

# EE16B

# Designing Information Devices and Systems II

Lecture 7A  
Stability of Linear State Models

# Last Time

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- Described linearization about an equilibrium point using Taylor approximation
  - Continuous time
  - Discrete time
- Started: Conditions for stability of linear systems

# Stability of Linear State Models

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Start with scalar system 1<sup>st</sup> order system:

$$x(t + 1) = ax(t) + bu(t)$$

Given initial condition  $x(0)$ :

$$x(1) = ax(0) + bu(0)$$

$$x(2) = ax(1) + bu(1)$$
$$= a^2x(0) + abu(0) + bu(1)$$

$$x(3) = a^3x(0) + a^2bu(0) + abu(1) + bu(2)$$

$$x(t) = a^t x(0) + a^{t-1} bu(0) + a^{t-2} bu(1) + \dots + a^0 bu(t-1)$$

# Stability of Linear State Models

Start with scalar system:

$$x(t+1) = ax(t) + bu(t)$$

Given initial condition  $x(0)$ :

$$x(t) = a^t x(0) + \boxed{a^{t-1} bu(0)} + a^{t-2} bu(1) + \dots + \boxed{a^0 bu(t-1)}$$

$$x(t) = \underbrace{a^t x(0)}_{\text{Initial condition}} + \underbrace{\sum_{k=0}^{t-1} a^{t-k-1} bu(k)}_{\text{input}}$$

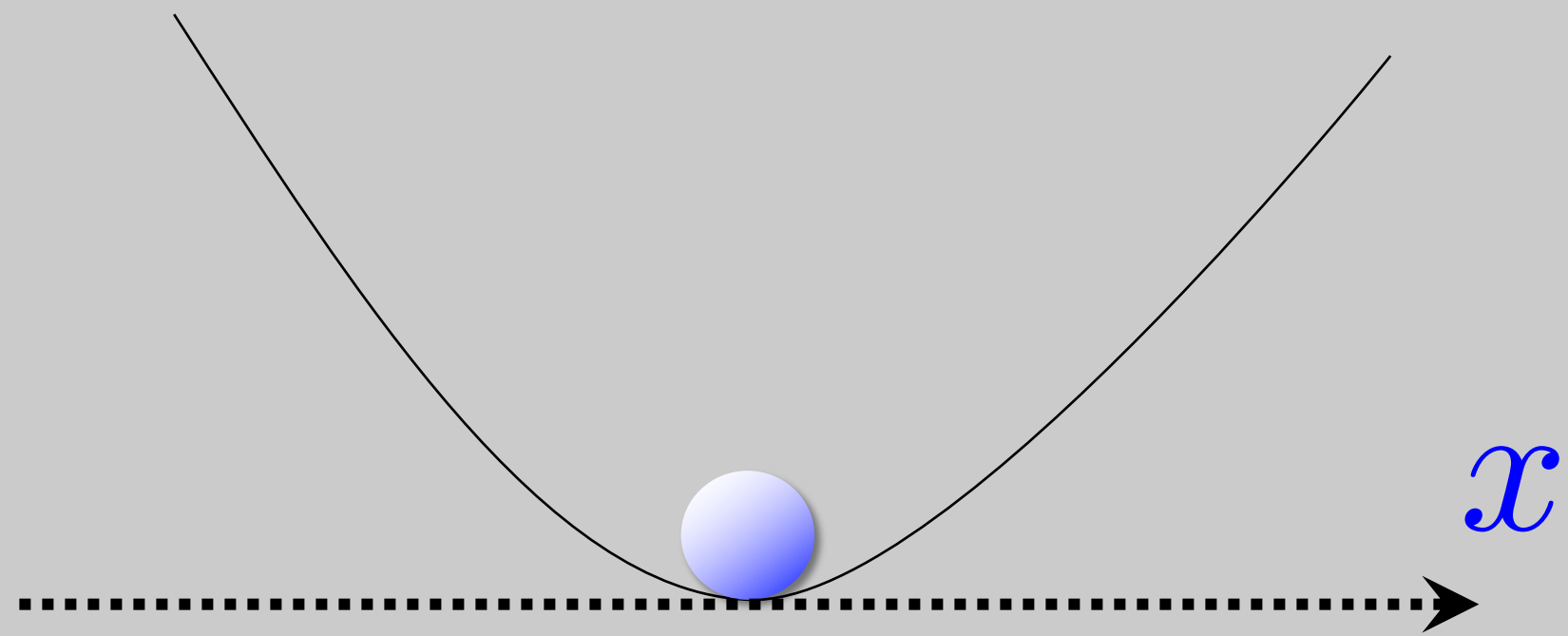
- A system is stable if  $\vec{x}(t)$  is bounded for any initial condition  $\vec{x}(0)$  and any bounded input sequence

$$u(0), u(1), \dots$$

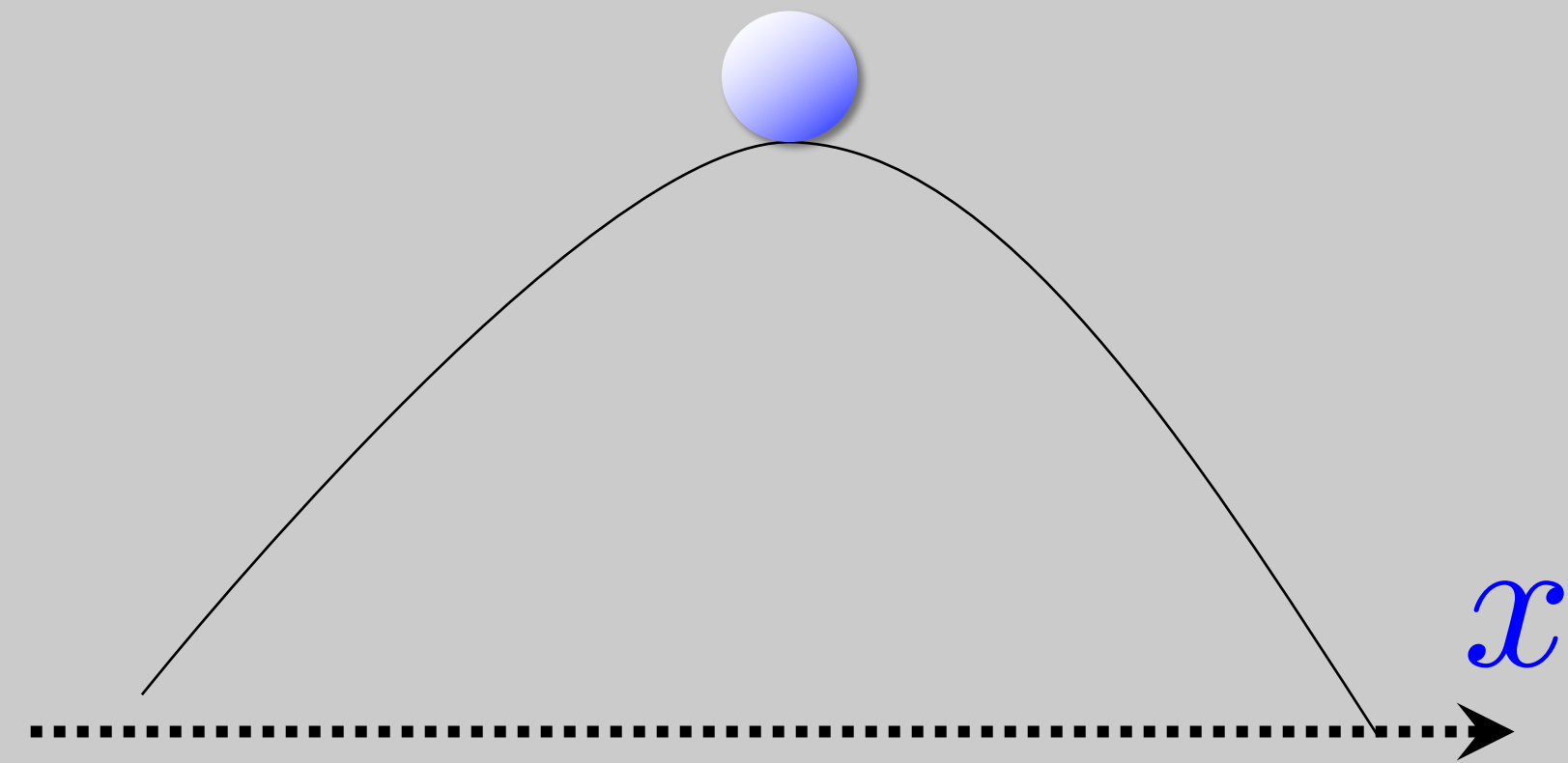
- A system is unstable if there is an  $\vec{x}(0)$  or a bounded input sequence for which

$$|\vec{x}(t)| \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty$$

# Example



stable



unstable

Q) Is this system stable?

$$x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$

A) Depends on  $|a|$

# Stability Proof

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$$x(t) = a^t x(o) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$

Claim 1: if  $|a| < 1$  then the system is stable

# Stability Proof

$$x(t) = a^t x(o) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$

Claim 1: if  $|a| < 1$  then the system is stable

Proof:  $a^t \rightarrow 0$  as  $t \rightarrow \infty$  because  $|a| < 1$  so,  
initial condition always bounded

Sequence is bounded – there exists  $M$  s.t.  $|u(t)| \leq M \forall t$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| \underbrace{|u(k)|}_{\leq M}$$

$|a_1 + a_2| \quad ? \quad |a_1| + |a_2|$



# Stability Proof Cont.

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| \underbrace{|u(k)|}_{\leq M}$$

Define:  $s = t - k - 1$

$$\leq \sum_{s=0}^{t-1} |a^s| |b| M = |b| M \sum_{s=0}^{t-1} |a|^s \leq |b| M \frac{1}{1 - |a|}$$

$$\sum_{s=0}^{\infty} |a|^s = \frac{1}{1 - |a|}, \quad |a| < 1$$

# Stability Proof Cont.

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Claim 2: unstable when  $|a| > 1$

Proof: if  $x(0) \neq 0$  (even  $u(t)=0 \forall t$ )

$$x(t) = a^t x(0) \rightarrow \infty$$

Q: What if  $|a| = 1$ , i.e.,  $a=1$  or  $a=-1$

A: Without input:  $x(t) = a^t x(0)$

$$x(t) = x(0), \quad \text{or} \quad x(t) = (-1)^t x(0)$$

With input  $u(t)=M$ ,  $a=1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} b M \right| \rightarrow \infty \quad \text{Not stable!}$$

# Quiz

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With input  $u(t)=M$ ,  $a=-1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b M \right| \leq bM$$

# Quiz

---

With input  $u(t)=M$ ,  $a=-1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b M \right| \leq bM$$

Q: what  $|u(t)| \leq M$  will make it unstable?

# Quiz

---

With input  $u(t)=M$ ,  $a=-1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b M \right| \leq bM$$

Q: what  $|u(t)| \leq M$  will make it unstable?

