1. **Interpolation with DFT**

In this question, we want to show you how to perform interpolation by adding "zeros" on DFT frequency domain.

Remember we have that $\vec{x} = U\vec{X}$ or more explicitly

$$\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1} \tag{1}$$

That is, $\vec{x}$ is a linear combination of the (normalized) complex exponentials $\vec{u}_i$ with coefficients $X[i]$.

(a) Compute the DFT coefficients $\vec{X}$ for the following signal:

$$\vec{x} = \begin{bmatrix}\cos\left(\frac{2\pi}{6}(0)\right) & \cos\left(\frac{2\pi}{6}(1)\right) & \cos\left(\frac{2\pi}{6}(2)\right) \cos\left(\frac{2\pi}{6}(3)\right) & \cos\left(\frac{2\pi}{6}(4)\right) & \cos\left(\frac{2\pi}{6}(5)\right)\end{bmatrix}^T.$$

(b) As we can see from (a), the DFT coefficients of $\vec{x}$ satisfy with $X[m] = 0$ for all $|m| > 1$. Here, we use the convention of treating $X[n-k]$ as $X[-k]$ for $k < \frac{n}{2}$ where $n$ is the length of the $\vec{x}$ vector. Let's create another set of DFT coefficients:

$$\vec{Y} = [X[0], X[1], X[2], \frac{X[3]}{2}, 0, 0, 0, 0, 0, \frac{X[3]}{2}, X[4], X[5]]^T.$$

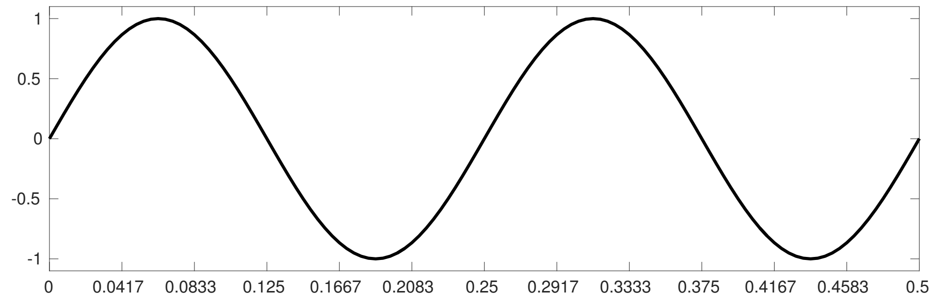
Now use the DFT basis with $n = 12$ to perform an inverse DFT and compute a time domain signal $\vec{y}$. How are $\vec{x}$ and $\vec{y}$ related to each other? Remember that $\vec{x}$ is sampled from $x(t) = \cos\frac{2\pi}{6}t$ at 1Hz.

(c) Based on the above example, how can we perform interpolation for a sampled signal $\vec{x}$ with its DFT coefficients $\vec{X}$ if we know $X[m] = 0$ for all $|m| > k$?

2. **The wagon wheel effect**

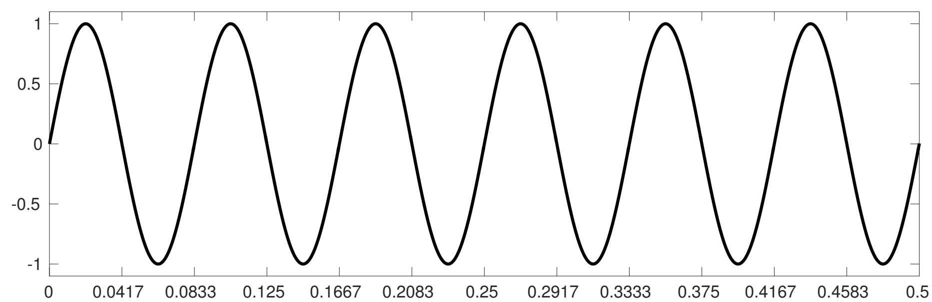
Suppose that we are recording a video of a wheel spinning. Our video camera captures an image of the wheel at a rate of 24 frames per second. Furthermore, let's assume that the wheel is spinning in the XY plane, and the camera is pointed down the Z axis such that it is looking at the wheel straight on. The wheel has a radius of 1 unit, and each image of the wheel is centered such that it can be represented as a plot of the unit circle about the origin of the XY plane.

(a) We want to track the $y$ position of a point on the wheel through time based on what we see in the recorded video. The point starts at (1,0) at time $t = 0$. The wheel spins counter clockwise from the persepctive of the camera at a rate of 4 revolutions per second, and the $y$ position of the point is given by $y(t) = \sin(8\pi t)$. Indicate on the plot below where we will sample the $y$ position of the point from $t = 0$ to $t = 1/2$.



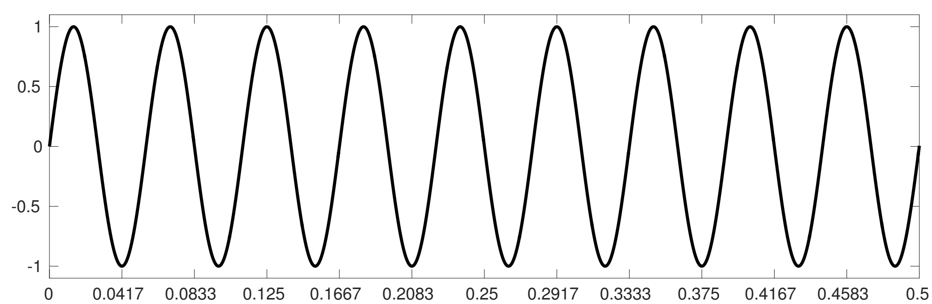

(b) What does this sampled signal look like in the frequency domain when we apply the DFT on the points we sampled from time $0 \leq t < 1/2$?

(c) Now, suppose that the wheel spins at 12 revolutions per second. Indicate on the plot where we will sample the *y* position of the point.




(d) What does the sampled signal look like now in the frequency domain when we apply the DFT on the points we sampled from time $0 \leq t < 1/2$?

(e) Now, let's spin the wheel at 18 revolutions per second. Indicate where the *y* position is sampled.




(f) What does the sampled signal look like in the frequency domain now?

(g) How many frames per second would our camera need to record the wheel spinning at a rate of *n* revolutions per second so that we can accurately represent the *y* position of the point on the wheel over time in the frequency domain?