EE16B
Designing Information Devices and Systems II

Lecture 9B
Computing the SVD

Magnetic Susceptibility Artifacts on MRI: A Hairy Situation

A 15-year-old black boy was admitted to the hospital with headache, fever, proptosis of the left eye, and restricted gaze. He had presented to an emergency department the previous day with a 5-day history of a headache. After physical examination had revealed no significant findings and CT of the head reportedly had shown normal findings, the patient had been discharged home. MRI and repeated head CT performed at admission (Fig. 1A and 1B) revealed acute sinusitis complicated by a left periorbital abscess and multiple subdural empyemas. The patient then underwent bur hole drainage of the subdural empyemas, incision and drainage of the periorbital abscess, and endoscopic sinus surgery. After these procedures, the patient made a full recovery.

Of particular interest in this case was the unusual artifact seen on gadolinium-enhanced MRI. The appearance of the artifact suggested that it was caused by the patient's hair "twists," a style popular in the black community. Twisting the hair requires the use of products such as gel or beeswax to hold the hair in place. Hair stylists use either untinted beeswax or black beeswax tinted with pigments containing iron oxide. Further investigation revealed that our patient used black beeswax in his hair, causing the observed paramagnetic effect. Similar artifacts are commonly known to be caused by cosmetics containing iron and cobalt pigments.

This case is similar to one that Duncan [1] previously described in *AJR* as a culturally linked imaging artifact: an MR artifact was seen in a traditional healer in South Africa who used a clay paste containing iron oxide to dress her braids. However, cases such as ours are more likely to be encountered by radiologists in the United States, especially considering the growing popularity among American blacks of hair styles—such as twists and dreadlocks—that require the use of beeswax.

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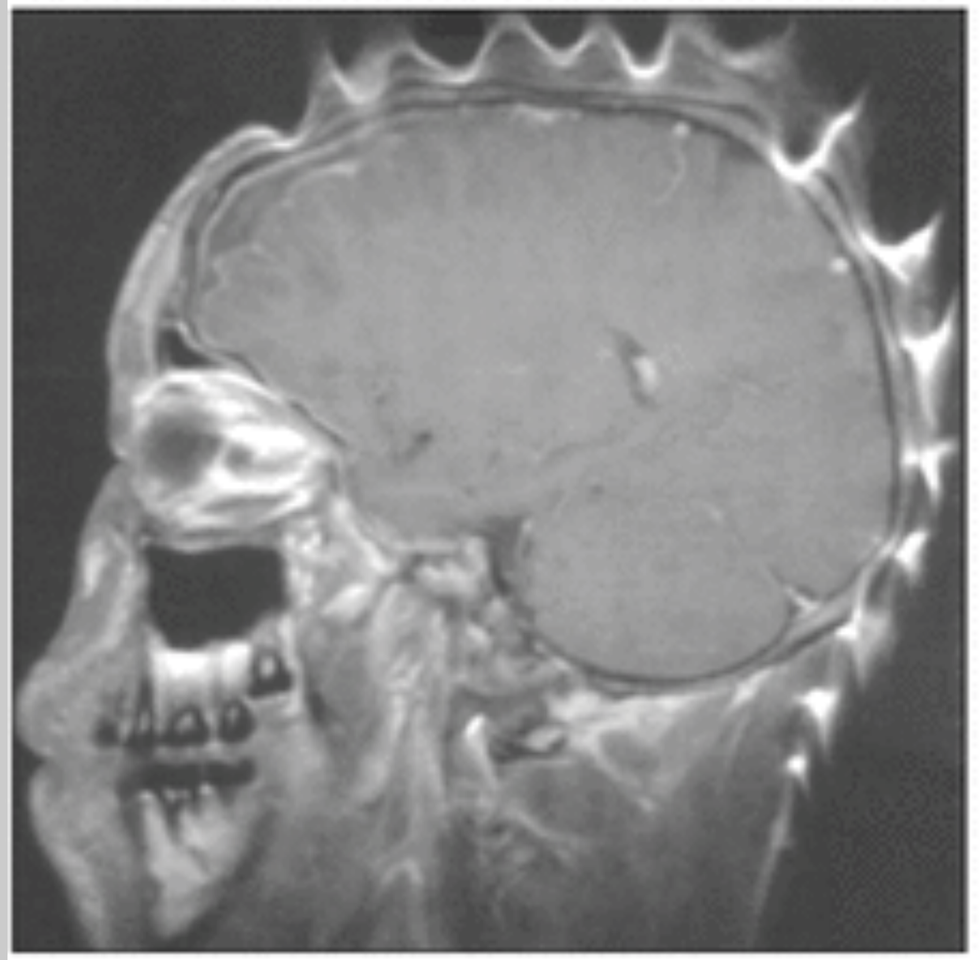
Reference

