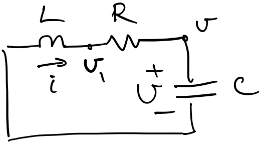
LRC Circuits and Intro to Phasors

Thursday, September 24, 2020 9:25 AM

Last Time

Ex 3



(i) Node eqn's :

(ii) Simplify ⇒ vector diff eq [aka state-space model]

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}}_{A}\begin{bmatrix} v \\ i \end{bmatrix}$$

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

Analysis

$$0 = \det(\lambda I - A) = \lambda^2 + \lambda\frac{R}{L} + \frac{1}{LC}$$

roots: $\lambda_{1,2} = -\frac{1}{2}\frac{R}{L} \pm \sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}}$

How things scale as R increases:

1) $R = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{-\frac{1}{LC}} = \pm j\sqrt{\frac{1}{LC}}$ ✓

purely imag.

$\updownarrow$

2) $\left(\frac{1}{2}\frac{R}{L}\right)^2 = \frac{1}{LC} \Rightarrow \sqrt{(\quad)} = 0$

$\lambda_{1,2} = -\frac{1}{2}\frac{R}{L}, \; -\frac{1}{2}\frac{R}{L}$

purely real, repeated roots

3) $\left(\frac{1}{2}\frac{R}{L}\right)^2 > \frac{1}{LC} \Rightarrow \lambda_{1,2}$ real and distinct

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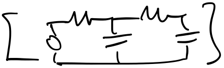
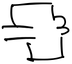
Already looked at case with

- 2 real roots  ✓

- pure L-C  , we get this with $R = 0$ ✓

Interesting case $R \neq 0$

$$\left(\frac{1}{2}\frac{R}{L}\right)^2 < \frac{1}{LC}$$

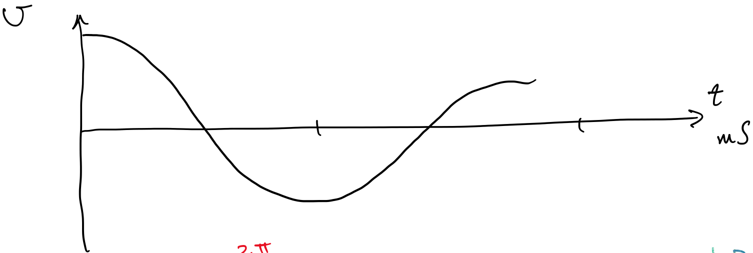
We get 2 roots, complex conjugate

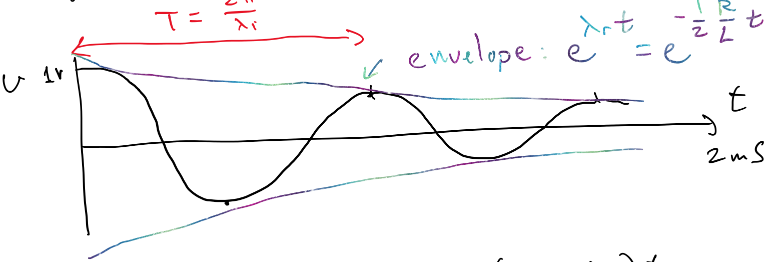
$$\lambda_{1,2} = \underbrace{-\frac{1}{2}\frac{R}{L}}_{\lambda_r} \pm j \underbrace{\sqrt{\frac{1}{LC} - \left(\frac{1}{2}\frac{R}{L}\right)^2}}_{\lambda_i}$$

$$= \lambda_r \pm j \lambda_i$$

What kind of time respones occur?

$v(0) = 1\,V$; $i(0) = 0\,A$





$T = \frac{2\pi}{\lambda_i}$

envelope: $e^{\lambda_r t} = e^{-\frac{1}{2}\frac{R}{L}t}$

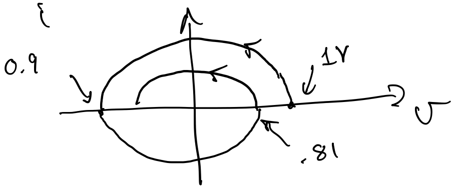
time response: $e^{\lambda t} = e^{(\lambda_r + j\lambda_i)t}$

$$= e^{\lambda_r t}\, e^{j\lambda_i t}$$

$$= e^{\lambda_r t}\left[\cos\lambda_i t + j\sin\lambda_i t\right]$$

$$\Rightarrow \quad \mathrm{Re}\{\quad\} = e^{\lambda_r t}\cos\lambda_i t$$

- called a damped oscillation

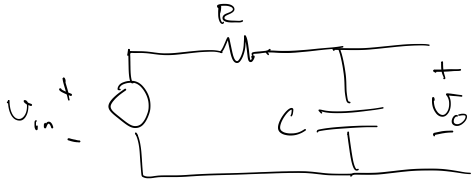


o-scope in x-y mode

Intro to Phasors:

