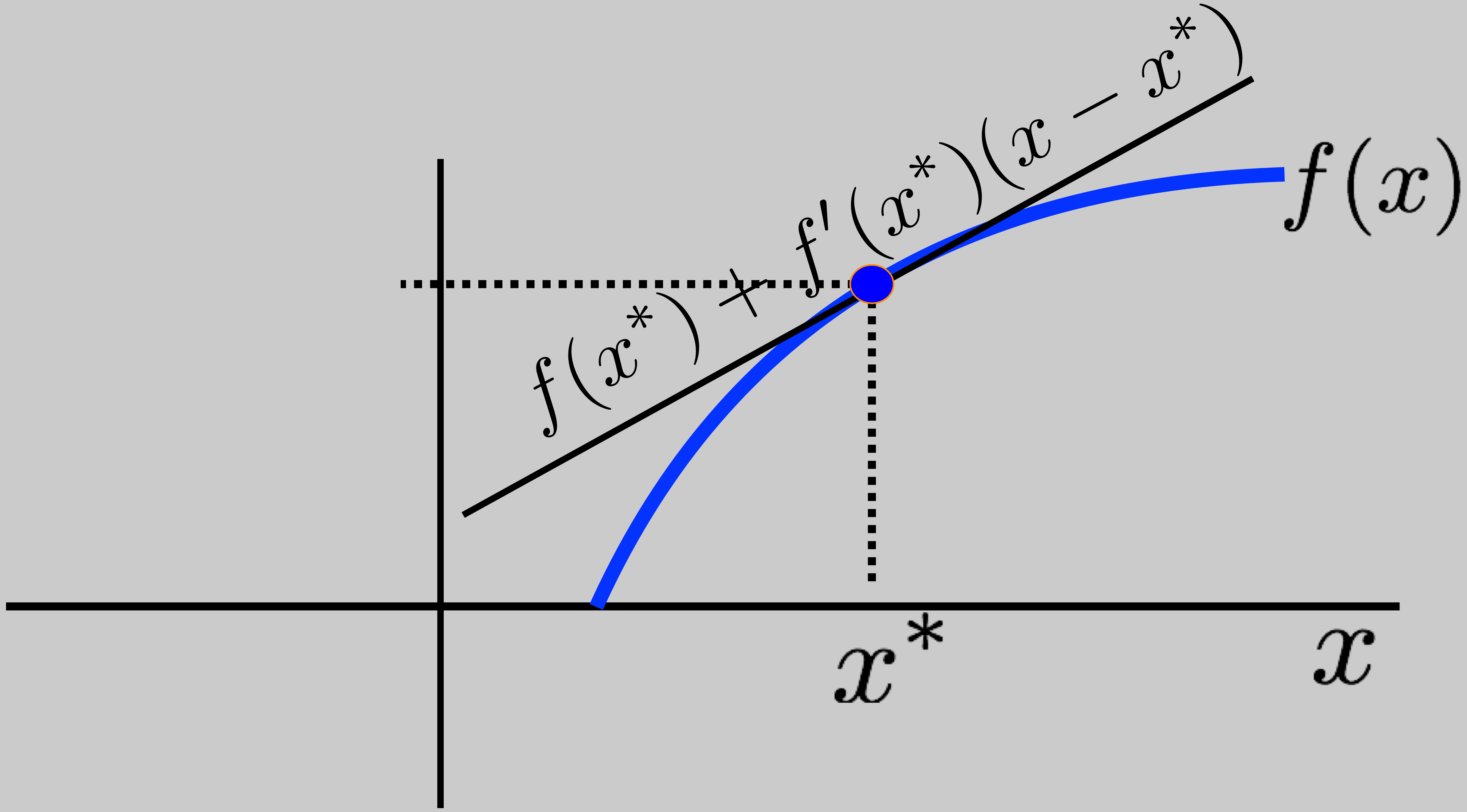
First for N=2 and slow derivation

Then – show math syntax sugar (gradient)

Generalize to $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ (Jacobian)

