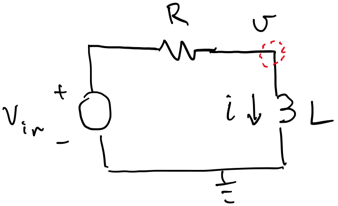
Inductor Circuits

Tuesday, September 22, 2020 9:14 AM

Ex 1:



Nodal Analysis

$$\left.\begin{aligned} \frac{v - V_{in}}{R} + i &= 0 \\ v - L\frac{d}{dt} i &= 0 \end{aligned}\right\} \text{2 eq's in 2 unknowns}$$

$$\frac{d}{dt} i = -\underbrace{\frac{R}{L}}_{\lambda} i + \frac{1}{L} v_{in}$$

- familiar as 1st order diff eqn.
- eigenvalue is $\lambda = -\frac{R}{L}$ [has units $\frac{rad}{s}$]
- time constant

$$\tau = \frac{L}{R} \underset{\text{units}}{\overset{}{=}} \frac{V\text{-}s/A}{V/A} = s \checkmark$$

$$L = \text{Henry's} = \frac{\lambda}{i} = \frac{\text{Volt-sec}}{\text{Amps}}$$

$$R = \text{ohms} : \Omega = \frac{V}{A}$$

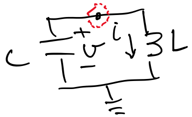
- Solution of problem, i.e. the diff eqn.

Ex 2 L - C ckt

Special Case

$L = 1\ H$

$C = 1\ F$



$$\begin{cases} C\frac{d}{dt}v + i = 0 \\ L\frac{d}{dt}i = v \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix} \underbrace{\begin{bmatrix} v \\ i \end{bmatrix}}_{\vec{x}} \; ; \quad \begin{bmatrix} v \\ i \end{bmatrix}(t_0) \ ✓$$

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x} \; ; \quad \vec{x}(0) \ ✓$$

Analysis - use eigen structure of A to solve

eigen-analysis : char. poly $= \det[\lambda I - A]$

$$\det\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0 \; ; \quad \sqrt{-1} = j$$

$$\lambda = \pm\sqrt{-1}$$

roots: $\lambda_1 = j$; $\lambda_2 = -j$

eigenvectors

$$\lambda_1 \leftrightarrow \begin{bmatrix} j & 1 \\ -1 & j \end{bmatrix} \vec{v}_1 = \vec{0} \; ; \; \vec{v}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

$$\lambda_2 \leftrightarrow \begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix} \vec{v}_2 = \vec{0} \; ; \; v_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

Observe: $\lambda_2 = \overline{\lambda_1}$; over bar means complex conj.

seems: eigenvalues & eigenvectors occur in complex conjugate pairs

true: why?

With real-valued matrix A:

$$\det(\lambda I - A)$$

$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

$\{a_0, \ldots, a_{n-1}\}$ real-valued since coefficients are sums of products of entries of matrix A

Fundamental Thm of Algebra

- n^{th} order polynomial has n roots over the complex numbers $\{\mathbb{C}\}$
- With a real-valued poly., roots are real or occur in conjugate pairs

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product rule for conjugation

$$Z = Z_r + jZ_i$$

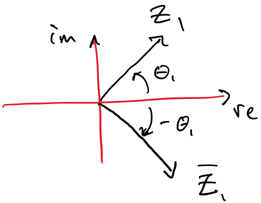
$$Z = |Z| e^{j\angle Z}$$

$$Z_1 = |Z_1| e^{j\theta_1} \qquad \theta_1 = \angle Z_1$$

$$Z_2 = |Z_2| e^{j\theta_2}$$

$$Z_1 \cdot Z_2 = |Z_1| e^{j\theta_1} \cdot |Z_2| e^{j\theta_2}$$

$$= |Z_1| \cdot |Z_2| e^{j(\theta_1+\theta_2)}$$



$$\bar{Z}_1 \cdot \bar{Z}_2 = |Z_1| e^{-j\theta_1} \cdot |Z_2| e^{-j\theta_2}$$

$$= |Z_1| \cdot |Z_2| e^{-j(\theta_1+\theta_2)}$$

$$= \overline{(Z_1 \cdot Z_2)}$$

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return to char. poly:

$p(\lambda) = 0$ has all real coeff's but λ_1 is complex

$$\lambda_1^n + a_{n-1}\lambda_1^{n-1} + \cdots + a_1\lambda_1 + a_0 = 0$$

$$\overline{p(\lambda)} = 0$$

$$\overline{\lambda_1^n} + \overline{a_{n-1}\lambda_1^{n-1}} + \cdots + \overline{a_1\lambda_1} + \overline{a_0} = 0$$

$$(\overline{\lambda_1})^n + a_{n-1}(\overline{\lambda_1})^{n-1} + \cdots + a_1\overline{\lambda_1} + a_0 = 0$$

$\Rightarrow$ $\overline{\lambda_1}$ is also a root, thus
$\overline{\lambda_1}$ also an eigenvalue

What about eigenvectors?

