Phasors, Frequency Response, and Bode Plots

Tuesday, September 29, 2020 9:17 AM

Phasor Summary

1) Analysis at single freq, usually in rad/s

$$\omega = 2\pi f \longleftrightarrow T = \frac{1}{f}$$

Aside: some examples may involve multiple frequencies:

$$u = U_1 \cos(\omega_1 t + \phi_1) + U_2 \cos(\omega_2 t + \phi_2)$$

Method still works —

we use superposition

(i) uniqueness

$$x(t) = A\cos(\omega t + \phi) \longleftrightarrow X = Ae^{j\phi}$$

↖ pos., real

(ii) linearity

$$x_1(t) = A_1 \cos(\omega t + \phi_1) \longleftrightarrow X_1 = A_1 e^{j\phi_1}$$

$$x_2(t) = A_2 \cos(\omega t + \phi_2) \longleftrightarrow X_2 = A_2 e^{j\phi_2}$$

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1 + a_2 X_2$$

KVL $\leftrightarrow$ KVL phasor domain

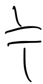
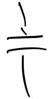
KCL $\leftrightarrow$ KCL phasor domain

(iii) Differentiation

$$x(t) = A\cos(\omega t + \phi) \leftrightarrow X = A e^{j\phi}$$

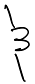
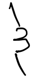
$$\frac{d}{dt} x(t) = -\omega A \sin(\omega t + \phi) \leftrightarrow j\omega X = j\omega A e^{j\phi}$$

Circuit elements


$i = C \frac{d}{dt} v \longleftrightarrow$

$\frac{1}{j\omega C}$ = impedance

$$I = j\omega C V$$

$$V = \frac{1}{j\omega C} I$$


$v = L \frac{d}{dt} i \leftrightarrow$

$j\omega L$ = impedance

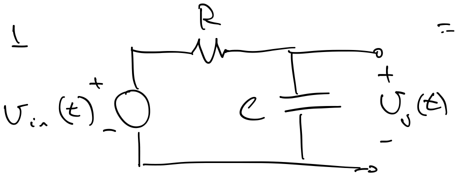
$$V = j\omega L I$$

Last Time

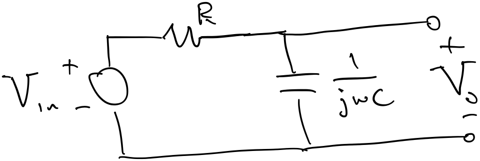
$v_{in}(t) = V_{in}\cos(\omega t)$

$= \mathrm{Re}(V_{in}e^{j\omega t})$

Ex 1



"Phasorize"



Circuit Analysis – voltage divider

$$\frac{V_o}{V_{in}} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+R} = \boxed{\frac{1}{1+j\omega CR}}$$

$$\frac{V_o}{V_{in}}(j\omega) = H(j\omega) =$$ "Transfer Function"

"System Function"

- Relates Output to Input at freq. ω:
  - scales input amplitude by $|H(j\omega)|$

- adjusts phase of input by $\angle H(j\omega)$

$$H(j\omega) = |H(j\omega)| e^{j \angle H(j\omega)}$$

- Look at many frequencies. Can basically scan over ω ⇒ "Frequency Response"

Audio:

- people say audible range $\overbrace{20\text{ Hz} - 20\text{ kHz}}^{3\text{ decades}}$
- "Dynamic Range" = range of amplitude of signal

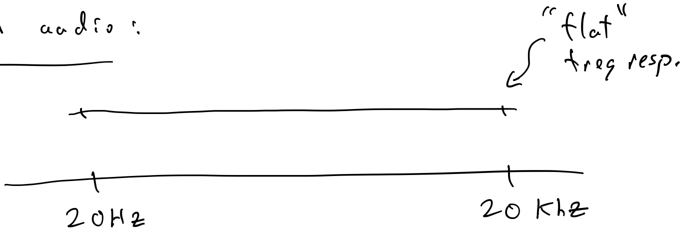
meaning: ratio of largest amplitude to smallest amplitude in signal of interest

~ 20 bit : 10 bits ⟷ $1024 \cong 10^3$

└ 10^6 = 6 decades

* Freq Response data represented on log-log scale, (almost always)