Taylor Approximation - scalar

$$f : \mathbb{R} \to \mathbb{R}$$



$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)$$

$$x^* = 0\Big| \Rightarrow \sin(x) \approx \sin(0) + \cos(0)(x - 0)$$

$$\sin x \approx x$$

Taylor Approximation - scalar

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)$$

Example:

Watch class for notes

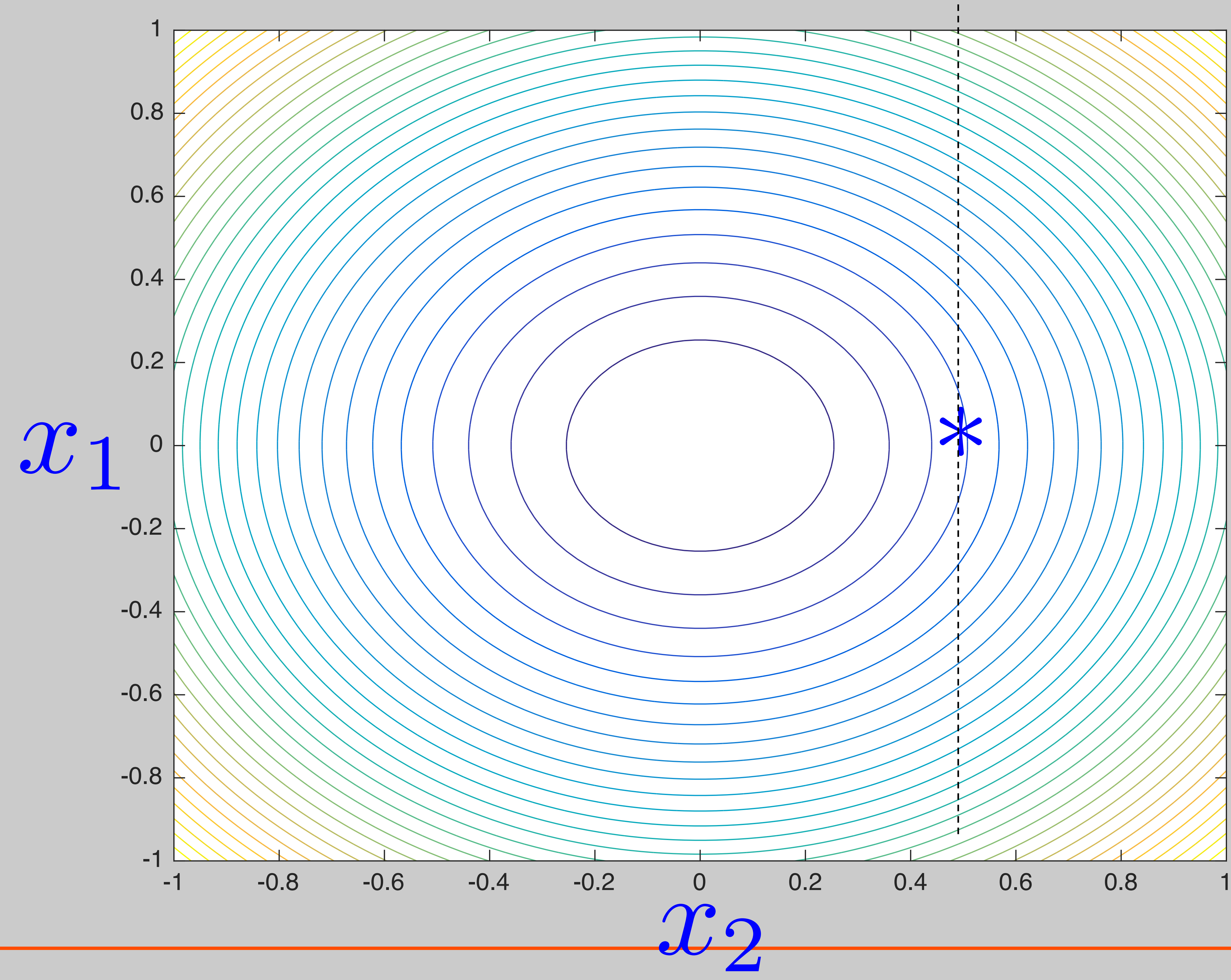
Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}$$