A:  $u(t) = (-1)^t M$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b (-1)^k M \right| = \left| \sum_{k=0}^{t-1} b M \right| \rightarrow \infty$$

# Stability of Linear State Models

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Previously, the scalar case:

$$x(t+1) = ax(t) + bu(t)$$

$$|a| < 1 \quad \Rightarrow \text{stable}$$

$$|a| \geq 1 \quad \Rightarrow \text{unstable}$$

Vector case:

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Solve with recursion:

$$\vec{x}(1) = A\vec{x}(0) + Bu(0)$$

$$\vec{x}(2) = A^2\vec{x}(0) + ABu(0) + Bu(1)$$

$$\vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} Bu(k)$$

# Stability – The Vector Case

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$$\vec{x}(t) = A^t \vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} B u(k)$$

Q: How do we determine stability?

A (partial): Not simple as in the scalar case – the state variables and inputs are coupled.

Approach: Let's change variables, to decouple them

# Change of Variables - Diagonalization

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

$$\vec{z}(t) = T\vec{x}(t)$$

$$\vec{z}(t+1) = T\vec{x}(t+1)$$

$$= T \underbrace{A\vec{x}(t)}_{T^{-1}\vec{z}(t)} + TBu(t)$$

$$A_{\text{new}} = TAT^{-1} \qquad B_{\text{new}} = TB$$

Q: What  $T$  to choose?

Similarity transformation

A: Choose  $T$  s.t.  $A_{\text{new}}$  is diagonal

Remember eigen values of new system same as original!



# Diagonalization

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\vec{z}(t+1) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \vec{z}(t) + \underbrace{B_{\text{new}} u(t)}_{\vec{v}(t)}$$

$$\vec{v}(t) = B_{\text{new}} u(t)$$

$$z_1(t+1) = \lambda_1 z_1(t) + v_1(t)$$

# Diagonalization

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Diagonalization = decoupling!

$$z_1(t+1) = \lambda_1 z_1(t) + v_1(t)$$

$$z_2(t+1) = \lambda_2 z_2(t) + v_2(t)$$

⋮

$$z_n(t+1) = \lambda_n z_n(t) + v_n(t)$$

Stable if:  $|\lambda_i| < 1$ ,  $i = 1, 2, \dots, n$

unstable if:  $|\lambda_i| \geq 1$ ,  $i = 1, 2, \dots, n$

Remember eigen values of new system same as original!

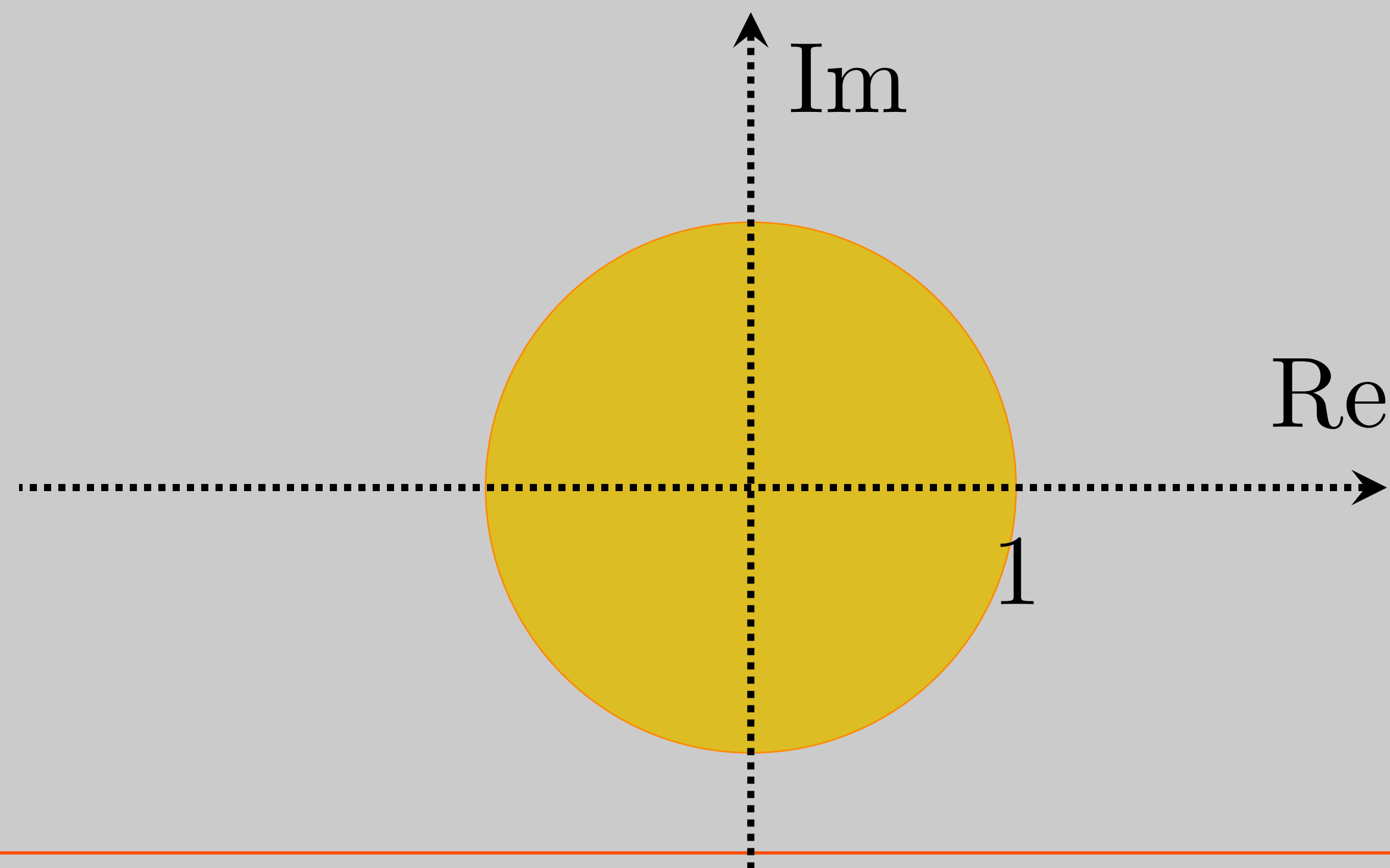
# Stability Cont.

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What if eigenvalues are complex valued?

Stable if:  $|\lambda_i| < 1, \quad i = 1, 2, \dots, n$

unstable if:  $|\lambda_i| \geq 1, \quad i = 1, 2, \dots, n$



# Non-Diagonalizable Systems

Q: What if  $A$  is not diagonalizable?

A: Transform to upper diagonal form (always possible)

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{beyond 16B material}$$

Stable if:  $|\lambda_i| < 1, \quad i = 1, 2, \cdots, n$

unstable if:  $|\lambda_i| \geq 1, \quad i = 1, 2, \cdots, n$

# Non-Diagonalizable Proof

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ & & \ddots & \\ & & & \lambda_{n-1} & \star \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Show stability for  $z_n$  :

$$z_n(t+1) = \lambda_n z_n(t) + v_n(t) \quad |\lambda_n| < 1$$

$z_n$  is bounded, show stability for  $z_{n-1}$  :

$$z_{n-1}(t+1) = \lambda_{n-1} z_{n-1}(t) + \underbrace{\star z_n(t) + v_{n-1}(t)}_{\text{treat as bounded input}}$$

Bounded if  $|\lambda_{n-1}| < 1$

show stability for  $z_i$  recursively !

Example of non-diagonalizable:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

# Stability of Cont.-Time Linear Systems

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$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Start with scalar  $x(t)$ :

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b \int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

# Stability of Cont.-Time Linear Systems

---

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Start with scalar  $x(t)$ :

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b \int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system stable?

# Stability of Cont.-Time Linear Systems

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b \int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system stable?


A: For  $a < 0$

Proof outline:

Show:

$$e^{at} \rightarrow 0, \quad t \rightarrow \infty$$

if  $|u(s)| \leq M \quad \forall s \Rightarrow \int \{ \} < \text{Const}$





# Stability of Cont.-Time Linear Systems

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$$x(t) = \underbrace{e^{at} x(0)}_{\text{Initial condition}} + \underbrace{b \int_0^t e^{a(t-s)} u(s) ds}_{\text{Due to input}}$$

Q: When is the system unstable?

A: For  $a \geq 0$

Proof : choose  $x(0) \neq 0$  and  $u(t) = M$

either “due to input” or “due to initial condition” explodes

# Stability of Cont.-Time Linear Systems

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Summary:

$$a < 0 \Rightarrow \text{stable}$$

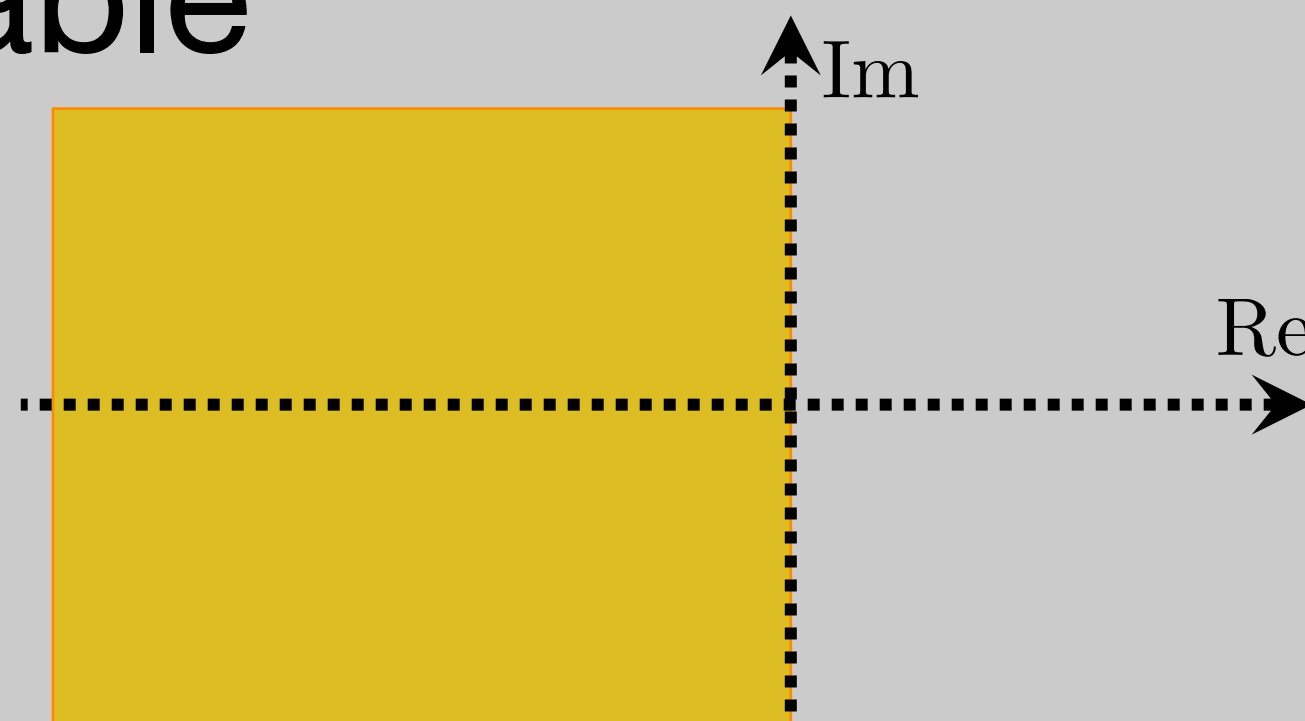
$$a \geq 0 \Rightarrow \text{unstable}$$