Ideal audio:



$$H(j\omega) = \frac{1}{1 + j\omega CR}$$

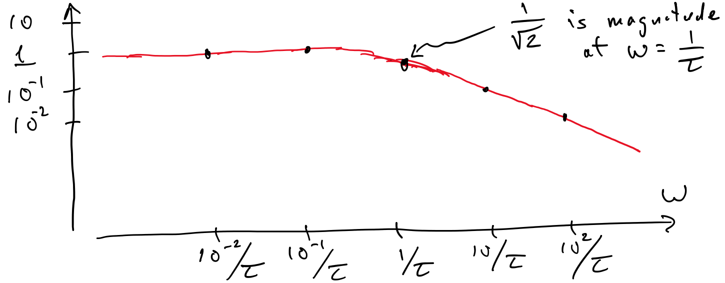
Interest in Magnitude and Angle

$$|H(j\omega)| = \left|\frac{1}{1+j\omega RC}\right| = \frac{1}{|1+j\omega RC|}$$

$$= \frac{1}{\sqrt{1 + \omega^2 (RC)^2}}$$

$$\boxed{\tau = RC}$$

Construct graph that has informative presentation [Magnitude part of Bode plot]



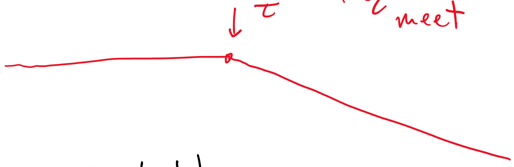
Freq $\omega = \frac{1}{\tau}$ = "cut-off" frequency

"corner" frequency

Cut-off freq: RC ckt is a low-pass filter.

- signals with freq $< \frac{1}{\tau}$ not attenuated very much
- signals with freq $> \frac{1}{\tau}$ are attenuated

Corner freq:



$\frac{1}{\tau}$ = freq where asymptotes meet

For magnitude sketch:

rule is: fill in asymptotes,

- identify "corners" where asymptotes meet. [Label these]
- remember for meeting of 2 asymptotes, amplitude factor is $1/\sqrt{2}$ or $\sqrt{2}$, etc.

Other Part of Freq Response in ∡ plot:

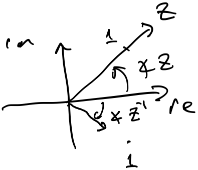
$$H(j\omega) = \frac{1}{1+j\omega RC} = \frac{1-j\omega RC}{1+\omega^2(RC)^2}$$

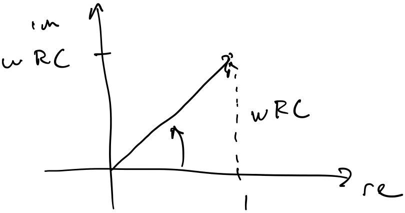
$$= \frac{1-j\omega\tau}{\underbrace{1+\omega^2\tau^2}_{\text{denom is real}}}$$

$$\measuredangle H(j\omega) = \measuredangle(1 - j\omega\tau)$$

Could also have looked at denominator directly:

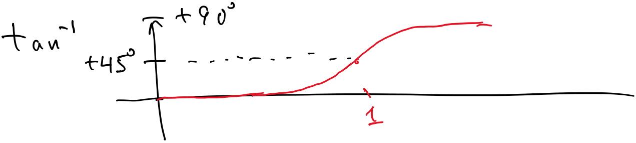
$$\measuredangle \frac{1}{1+j\omega RC} = -\measuredangle(1+j\omega RC)$$

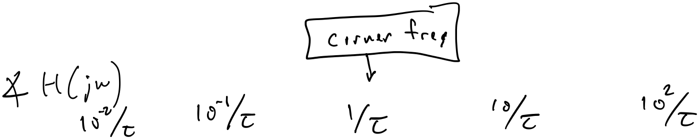


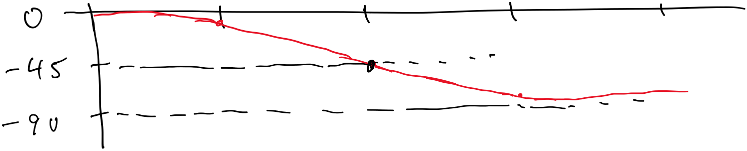


$$\measuredangle(1+j\omega RC) = \tan^{-1}(\omega RC)$$

$$\measuredangle H(j\omega) = \measuredangle \frac{1}{1+j\omega RC} = -\tan^{-1}(\omega RC)$$





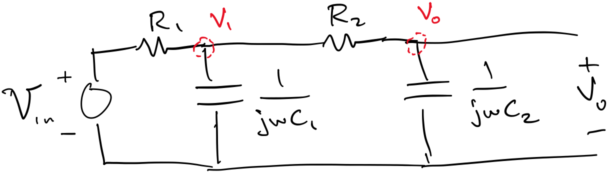


What is a Bode Plot?