Example: $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ $\quad f(\vec{x}) = ||\vec{x}||^2 = x_1^2 + x_2^2$

Let's look at

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$



Partial Derivative

Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$

$$\frac{d}{dx_1} f(x_1, x_2^*) = \frac{d}{dx_1} x_1^2 + \frac{d}{dx_1} {x_2^*}^2 = 2x_1$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2) = 2x_1$$

$$\frac{\partial}{\partial x_2} f(x_1, x_2) = 2x_2$$

Taylor Approximation - vector

Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

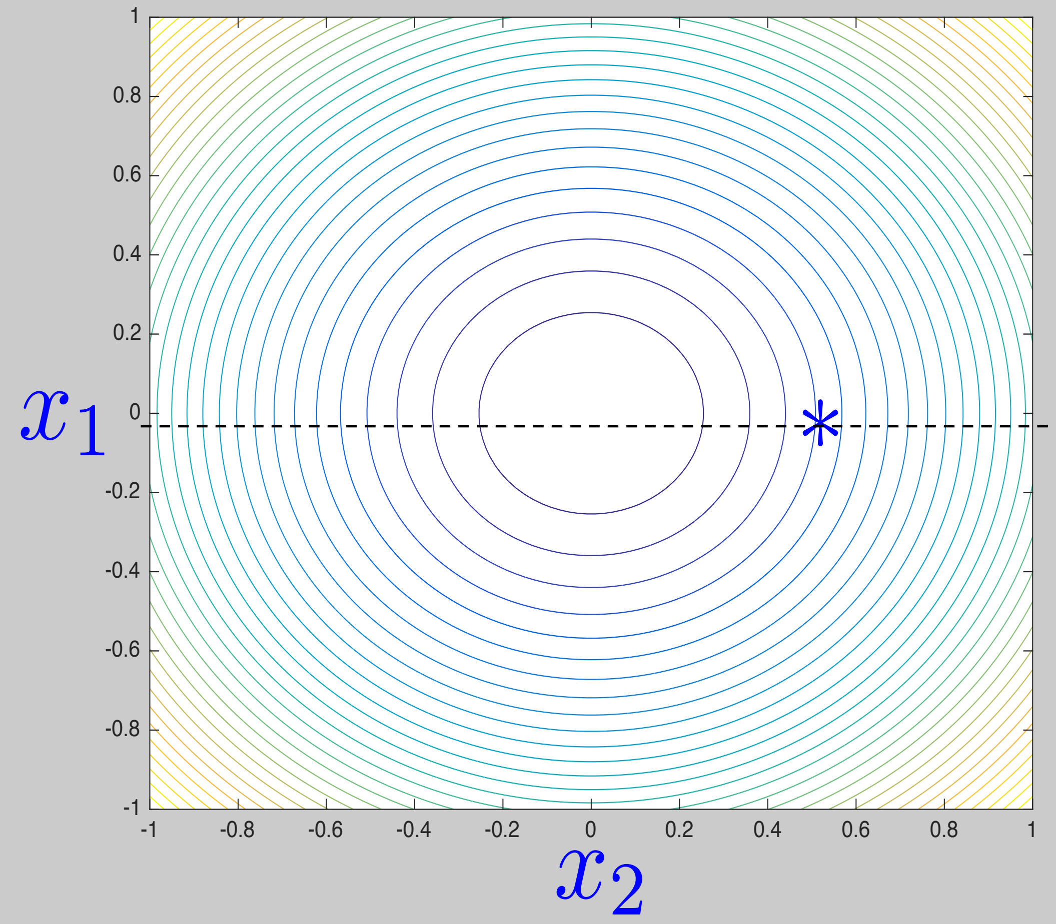
$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$

$$\left.\frac{\partial}{\partial x_1} f(x_1, x_2)\right|_{x_1^*, x_2^*} = 2x_1^*$$

$$f(x_1, x_2^*) \approx {x_1^*}^2 + {x_2^*}^2 \quad + 2x_1^*(x_1 - x_1^*)$$

Similarly:

$$f(x_1^*, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_2^*(x_2 - x_2^*)$$