For complex-valued pair $\{\lambda_1, \vec{v}_1\}$

$$A\vec{v}_1 = \lambda_1\vec{v}_1$$

$$\overline{Av_1} = \overline{\lambda_1\vec{v}_1}$$

$$A\overline{v}_1 = \overline{\lambda_1}\cdot\overline{\vec{v}_1}$$

Yes: $\overline{\vec{v}_1}$ is an eigenvector, and
corresp to $\overline{\lambda_1}$

Back to L-C example:

$$\lambda_1 = j \quad \vec{v}_1 = \begin{bmatrix}1\\-j\end{bmatrix} \quad ; \quad \lambda_2 = -j \quad \vec{v}_2 = \begin{bmatrix}1\\j\end{bmatrix}$$

init. condition: $\vec{x}(0) = \begin{bmatrix}v\\i\end{bmatrix}(0) = \begin{bmatrix}1\text{V}\\0\text{A}\end{bmatrix}$

Homog. Solution: sum of the two "modes"

$$\vec{x}(t) = \vec{v}_1 \underset{\tilde{x}_1(0)}{e^{\lambda_1 t}} + \vec{v}_2 e^{\lambda_2 t} \tilde{x}_2(0)$$

$$\vec{x}(0) = \underset{\begin{bmatrix}1\\-j\end{bmatrix}}{\vec{v}_1} \tilde{x}_1(0) + \underset{\begin{bmatrix}1\\j\end{bmatrix}}{\vec{v}_2} \tilde{x}_2(0)$$

$\tilde{x}_1(0) = \frac{1}{2}$ and $\tilde{x}_2(0) = \frac{1}{2}$ will work

$$\vec{x}(0) = \frac{1}{2}\begin{bmatrix}1\\-j\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\j\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

for other times

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix}1\\-j\end{bmatrix} e^{jt} + \frac{1}{2}\begin{bmatrix}1\\j\end{bmatrix} e^{-jt}$$

Euler identity:

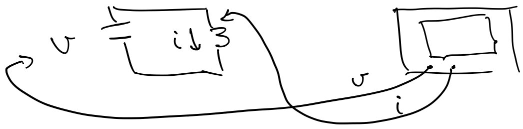
$$e^{j\theta} = \cos\theta + j\sin\theta$$

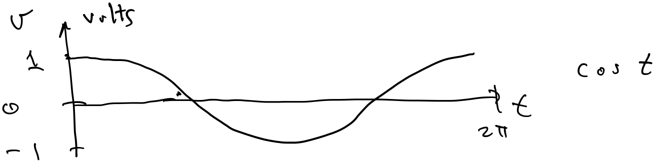
$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

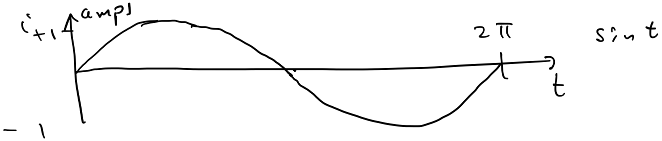
$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} e^{jt} + e^{-jt} \\ -je^{jt} + je^{-jt} \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

Lab exp. - perfect 1F super cap
perfect 1H inductor [super-conducting]







trig: $\cos^2 t + \sin^2 t = 1$

Not a co-incidence:

$v = \cos t$; $i = \sin t$

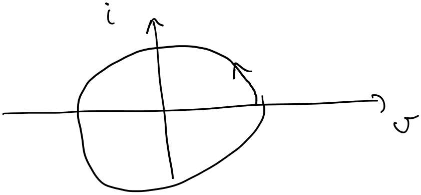
$C = 1\,F$; $L = 1\,H$

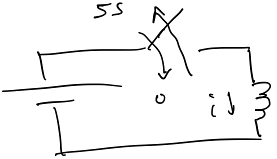
$\frac{1}{2} C v^2$ = energy in cap

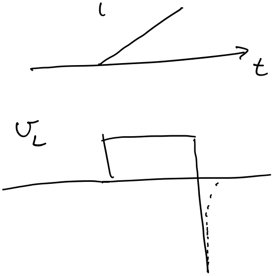
$\frac{1}{2} L i^2$ = energy in inductor

Then, what physical quantity is important, and what physics principle is illustrated?

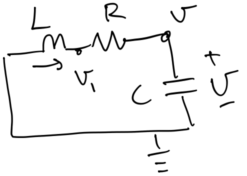
Oscilloscope in "x-y" mode







Ex 3 L-R-C ckt



Node eqns

(v_1) $-i + \frac{v_1 - v}{R} = 0$

(v) $\frac{v - v_1}{R} + C\frac{d}{dt}v = 0$

③ $L\frac{d}{dt}i + v_1 = 0$

use 1st eq. to eliminate v_1 in 2nd & 3rd eqns

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}}_{A}\begin{bmatrix} v \\ i \end{bmatrix}$$

Analysis

$$0 = \det(\lambda I - A)$$

$$0 = \det\begin{bmatrix} \lambda & -1/C \\ 1/L & \lambda + R/L \end{bmatrix}$$

$$0 = \lambda^2 + \lambda\frac{R}{L} + \frac{1}{LC}$$

roots : quadratic formula

$$\lambda_{1,2} = -\frac{1}{2}\frac{R}{L} \pm \sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}}$$