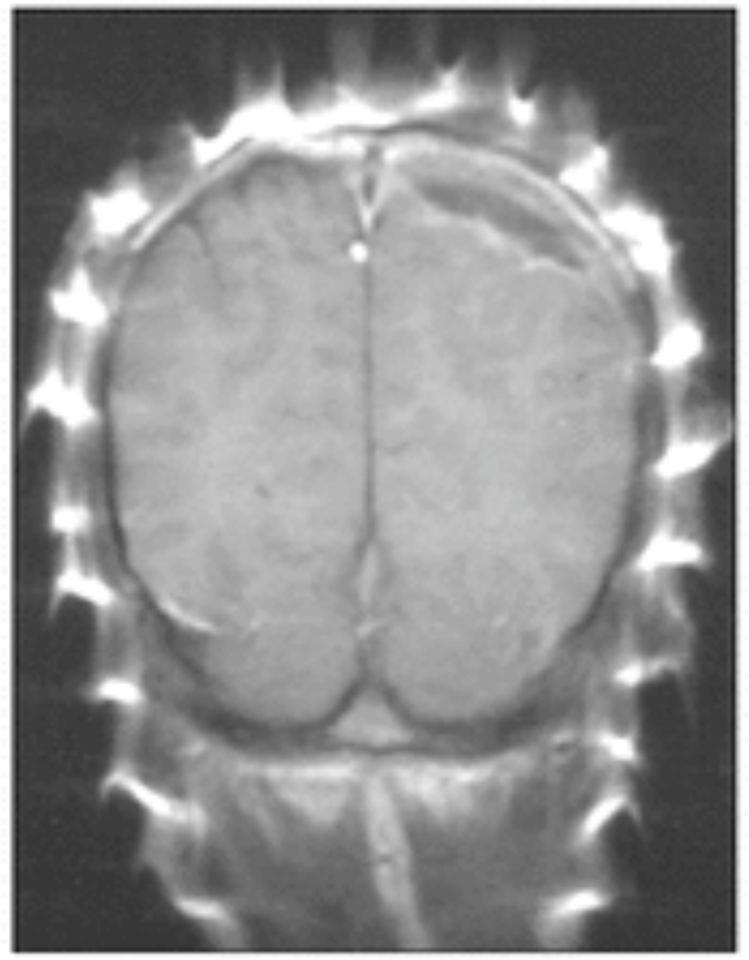
1. Duncan IC. The "aura" sign: an unusual cultural variant affecting MR imaging. (letter) *AJR* 2001;177:1487

Pyelovenous Backflow Seen on CT Urography

A 32-year-old woman who underwent a cesarean-section delivery and subsequent hysterectomy presented 2 weeks after surgery with fever, tachycardia, lower back pain, diffuse abdominal pain, and hematuria. Contrast-enhanced CT of the abdomen and pelvis revealed a large abdominopelvic abscess with thrombus in the right renal vein



A



B

Fig. 1.—15-year-old boy with acute sinusitis and subdural empyemas.
A and **B**, Gadolinium-enhanced sagittal (**A**) and coronal (**B**) T1-weighted images show left subdural empyema and image distortion from susceptibility artifacts caused by iron oxide particles suspended in beeswax dressing in patient's hair.

- Last time:
  - Described the SVD
  - Showed examples
- Today
  - Compact matrix form: $U_1 S V_1^T$
  - Full form: $U \Sigma V^T$
  - Show a procedure to SVD via $A^T A$
  - Maybe show procedure via AA^T

"I think you'll get a kick out of our 'haunted' MRI, Mrs. Hanratty."

SVD

SVD decomposes a rank r matrix $A \in \mathbb{R}^{m \times n}$
into a sum of r rank-1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

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1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \Rightarrow ||\vec{u}_i|| = 1 \quad \vec{u}_i \perp \vec{u}_j$

2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \Rightarrow ||\vec{v}_i|| = 1 \quad \vec{v}_i \perp \vec{v}_j$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Matrix Form of SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

Matrix Form of SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \quad S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \quad V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$

$m \times r$ $\qquad$ $r \times r$ $\qquad$ $n \times r$

$$A = U_1 S V_1^T$$

$$U_1^T U_1 = I_{r \times r}$$

$$V_1^T V_1 = I_{r \times r}$$

$$S \succ 0 \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Matrix Form of SVD

$$A = U_1 S V_1^T$$

"I tell ya, work has gotten to be so much more fun since we hung that rubber spider in there."