So,

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*) + 2x_2^*(x_2 - x_2^*)$$

Taylor Approximation - vector

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*) + 2x_2^*(x_2 - x_2^*)$$

Write in vector form:

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix}(\vec{x} - \vec{x}^*)$$

Taylor Approximation - vector

Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix}(\vec{x} - \vec{x}^*)$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \begin{bmatrix} \frac{\partial}{\partial x_1} f(\vec{x}^*) & \frac{\partial}{\partial x_2} f(\vec{x}^*) \end{bmatrix} (\vec{x} - \vec{x}^*)$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*)$$

Q: What are the dimensions of ∇f(x*) ? (gradient / Jacobian)

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N \qquad \frac{d}{dt}\vec{x}(t) = f\left(\vec{x}(t)\right)$$

$$\underbrace{f(\vec{x})}_{\text{Nx1}} \approx \underbrace{f(\vec{x}^*)}_{\text{Nx1}} + \nabla f(\vec{x}^*)\underbrace{(\vec{x} - \vec{x}^*)}_{\text{Nx1}}$$

Q: What are the dimensions of $\nabla f(\vec{x}^*)$? (Jacobian)

A: NxN ?

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$$f(\vec{x}) = \left[\quad \right]$$

$$\nabla f(\vec{x}) = \left[\quad \right]$$

i,j th entry:

$$\frac{\partial f_i(x)}{\partial x_j}$$

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$$f(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\ f_2(x_1, \cdots, x_N) \\ \vdots \\ f_N(x_1, \cdots, x_N) \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ & \vdots & \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

i,j th entry:

$$\frac{\partial f_i(x)}{\partial x_j}$$

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N \qquad f(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\ f_2(x_1, \cdots, x_N) \\ \vdots \\ f_N(x_1, \cdots, x_N) \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ & \vdots & \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

i,j th entry:

$$\frac{\partial f_i(x)}{\partial x_j}$$

Linearization of State-Space

Linearize around an equilibrium, a point s.t.:

$$f(\vec{x}^*) = 0$$

Q: why?

A: no change!

$$\frac{d}{dt}\vec{x} = f(\vec{x})$$

$$\approx \underbrace{f(\vec{x}^*)}_{=0} + \nabla f(\vec{x}^*)\underbrace{(\vec{x} - \vec{x}^*)}_{\tilde{x}}$$

Which of the variables is a function of t?

write a state model for deviation!