Method for analyzing linear circuits in response to sinusoidal inputs

- all at one frequency
- can get brave and generalize to multiple freq's. *Not today.*



Diff Eqn:

$$\frac{d}{dt} v = -\frac{1}{RC} v + \frac{1}{RC} v_{in} \qquad \left[\lambda = -\frac{1}{RC}\right]$$

$\Rightarrow v_{in} = V_{in} \cos \omega t$; V_{in} is real amplitude in volts

Aside: $v_{in} = V_{in} e^{st}$

guess & check: $v_o = V_o e^{st}$

$$\underbrace{s V_o e^{st}}_{\frac{d}{dt} v_o} = -\frac{1}{RC} V_o e^{st} + \frac{1}{RC} V_{in} e^{st}$$

$$V_o = \frac{1}{1 + sRC} V_{in}$$

$$V_o = \frac{1}{1 - \frac{s}{\lambda}} V_{in}$$

time domain resp:

$$v_o(t) = V_o e^{st} + \underbrace{A}_{\leftarrow \text{ computed to match init. cond.}} e^{\lambda t}$$

homog. resp.: neg. real λ

$$e^{\lambda t} \rightsquigarrow 0 \quad \text{as} \quad t \to \infty$$

response converges to $v_o(t) = V_o e^{st}$

$$s = j\omega \Rightarrow e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$\cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \mathrm{Re}\{\cos\omega t + j\sin\omega t\}$$

Consider inputs like

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

This is the Phasor Method:

What is a Phasor?

$$x(t) = A\cos(\omega t + \phi)$$

general sine wave with freq: ω

char. by $\begin{cases} \text{amplitude } A \\ \text{phase } \phi \end{cases}$

$$= \mathrm{Re}\left[\boxed{Ae^{j\phi}}\, e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)\right]$$

Phasor =: complex number, often with units like voltage, current, etc, that has magnitude matching cos waveform of interest, and phase matching same waveform.

Facts

1) uniqueness
2) linearity
3) differentiation property

Uniqueness

$$x(t) = A\cos(\omega t+\phi) \longleftrightarrow Ae^{j\phi}$$

consider

$$x_1(t) = \mathrm{Re}\left[A_1 e^{j\omega t}\right] \longleftrightarrow A_1$$

$$\uparrow$$

$$A_1 = |A_1| e^{j\phi_1}$$

$$x_2(t) = \mathrm{Re}\left[A_2 e^{j\omega t}\right] \longleftrightarrow A_2$$

$$\uparrow$$

$$A_2 = |A_2| e^{j\phi_2}$$

(i) Let's say $A_1 = A_2$; then by def. $x_1(t) = x_2(t)$ ✓

(ii) Let's say $x_1(t) = x_2(t)$ for all time t

* check at $t = 0$

$$x_1(t=0) = \mathrm{Re}[A_1]$$

$$x_2(t=0) = \mathrm{Re}[A_2] \Rightarrow \boxed{\mathrm{Re}[A_1] = \mathrm{Re}[A_2]}$$

* check at $\frac{1}{4}\,\frac{2\pi}{\omega}$

$$x_1\left(t = \frac{1}{4}\,\frac{2\pi}{\omega}\right) = \mathrm{Re}\left[A_1 e^{j\omega\frac{2\pi}{4\omega}}\right]$$

$$= \mathrm{Re}\left[A_1 e^{j\frac{\pi}{2}}\right] = \mathrm{Re}[jA_1]$$

$$e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$$

$$= \mathrm{Re}\left[j\left(\mathrm{Re}(A_1) + j\,\mathrm{Im}(A_1)\right)\right]$$

$$= \mathrm{Re}\left[j\,\mathrm{Re}\,A_1 - \mathrm{Im}\,A_1\right]$$

$$= -\mathrm{Im}(A_1)$$

repeat for $x_2(t)$

$$x_2\left(t=\frac{1}{4}\,\frac{2\pi}{\omega}\right) = \cdots = -\mathrm{Im}(A_2)$$

$$X_1 = X_2 \;\Rightarrow\; \boxed{\mathrm{Im}(A_1) = \mathrm{Im}(A_2)}$$

$$\Rightarrow \boxed{A_1 = A_2} \qquad \checkmark$$

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(2) Linearity

$$x_1(t) = \mathrm{Re}\left[A_1 e^{j\omega t}\right]$$
$$x_2(t) = \mathrm{Re}\left[A_2 e^{j\omega t}\right]$$