Full Matrix Form of SVD

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix}$$

$m \times r$

$$S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

$r \times r$

$$V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$

$n \times r$

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

$m \times m$

Full Matrix Form of SVD

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \quad m \times r$$

$$S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \quad r \times r$$

$$V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix} \quad n \times r$$

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \quad m \times m$$

$$\Sigma = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & 0 \end{array}\right] \quad m \times n$$

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \quad n \times n$$

$$A = U\Sigma V^T$$

$$U^T U = I_{m \times m}$$
$$V^T V = I_{n \times n}$$
$$\Sigma \succeq 0$$

Computing the SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ 2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

- What's the singular value of A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Computing the SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ 2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

- What's the singular value of A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \underbrace{\sqrt{2}}_{\sigma_1} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}_{\vec{u}_1} \underbrace{[1 \quad 0]}_{\vec{v}_1^T}$$

General Procedure for SVD

$$A \in \mathrm{R}^{m \times n}$$

1) Procedures based on A^TA (…and AA^T …later!)

General Procedure for SVD

$$A \in \mathrm{R}^{m \times n}$$

1) Procedures based on A^TA (…and AA^T …later!)

A^TA has only real eigenvalues, r of them are positive and the rest are zero
A^TA has orthonormal eigenvectors (to be proven next time)

Step1: Find eigenvalues of A^TA and order them from biggest to smallest $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \cdots 0$

Step2: Find orthonormal vectors: $\vec{v}_1, \cdots, \vec{v}_r : \; A\vec{v}_i = \lambda\vec{v}_i$

$$A = a \quad \Rightarrow \quad A^TA = a^2 \Rightarrow \lambda = a^2$$
$$A = a \quad \Rightarrow \quad \sigma = |a|$$

Step3: Set $\sigma_i = \sqrt{\lambda_i}$, and $\vec{u}_i = \dfrac{1}{\sigma_i}A\vec{v}_i$

Example

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 4 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1 \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sigma_1 = 2$$

$$\sigma_2 = 1$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Computing the SVD with A^TA

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$$

- Proof concept: let $A^T A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow A^T A V_1 = \Lambda V_1$

$$\sigma_i^2 = \lambda_i \qquad S^2 = \Lambda$$

Show that $A\vec{v}_i = \sigma_i \vec{u}_i$, where

$$\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow U_1^T U_1 = I_{r\times r}$$

Show that $A = U_1 S V_1^T$

Proof U_1 is orthonormal

- Let,

$$A\vec{v}_i = \hat{\sigma}_i \vec{u}_i \qquad i = 1, \cdots, r$$

$$(A\vec{v}_j)^T A\vec{v}_i = (A\vec{v}_j)^T \hat{\sigma}_i \vec{u}_i$$

$$(A\vec{v}_j)^T A\vec{v}_i = \hat{\sigma}_j \vec{u}_j^T \hat{\sigma}_i \vec{u}_i$$

$$\vec{v}_j^T \underbrace{A^T A\vec{v}_i}_{\sigma_i^2 \vec{v}_i} = \hat{\sigma}_j \hat{\sigma}_i \vec{u}_j^T \vec{u}_i$$

$$\sigma_i^2 \vec{v}_j^T \vec{v}_i = \hat{\sigma}_j \hat{\sigma}_i \vec{u}_j^T \vec{u}_i$$

Orthonormal!

$$\hat{\sigma}_j \hat{\sigma}_i \vec{u}_j^T \vec{u}_i = \begin{cases} \sigma_i^2 & i = j \\ 0 & i \neq j \end{cases}$$

Proof $A=U_1SV_1^T$

$$A\vec{v}_i = \sigma_i \vec{u}_i \qquad i = 1, \cdots, r$$

$$\Rightarrow AV_1 = U_1 S$$

$$AV_1V_1^T = U_1SV_1^T$$

← form we want!