Linearization of State-Space

$$\tilde{x} = \vec{x} - \vec{x}^*$$

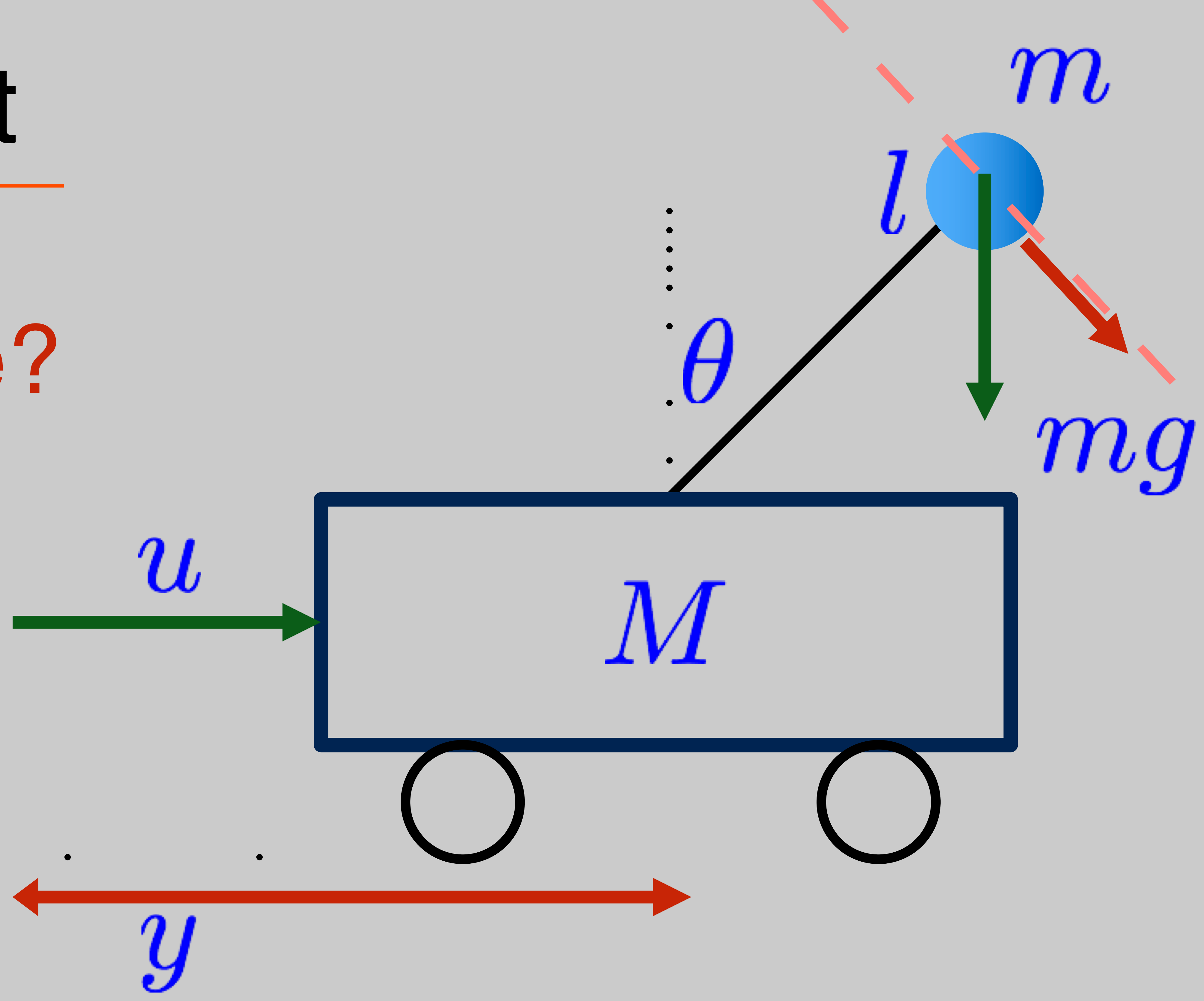
$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}\vec{x}(t) - \underbrace{\frac{d}{dt}\vec{x}^*}_{= 0}$$

$$= f(\vec{x}(t)) \approx \underbrace{f(\vec{x}^*)}_{= 0} + \nabla f(\vec{x}^*)\tilde{x}(t)$$

$$\frac{d}{dt}\tilde{x}(t) = [\nabla \underbrace{f(\vec{x}^*)}_{A}]\,\tilde{x}(t)$$