If  $a$  is complex, then:

$$\text{Re}\{a\} < 0 \Rightarrow \text{stable}$$

$$\text{Re}\{a\} \geq 0 \Rightarrow \text{unstable}$$

$$|e^{a_r + ia_i}| = |e^{a_r}| \cdot |e^{ia_i}| = |e^{a_r}|$$



# Stability of Cont.-Time Linear Systems

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Vector Case:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

Diagonalize:

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\vec{z}(t) = T\vec{x}(t)$$

$$\frac{d}{dt}z_i(t) = \lambda_i z_i(t) + v_i(t)$$

# Stability of Cont.-Time Linear Systems

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Stability test for

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$\operatorname{Re}\{\lambda_i(A)\} < 0 \quad \forall i \mid i = 1, 2, \dots, n \Rightarrow \text{stable}$$

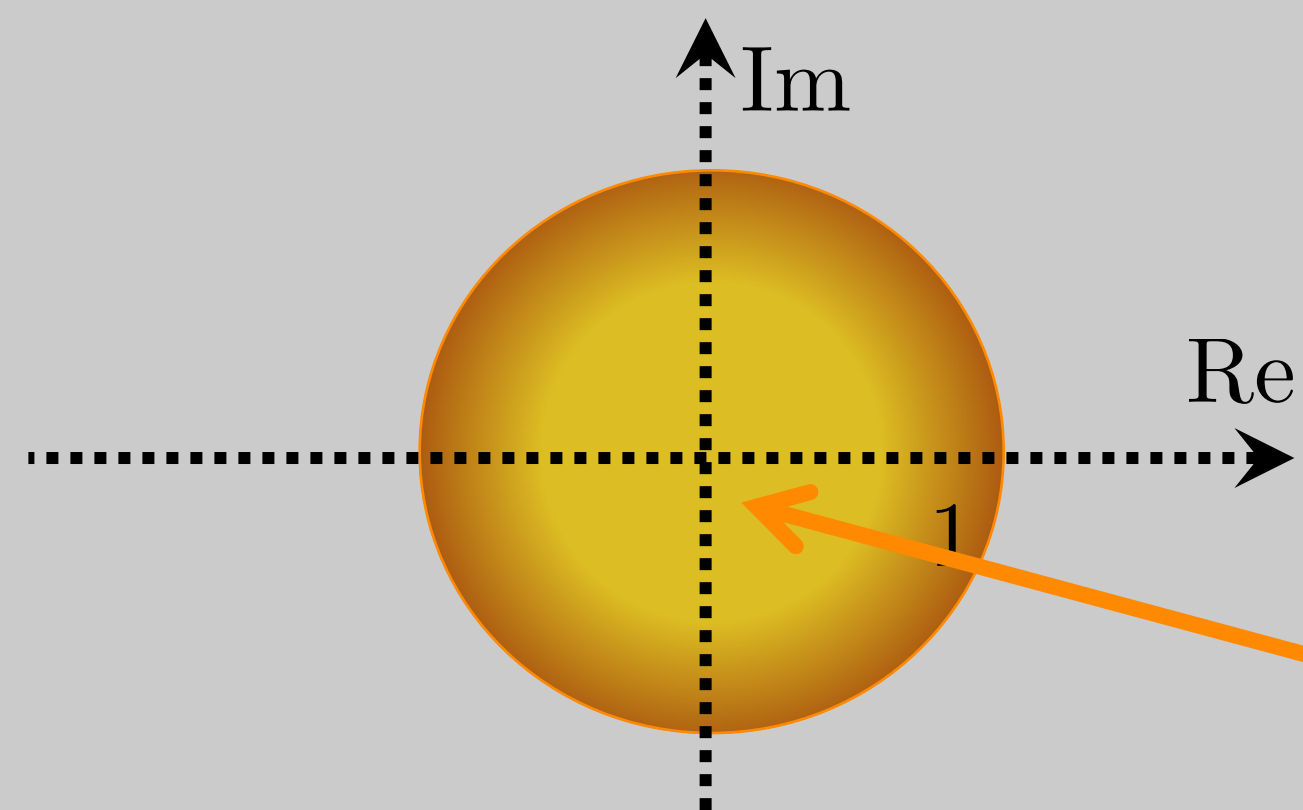
$$\operatorname{Re}\{\lambda_i(A)\} \geq 0 \quad \exists i \mid i = 1, 2, \dots, n \Rightarrow \text{unstable}$$

# Stability -- Summary

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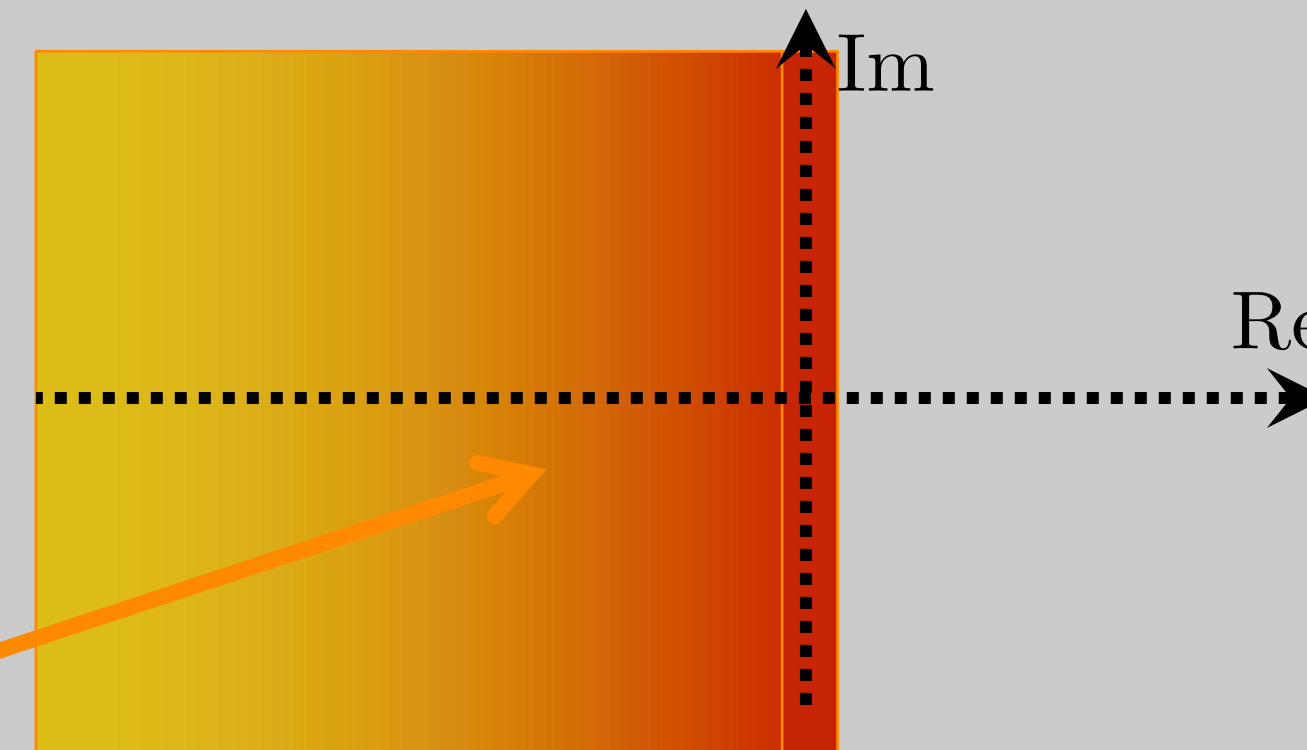
Discrete-Time

$$|\lambda_i(A)| < 1$$



Continuous-Time

$$\text{Real}\{\lambda_i(A)\} < 0$$



Stable regions

Stay away from boundaries! System uncertainty can  
Move you over to unstable region

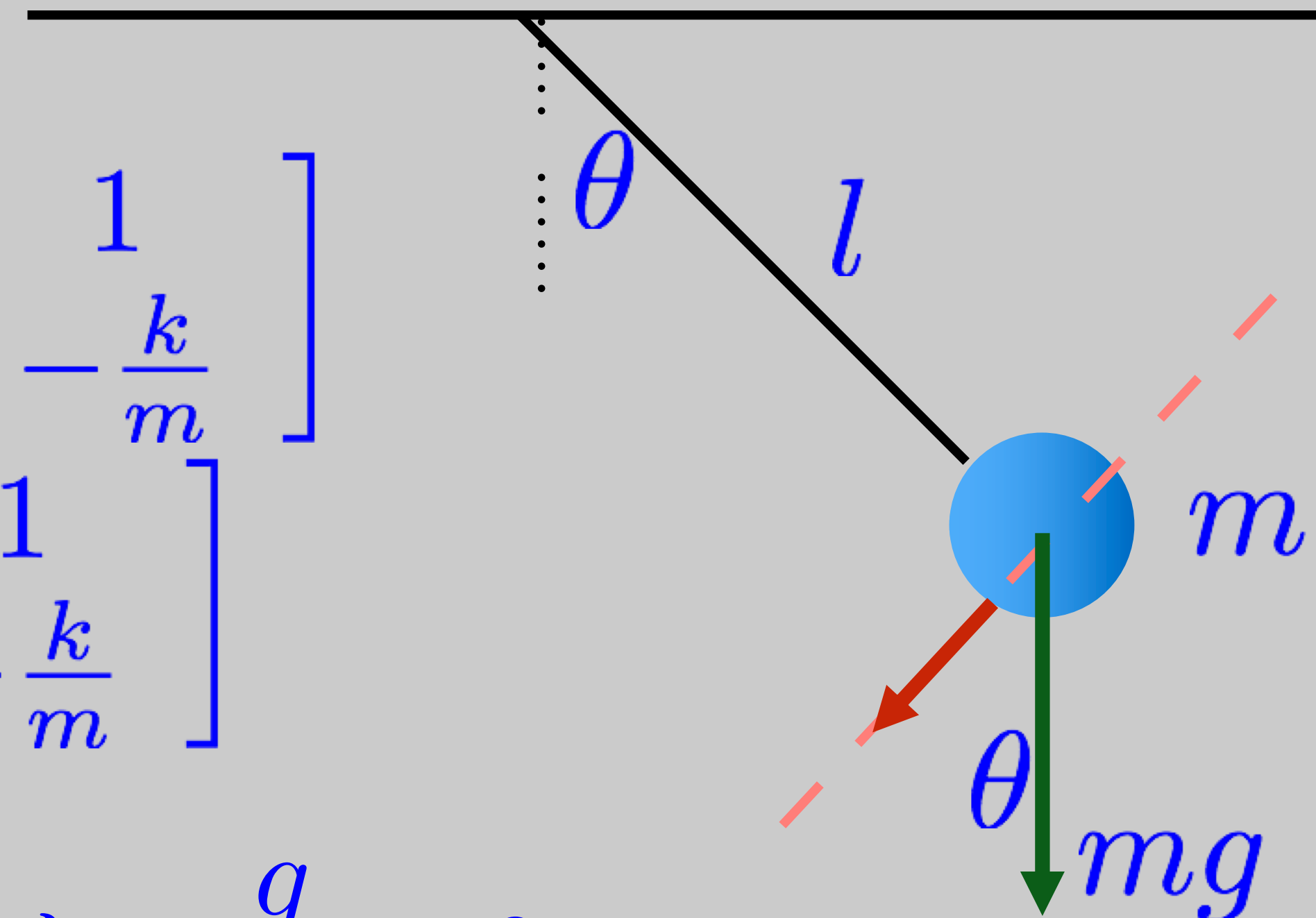
# Back to the Pendulum

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$|\lambda I - A_{\text{down}}| = \begin{vmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{k}{m} \end{vmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$