- Need to show:

$$AV_1V_1^T = A$$

- We know:

$$A[V_1 \quad V_2][V_1 \quad V_2]^T = A$$

$[V_1 \quad V_2][V_1 \quad V_2]^T$: $VV^T = I_{n\times n}$

$$AV_1V_1^T + AV_2V_2^T = A$$

$AV_2V_2^T$: Show =0

Proof $A=U_1SV_1^T$

$$AV_1V_1^T = U_1SV_1^T$$ ← form we want!

$$AV_1V_1^T + AV_2V_2^T = A$$ ($AV_2V_2^T$: Show =0)

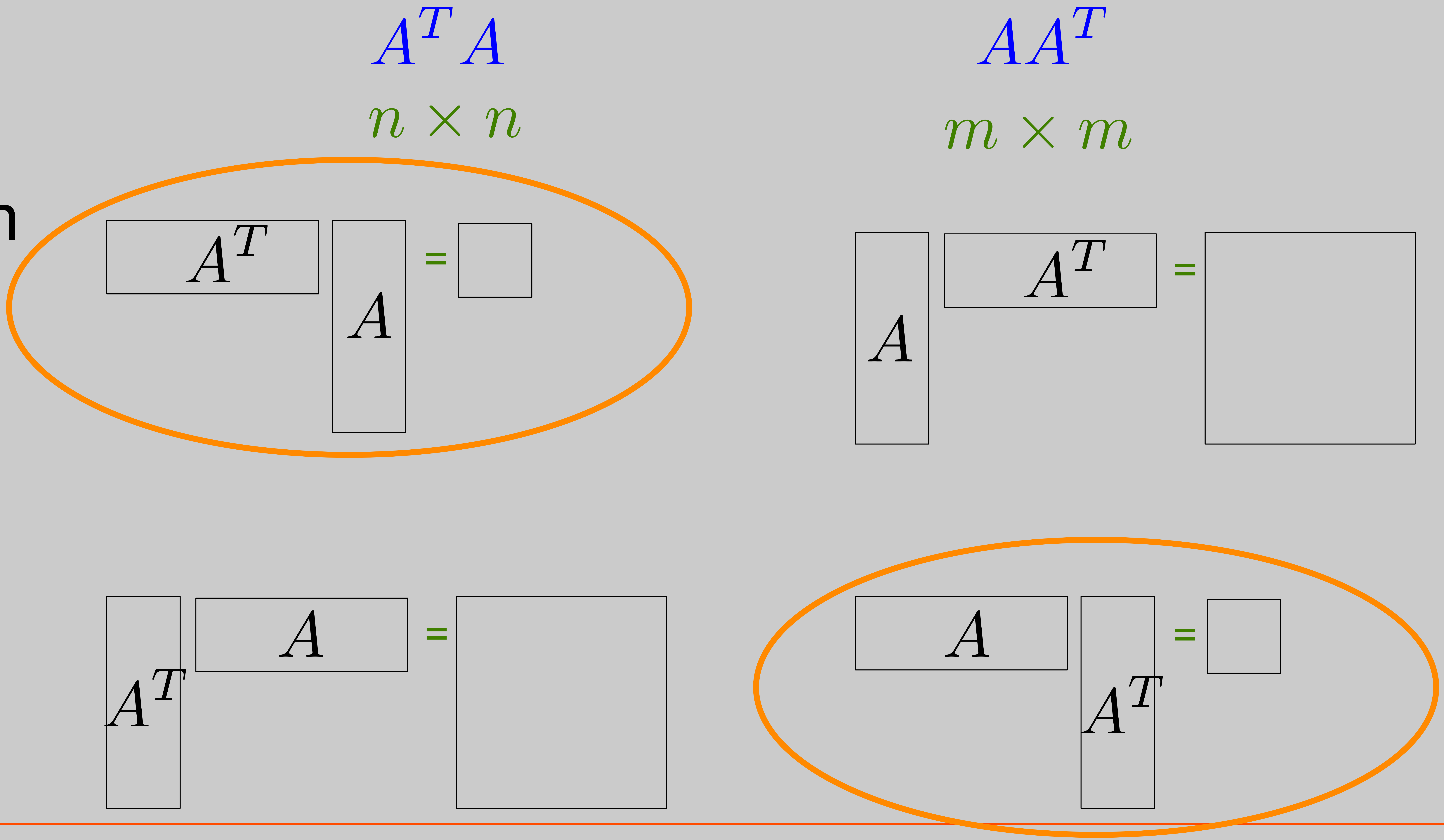
- We know: $A^TAV_2 = 0$
- If also, $AV_2 = 0$, we are done!