Scary Example: Pole on a Cart

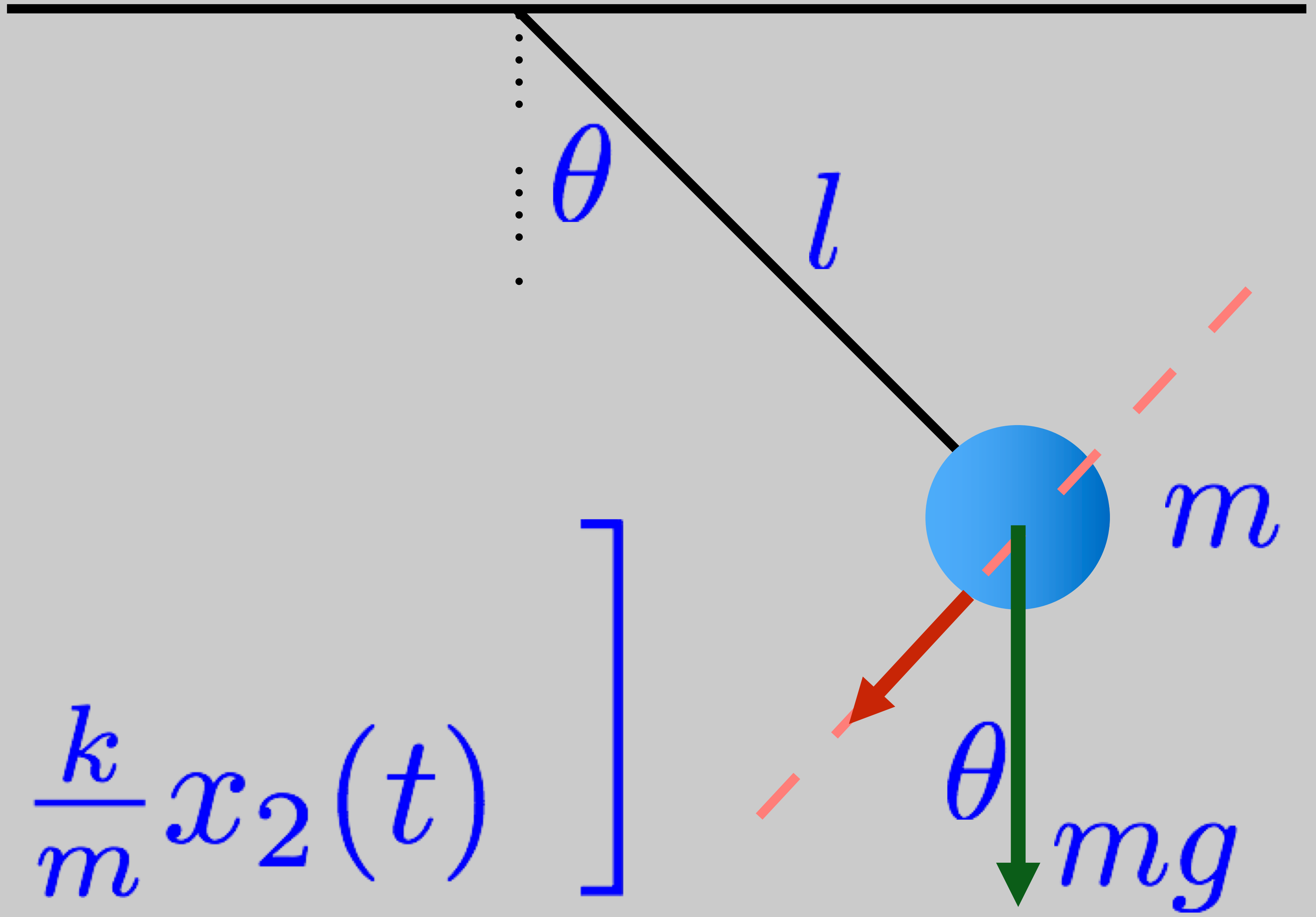
Q) Can you do it for this example?