(1) Plot of magnitude of $H(j\omega)$ vs ω on log-log scale

(2) Plot $\measuredangle H(j\omega)$ vs ω on angle vs. log scale freq.

Ex2: $v_{in} = \mathrm{Re}\left[V_{in} e^{j\omega t}\right]$



① $$\frac{V_1 - V_{in}}{R_1} + j\omega C_1 V_1 + \frac{V_1 - V_o}{R_2} = 0$$

② $$\frac{V_o - V_1}{R_2} + j\omega C_2 V_o = 0$$

eliminate V_1

$$V_{in} = \left[1 + j\omega\left(C_1R_1 + C_2R_2 + C_2R_1\right) + (j\omega)^2 C_1C_2R_1R_2\right] V_o$$

$$\frac{V_o}{V_{in}} = \frac{1}{1 + j\omega\left(C_1R_1 + C_2R_2 + C_2R_1\right) + (j\omega)^2 C_1C_2R_1R_2}$$

By inspection: denominator is 2nd order poly. in $(j\omega)$

- Can factor denominator
- Could use quadratic formula

$$\frac{V_o}{V_{in}} = \frac{1}{(1 + j\omega\tau_1)(1 + j\omega\tau_2)}$$

Make Bode Plot:

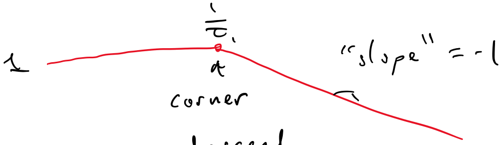
(1) Magnitude – will be on log-log scale

- can treat each of 2 factors separately, and then add together

[property of log fn]

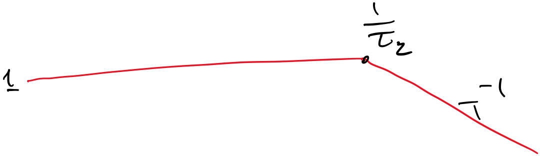
$$H(j\omega) = \frac{1}{1 + j\omega\tau_1} \circ \frac{1}{1 + j\omega\tau_2}$$

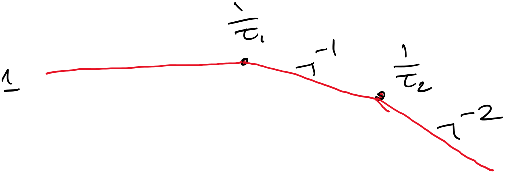
$$\left|\frac{1}{1+j\omega\tau_1}\right|$$



"slope" = -1 means we descend by factor of 10 for each factor of 10 in frequency

$$\left|\frac{1}{1+j\omega\tau_2}\right|$$



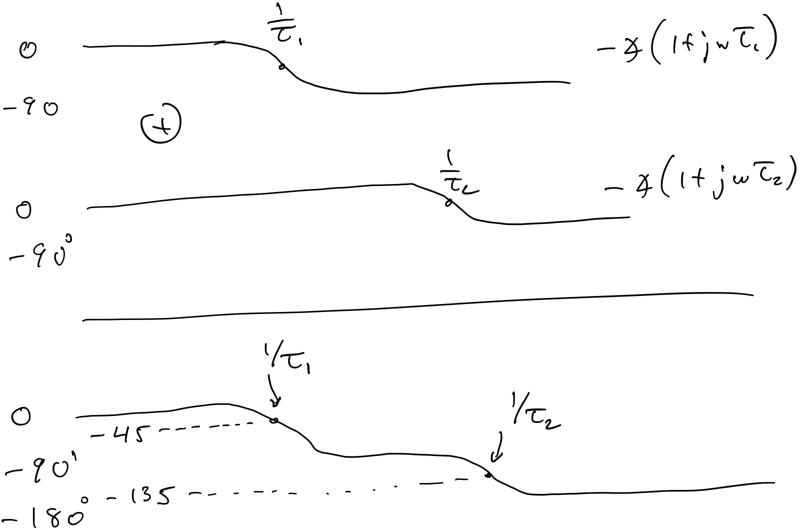


(2) Angle of $H(j\omega)$ vs ω on log scale

$$\measuredangle H(j\omega) = -\measuredangle(1+j\omega\tau_1) - \measuredangle(1+j\omega\tau_2)$$

product of complex #'s: obtain sum of angles

$$z_1 \cdot z_2 = |z_1| e^{j\measuredangle z_1} \cdot |z_2| e^{j\measuredangle z_2}$$

$$= |z_1 z_2| e^{j(\measuredangle z_1 + \measuredangle z_2)}$$