$$A\vec{v}_i \qquad i = r+1, r+2, \cdots, n$$

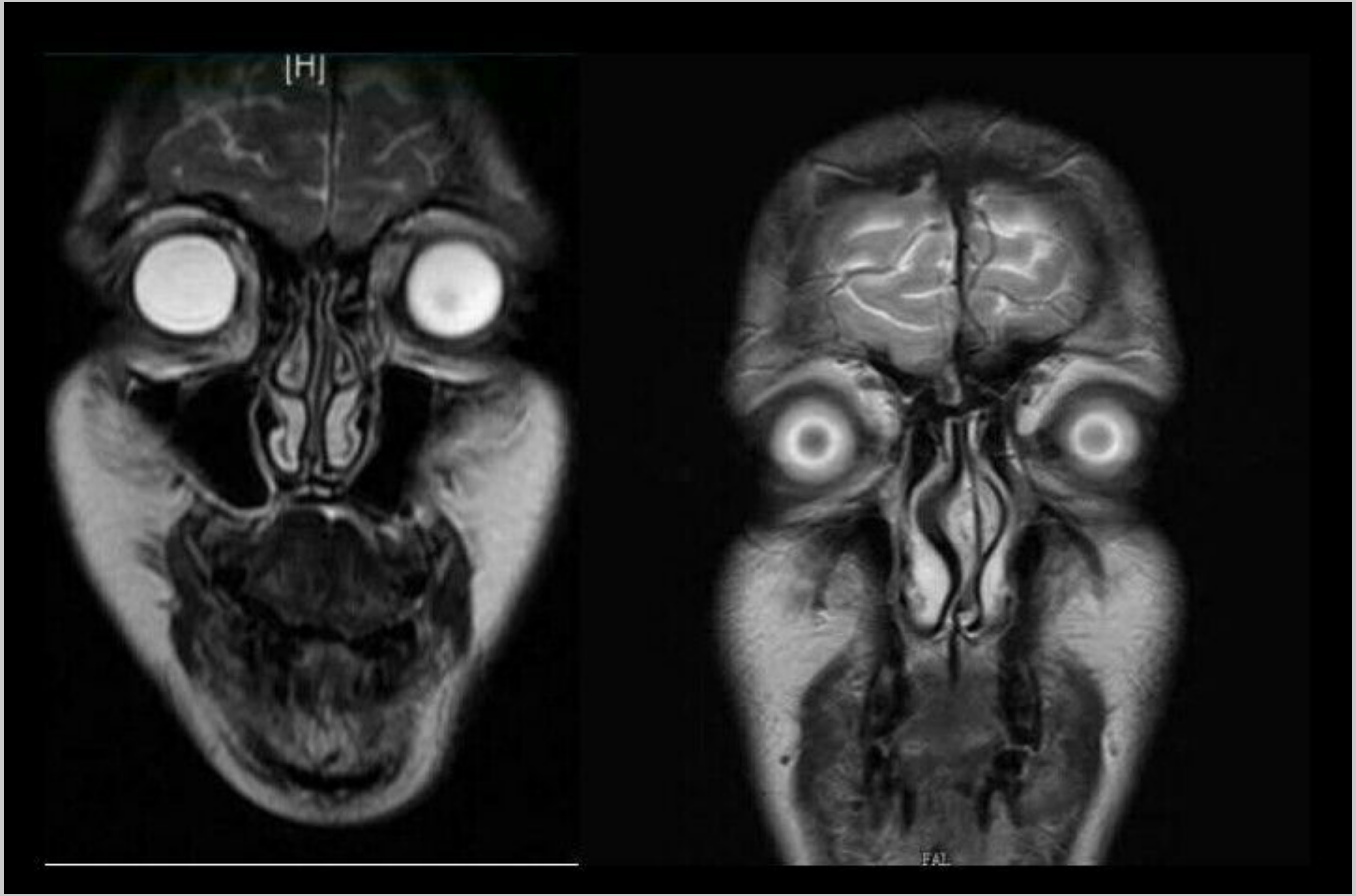
$$||A\vec{v}_i||^2 = (A\vec{v}_i)^T A\vec{v}_i$$
$$= \vec{v}_i^T A^T A\vec{v}_i = 0$$
$$\Rightarrow AV_2 = 0$$

Alternate Procedure using AA^T

- If, m > n
- If m < n



Smile!



Alternate Procedure using AA^T

Step 1: Find eigenvalues of AA^T and order s.t.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \cdots 0$$

Step 2: Find orthonormal eigenvectors of AA^T:

$$AA^T \vec{u}_i = \lambda_i \vec{u}_i \qquad i = 1, \cdots, r$$

Step 3: Set,

$$\sigma_i = \sqrt{\lambda_i} \qquad \vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i$$

Example

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \qquad r = 2$$

$$A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\lambda_1 = 32 \qquad \lambda_2 = 18$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i \qquad \vec{v}_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Signs of u_1,v_1 (u_2,v_2) can be flipped!