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g\sin\theta\right)$$

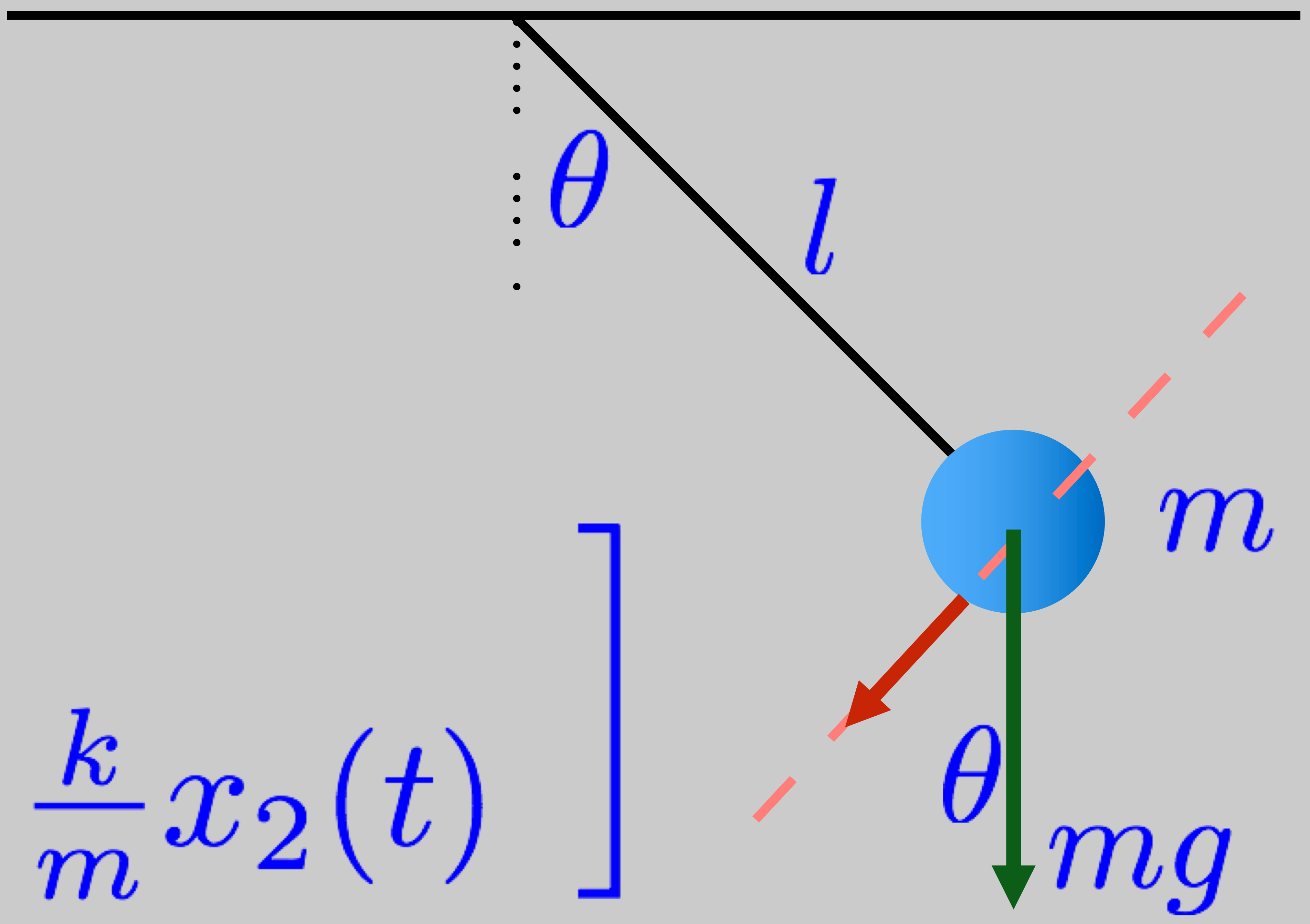
Back to the Pendulum



$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

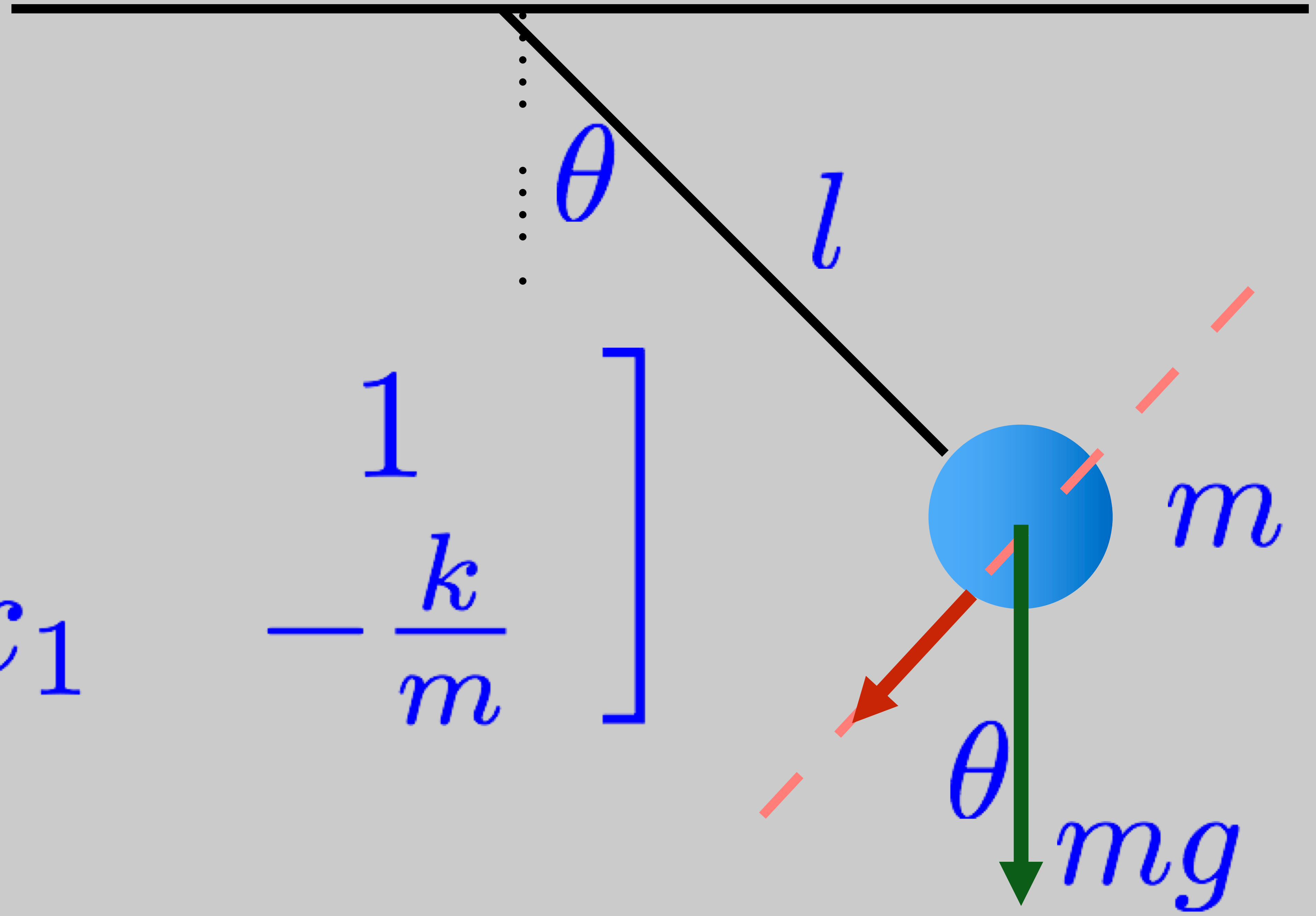
Back to the Pendulum



$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

Pendulum at Equilibrium



$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$x_1^* = 0,\ x_2^* = 0$, Downward equilibrium

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

This is the same as small signal analysis!

$x_1^* = \pi,\ x_2^* = 0$, Upward equilibrium

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

Talk about next lecture!

Discrete Time

$$\vec{x}(t+1) = f(\vec{x}(t))$$

$\vec{x} = \vec{x}*$ is an equilibrium <u>if:</u>

$$f(\vec{x}^*) = \vec{x}^*$$

(for cont. $f(\vec{x}^*) = 0$)

$$\tilde{x}(t) = \vec{x}(t) - \vec{x}^*$$

$$\begin{aligned}\tilde{x}(t+1) &= \vec{x}(t+1) - \vec{x}^* \\ &= f(\vec{x}(t)) - \vec{x}^* \\ &\approx \cancel{f(\vec{x}^*)} + \overbrace{\nabla f(\vec{x}^*)}^{A}\tilde{x}(t) - \cancel{\vec{x}^*}\end{aligned}$$

$$\boxed{\tilde{x}(t+1) = A\tilde{x}(t)}$$

Summary

- Described linearization about an equilibrium point
  - Continuous time
  - Discrete time
- Next time:
  - Conditions for stability of a linear systems
  - Discrete, First order and scalar
  - Vector case! (which leads to Eigen-value analysis)