# Back to the Pendulum

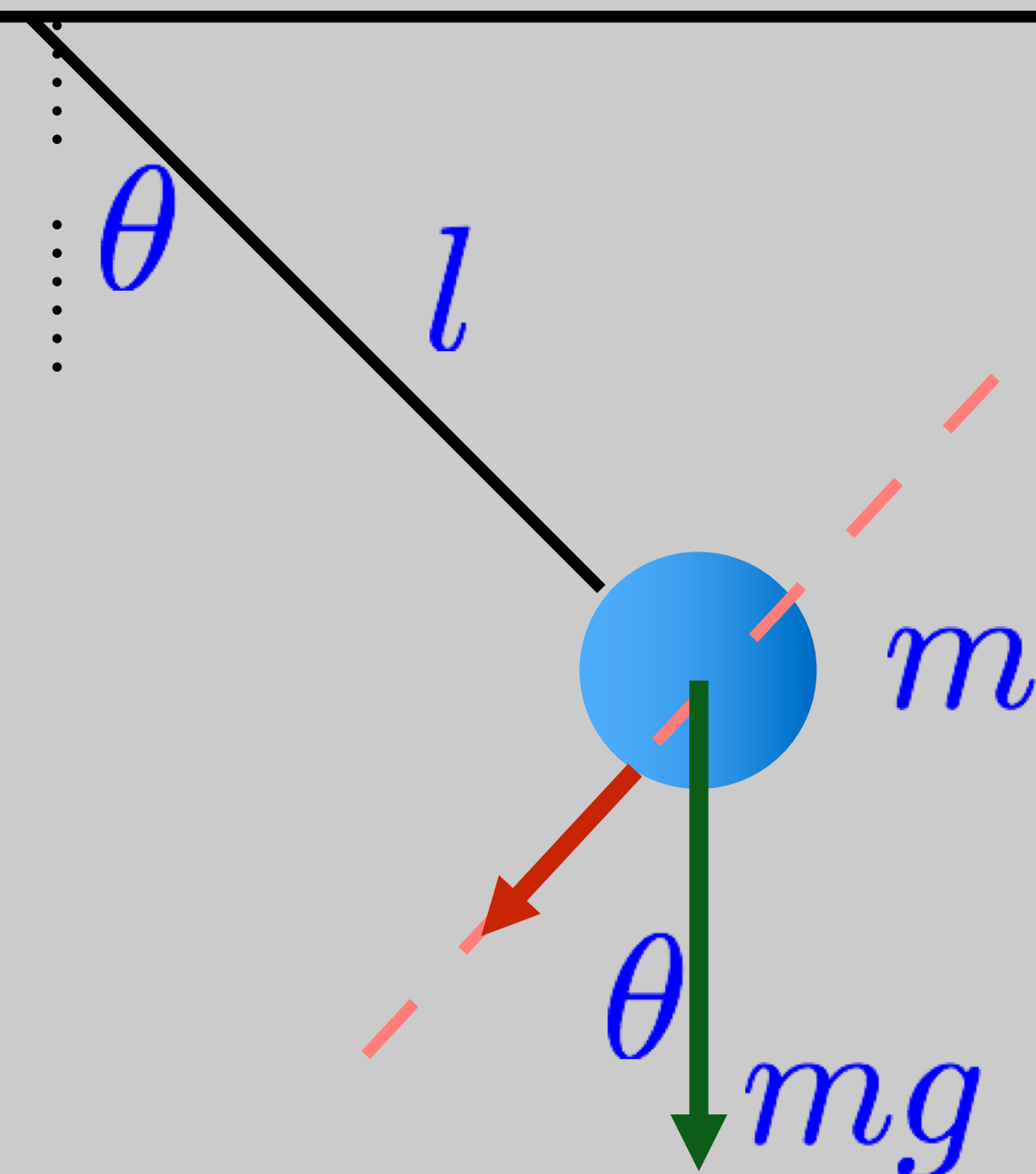
$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$

If  $\frac{k^2}{m^2} \geq 4\frac{g}{l}$ , i.e, sqrt is real, then  $\frac{k}{2m} \geq \frac{1}{2} \sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$

So,  $\lambda_{1,2}$  always negative -- stable!

If  $\frac{k^2}{m^2} < 4\frac{g}{l}$ , i.e, sqrt is imaginary, then  $\text{Re}\{\lambda_{1,2}\} = -\frac{k}{2m}$

So,  $\text{Re}\{\lambda_{1,2}\}$  always negative -- stable!



# Back to the Pendulum

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

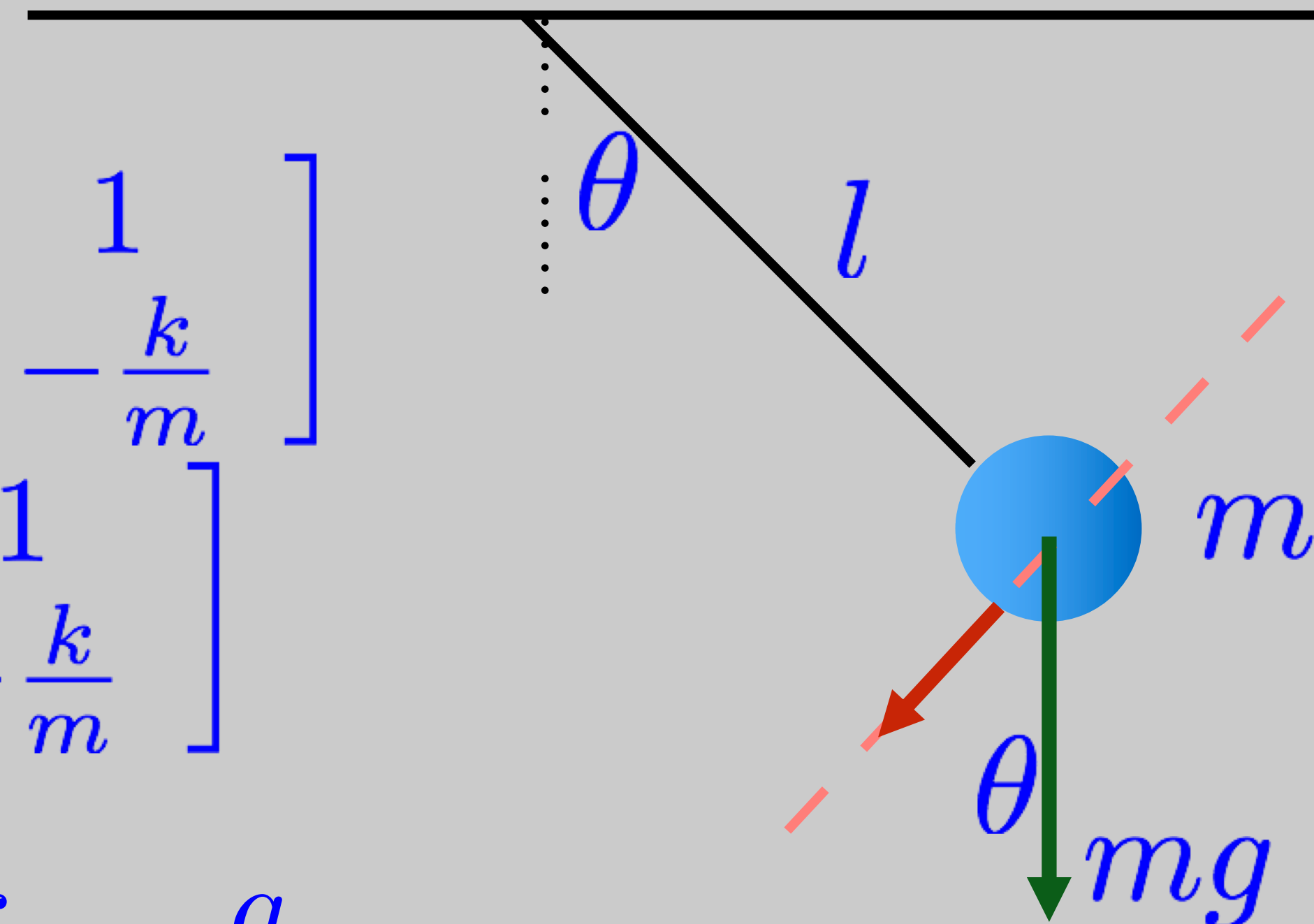
$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$|\lambda I - A_{\text{up}}| = \begin{vmatrix} \lambda & -1 \\ -\frac{g}{l} & \lambda + \frac{k}{m} \end{vmatrix} = \lambda^2 + \frac{k}{m}\lambda - \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$