- real scalar coeff.'s $\{a_1, a_2\}$

$$a_1 x_1(t) + a_2 x_2(t)$$
$$= a_1 \mathrm{Re}\left[A_1 e^{j\omega t}\right] + a_2 \mathrm{Re}\left[A_2 e^{j\omega t}\right]$$
$$= \mathrm{Re}\left[a_1 A_1 e^{j\omega t}\right] + \mathrm{Re}\left[a_2 A_2 e^{j\omega t}\right]$$
$$= \mathrm{Re}\left[a_1 A_1 e^{j\omega t} + a_2 A_2 e^{j\omega t}\right]$$
$$= \mathrm{Re}\left[\underbrace{(a_1 A_1 + a_2 A_2)}_{\text{phasor that corresp. to time-waveform } a_1 X_1(t) + a_2 X_2(t)} e^{j\omega t}\right] \qquad \checkmark$$

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(3) Derivatives

$$x(t) = \text{Re}\left[A e^{j\omega t}\right]$$

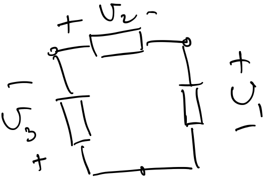
$$\frac{d}{dt} x(t) = \text{Re}\left[\underbrace{j\omega A}_{\text{phasor for } \frac{d}{dt}x(t)} e^{j\omega t}\right]$$

$$x(t) \longleftrightarrow A$$

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega A$$

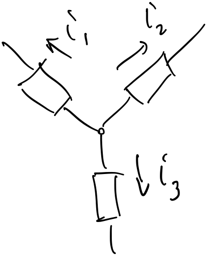
Appl. to Circuits

KVL :



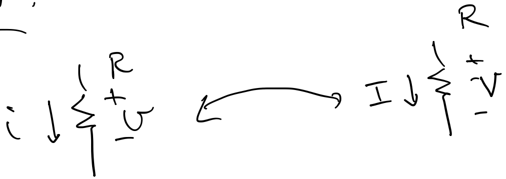
$$\sum_{l \text{ in loop}} v_l = 0 \underset{\underset{\text{linearity prop.}}{\Uparrow}}{\longleftrightarrow} \sum_{l \text{ in loop}} V_l = 0$$

KCL:



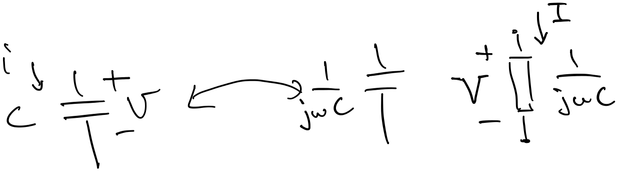
$$\sum_{k \text{ incident at node}} i_k = 0 \Longleftrightarrow \sum_{k \text{ incident at node}} I_k = 0$$

Elements:



$v = Ri$

$V = RI$



$i = C\frac{d}{dt}v$

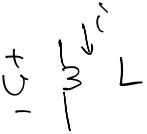
$I = j\omega C V$

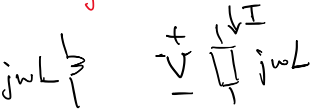
derivative rule

$V = \boxed{\frac{1}{j\omega C}} I$

Impedance

– generalizes resistance





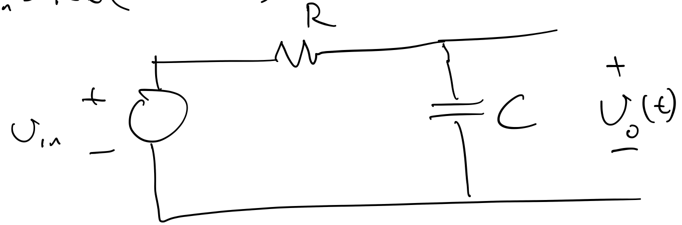
$$v = L\frac{d}{dt}i$$

$$V = \boxed{j\omega L}\, I$$

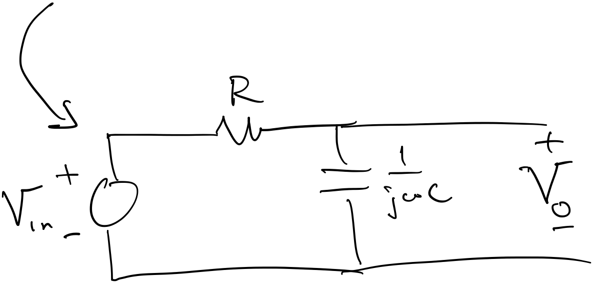
impedance

Ex 1 (again)

$$v_{in} = \mathrm{Re}\left(V_{in} e^{j\omega t}\right)$$



phasor domain



ckt analysis

- usual analysis methods apply
- voltage divider in this example:

$$V_o = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \cdot V_{in}$$

$$= \frac{1}{1 + j\omega CR} \cdot V_{in}$$

[already introduced, but not fully explained]