$$\lambda_1 > 0$$

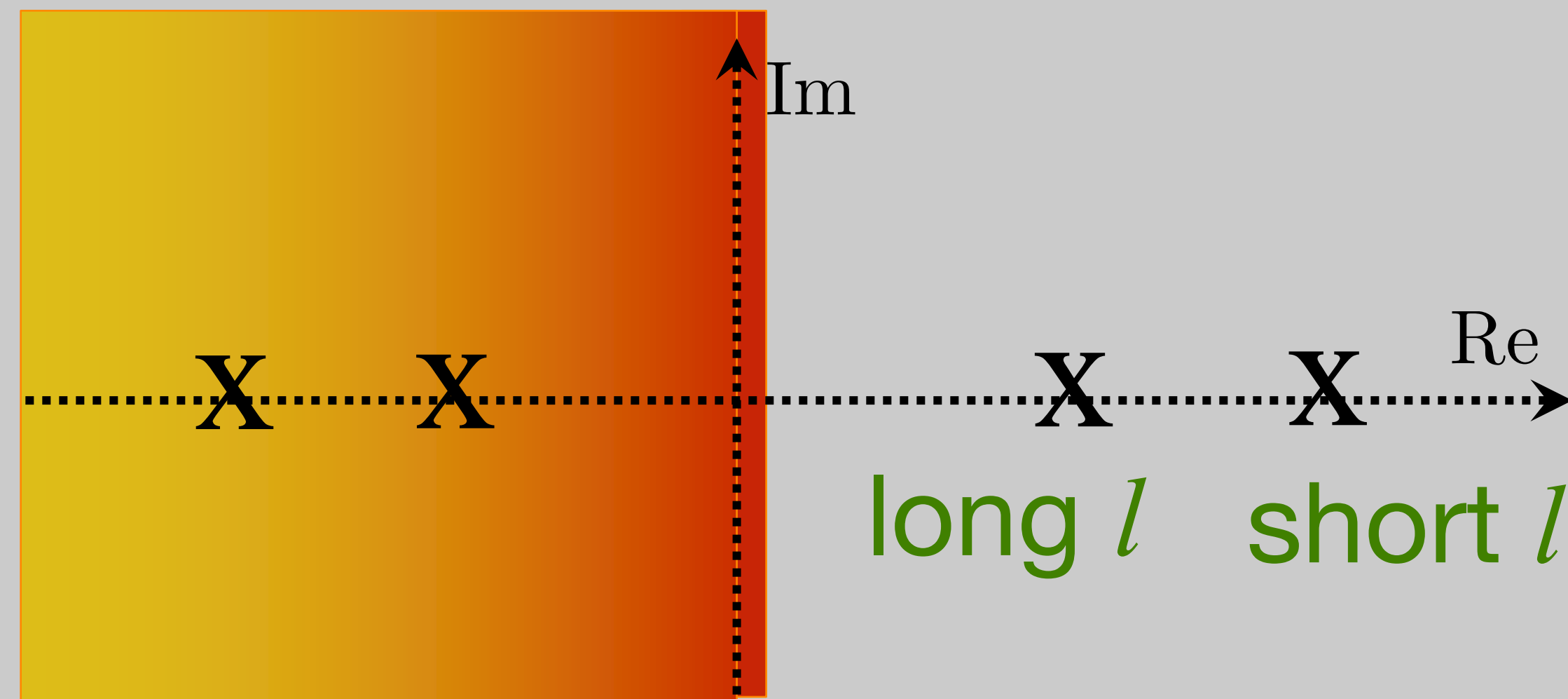
$$\lambda_2 < 0$$





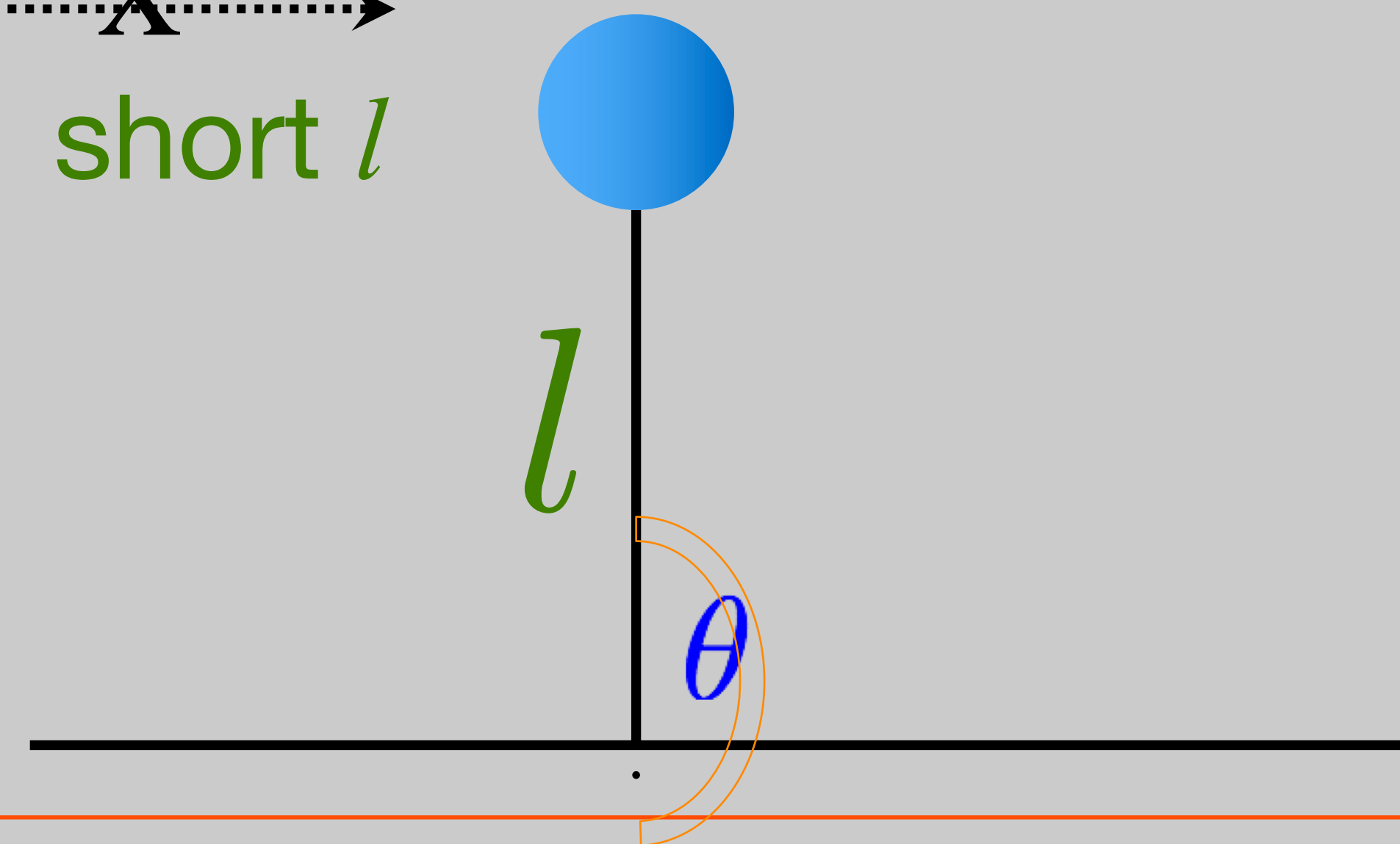
# Back to the Pendulum

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$



$$\lambda_1 > 0$$

$$\lambda_2 < 0$$

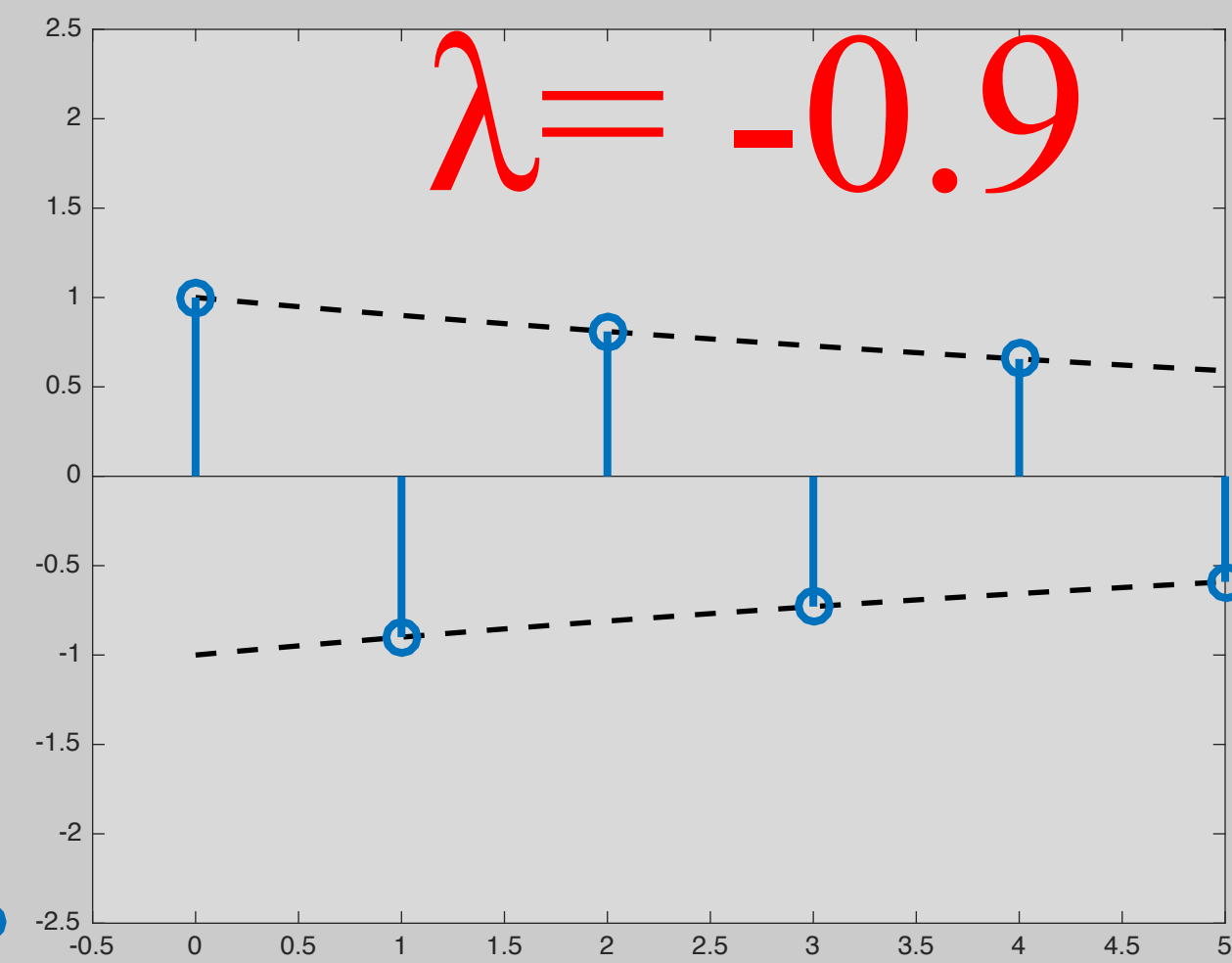
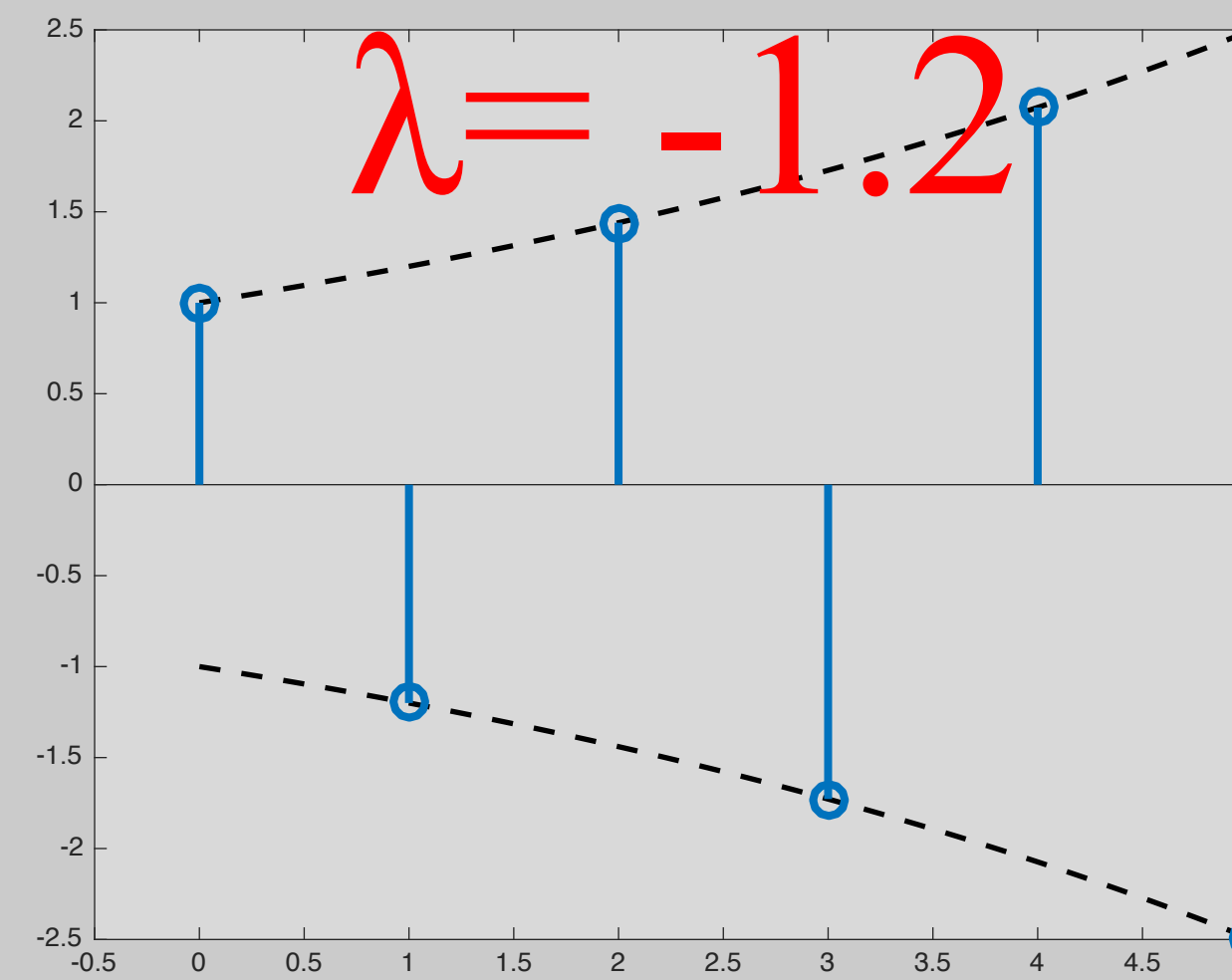
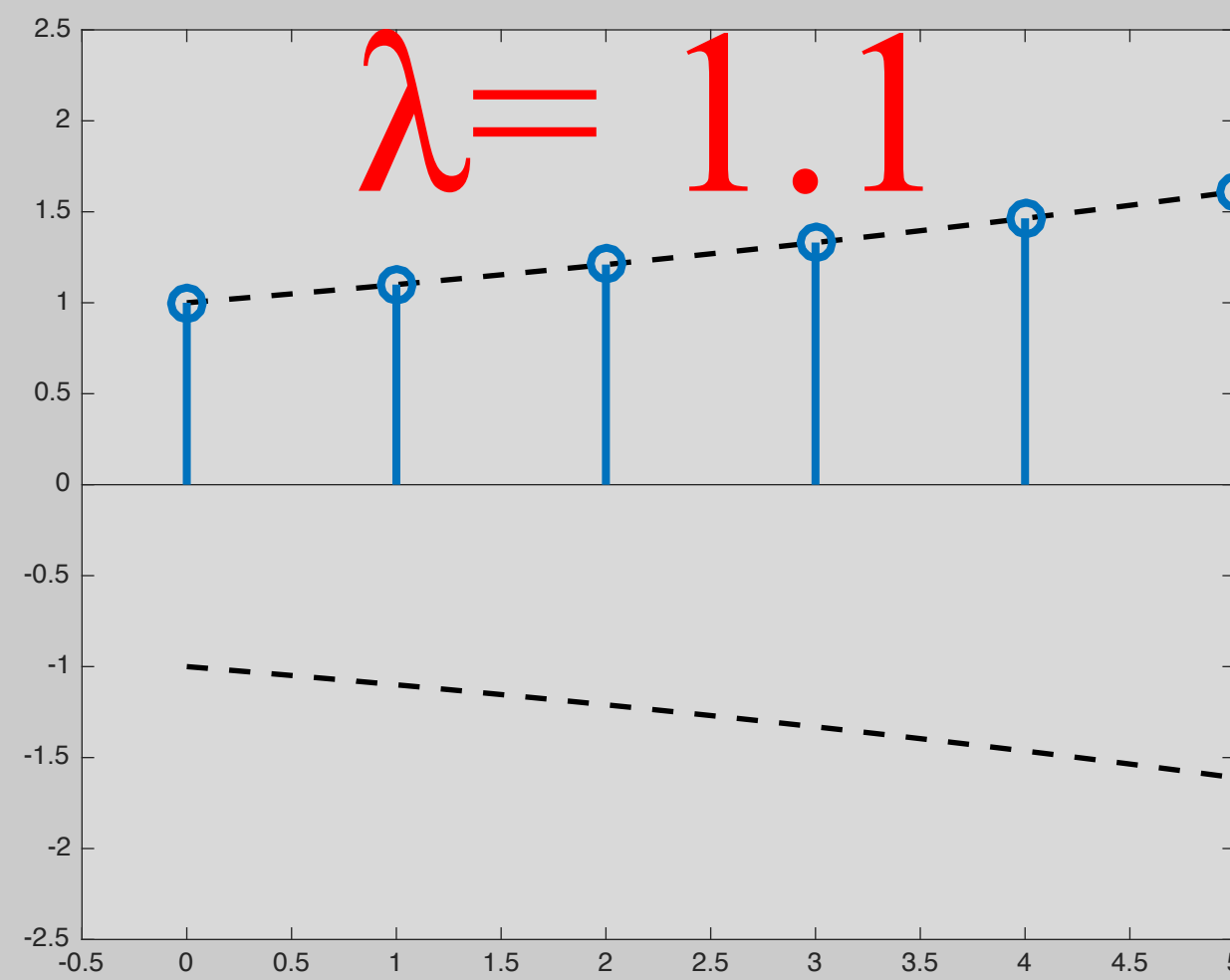
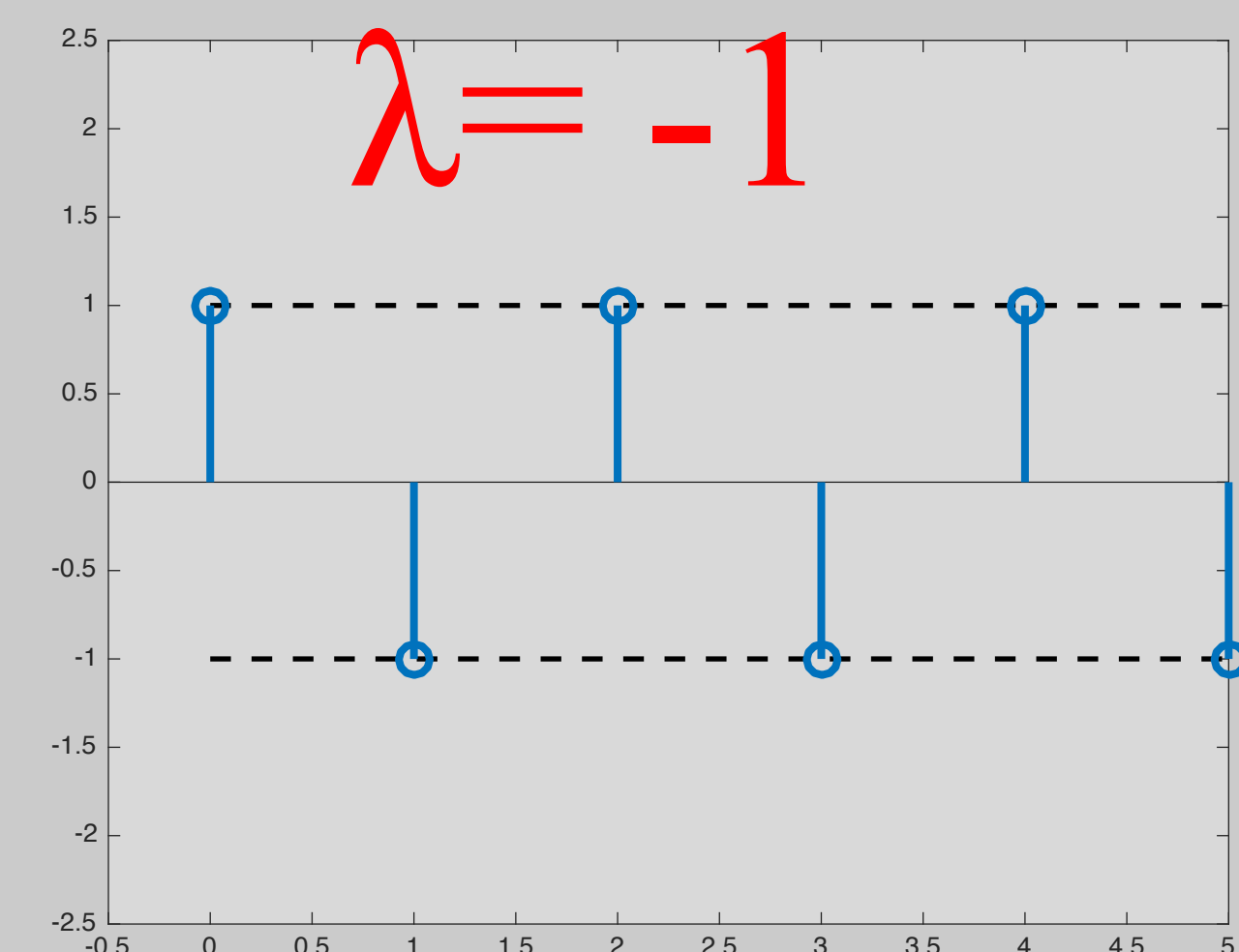
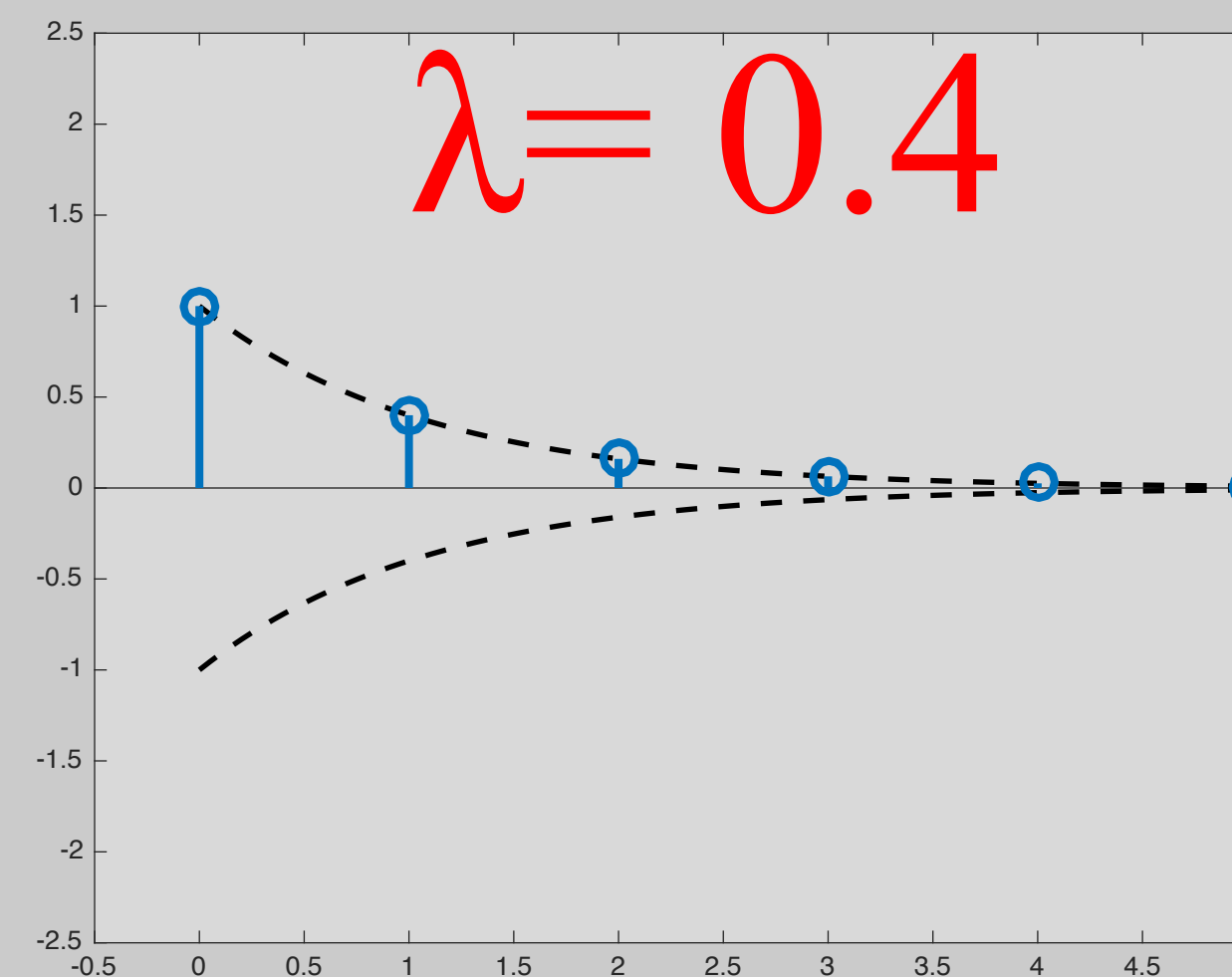
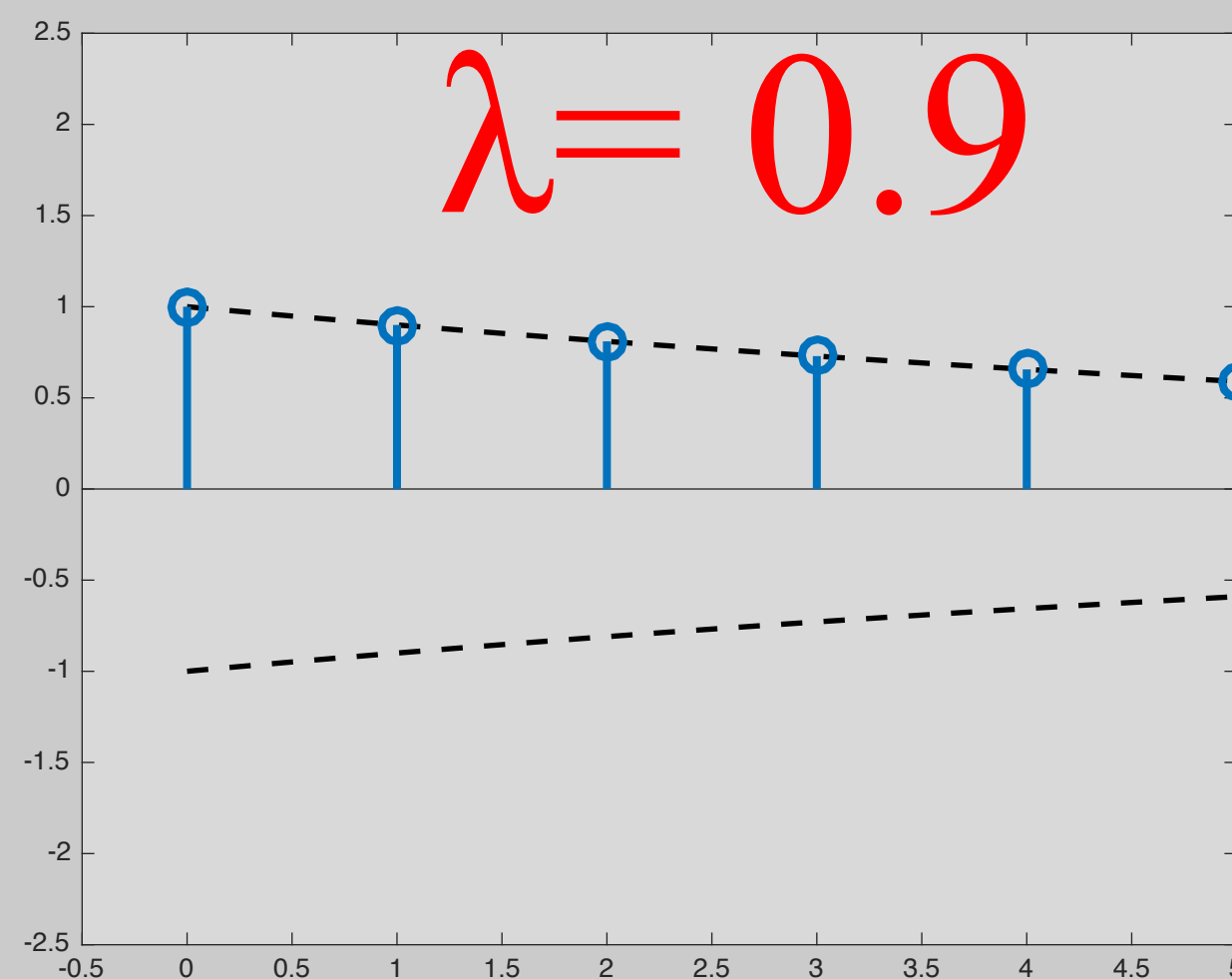


# Predicting System Behavior

$$z(t+1) = \lambda_i z(t)$$
$$\text{Soln : } \lambda_i^t z(0)$$

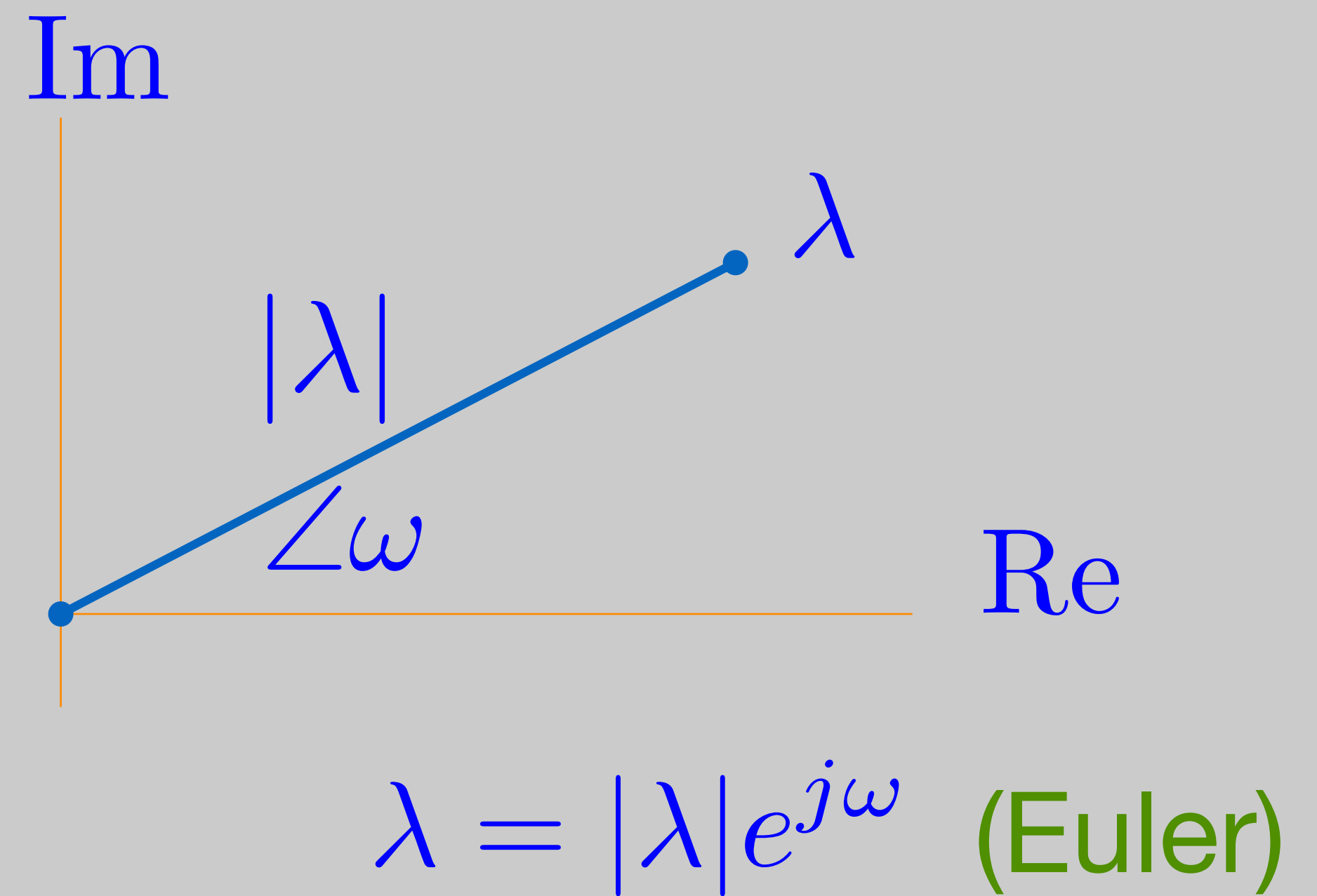
## Discrete Time

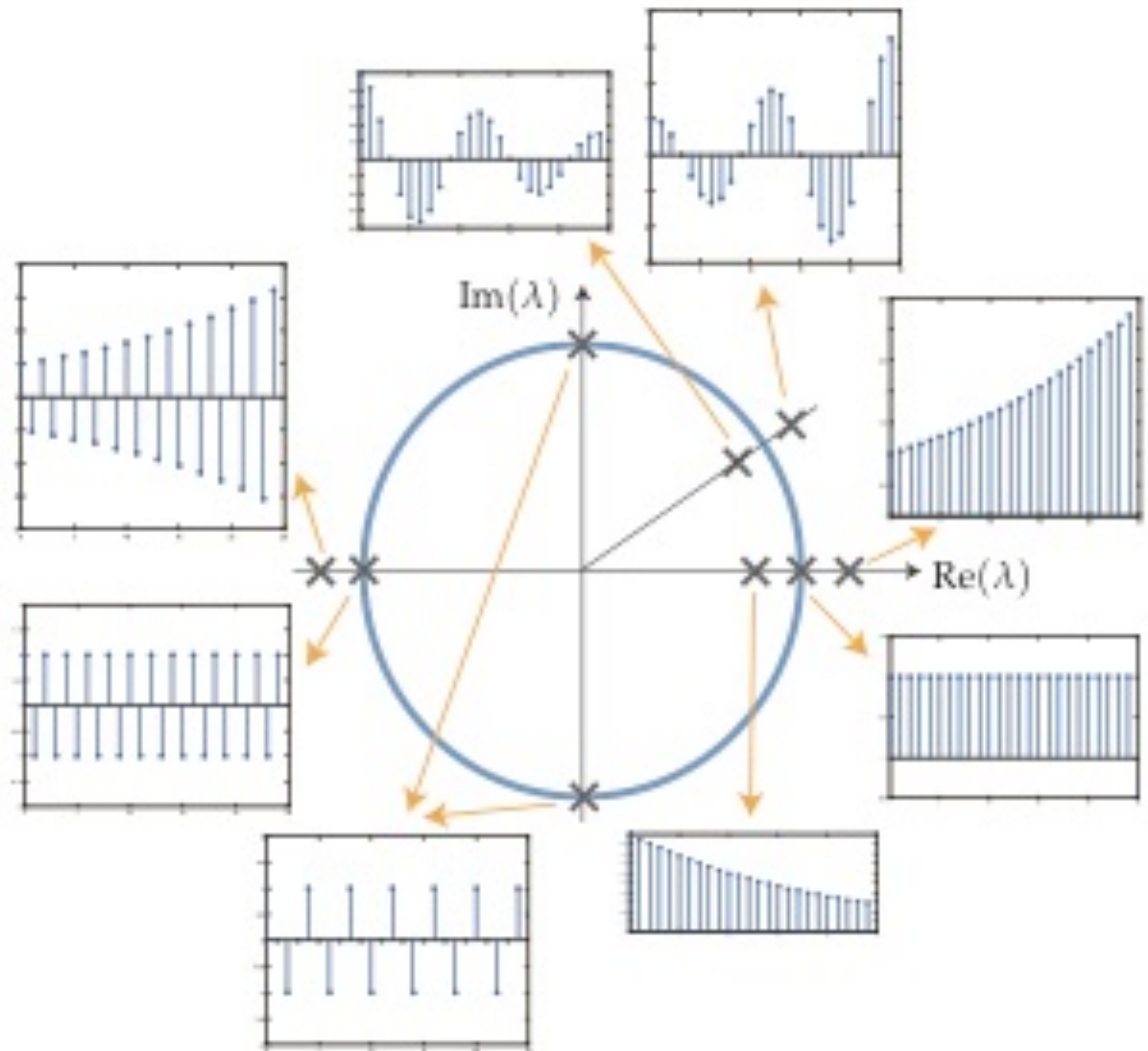
$$\lambda^t$$



- If  $\lambda$  is complex

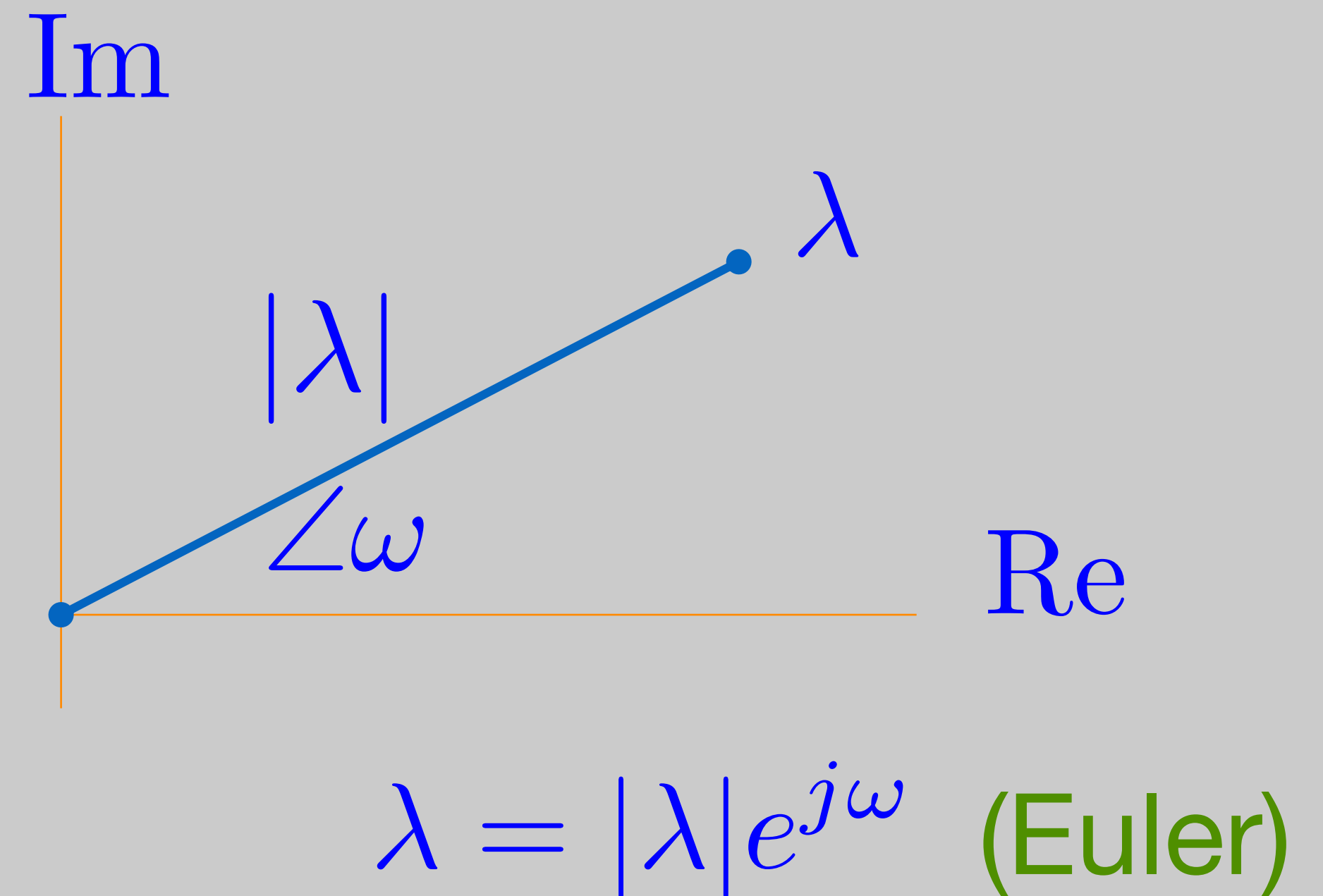
$$\begin{aligned}\lambda^t &= (|\lambda|e^{j\omega})^t \\ &= |\lambda|^t e^{j\omega t}\end{aligned}$$





- If  $\lambda$  is complex

$$\begin{aligned}\lambda^t &= (|\lambda|e^{j\omega})^t \\ &= |\lambda|^t e^{j\omega t}\end{aligned}$$



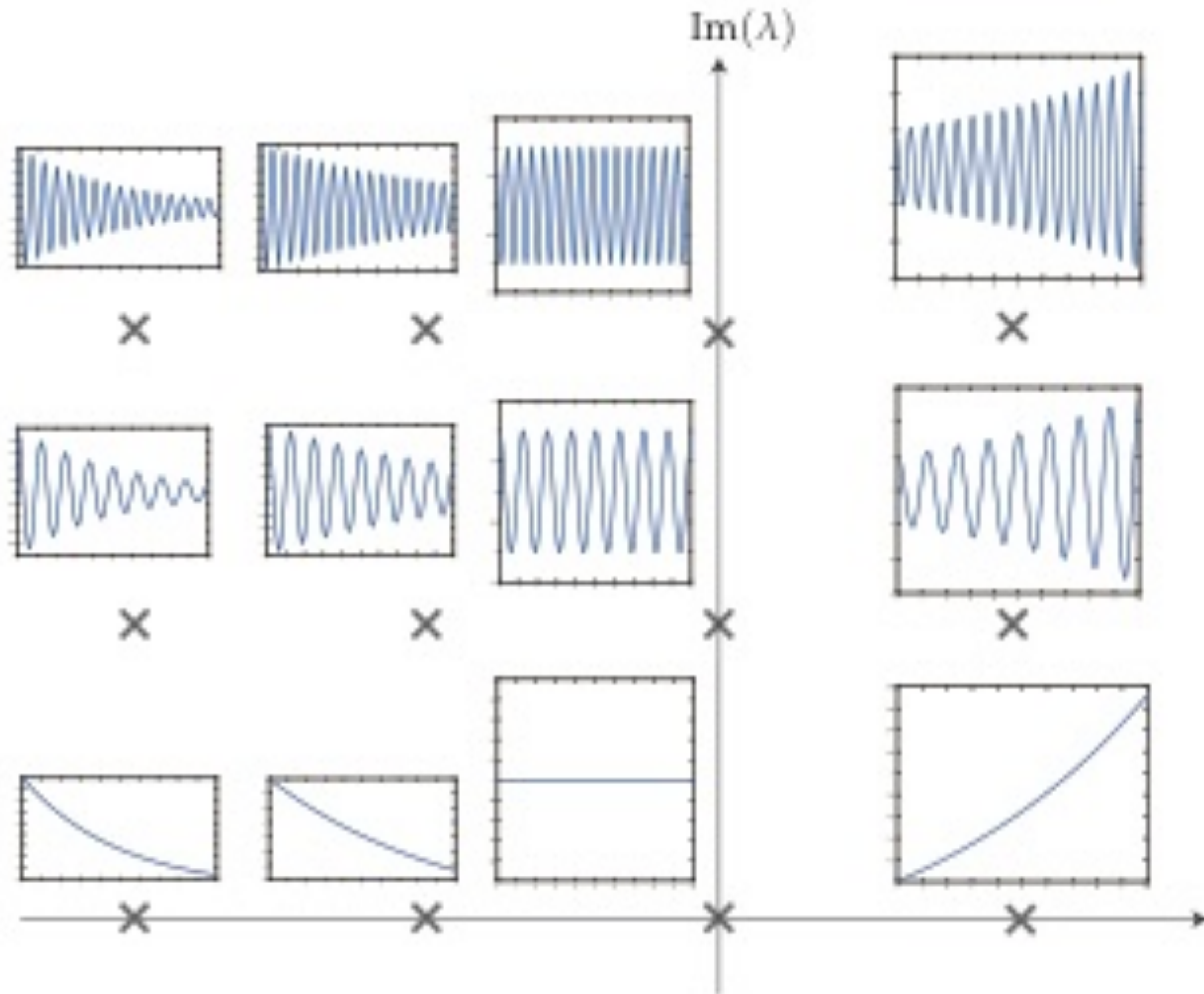
- Continuous time:

$$\frac{d}{dt}Z_i(t) = \lambda_i Z_i(t) \Rightarrow e^{\lambda_i t} Z_i(0)$$

Q) What does  $e^{\lambda t}$  look like for different choices of  $\lambda$ ?

A)  $\lambda = v + j\omega \Rightarrow e^{\lambda t} = e^{vt} e^{j\omega t}$





Re

# Summary

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- Derived stability conditions for vector discrete and continuous systems
- Showed that it is easy to analyze with change of variables!
- Prediction of system behaviour for different eigenvalues
  - For discrete – Phase (angle) determines frequency and magnitude determines relaxation
  - For continuous – Real part = relaxation, imaginary = frequency of oscillations
- Next time: Control design – putting the eigenvalues where we want them!